AN INTERESTING INEQUALITY FOR THE EULER'S GAMMA FUNCTION

NECDET BATIR

Abstract. In this short paper we derived new and interesting upper and lower bounds for the Euler's gamma function in terms of the digamma function $\psi(x) = \Gamma'(x)/\Gamma(x)$. The method we use is a nice application of mean value theorem for differentiation.

1. Introduction

It is well known that the Euler's gamma function $\Gamma(z)$ and the psi or digamma function, the logarithmic derivative of the gamma function, are defined as

$$\Gamma(z) = \int_0^\infty e^{-u} u^{z-1} du, \ \mathbf{Re} z > 0$$

and

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \ x > 0,$$

respectively. The derivatives ψ', ψ'', ψ''', ... are known as polygamma functions.

The gamma function has been investigated intensively by many authors even recent years. In particular many authors published numerous interesting inequalities for this important function (see [2]-[10]). In this note we presented new and interesting upper and lower bounds for this function. Throughout we denote by $c = 1.461632144968362...$ the only positive zero of ψ-function(see [1, p.259, 6.3.19]).

2. Main results

The following theorem is our main result.

Theorem 2.1. *For all $x \geq c = 1.461632144968362...$, the following inequality for the gamma function holds.*

$$\begin{aligned} \Gamma(c)\exp[\psi(x)e^{\psi(x)} - e^{\psi(x)} + 1] &\leq \Gamma(x) \\ &\leq \Gamma(c)\exp\left[k\left(\psi(x)\, e^{\psi(x)} - e^{\psi(x)} + 1\right)\right], \end{aligned} \tag{2.1}$$

where γ is Euler-Mascheroni constant, $\Gamma(c) = 0.885603194410889...$; see [1, p. 259; 6.3.9] and $k = 6e^\gamma/\pi^2 = 1.0827621932609...$.

Proof. Applying mean value theorem to the function $\log\Gamma(x)$ on $[u, u+1]$ with $u > 0$ and using the well known difference equation $\Gamma(u+1) = u\,\Gamma(u)$ for the gamma function, there exists a θ depending on u such that for all $u > 0$, $0 < \theta = \theta(u) < 1$ and

$$\psi(u + \theta\,(u)) = \log u. \tag{2.2}$$

Key words and phrases. Gamma function, psi function, inequalities, polygamma function.
2000 Subject Classification. Primary: 33B15; Secondary: 26D07.

First, we show that the function θ is strictly increasing and θ' is strictly decreasing on $(0,\infty)$. For this purpose put $u=e^{\psi(t)}$ with $t>0$ in (2.2) to obtain

$$\psi(e^{\psi(t)}+\theta(e^{\psi(t)}))=\psi(t).$$

Since the mapping $t\to\psi(t)$ from $(0,\infty)$ to $(-\infty,\infty)$ is bijective, we find that

$$\theta(e^{\psi(t)})=t-e^{\psi(t)},\ t>0. \tag{2.3}$$

Differentiating both sides of this equation, we get

$$\theta'(e^{\psi(t)})=\frac{1}{\psi'(t)\,e^{\psi(t)}}-1. \tag{2.4}$$

By [2, (4.34)] we have

$$\psi'(x)e^{\psi(x)}\quad<\quad 1 \tag{2.5}$$

for $x>0$, so that we conclude $\theta'(e^{\psi(t)})>0, t>0$. But since the mapping $t\to e^{\psi(t)}$ from $(0,\infty)$ to $(0,\infty)$ is also bijective this implies that $\theta'(t)>0$ for all $t>0$. Now differentiate once both sides of (2.3) to obtain

$$\theta''(e^{\psi(t)})=-\frac{e^{-2\psi(t)}}{\psi'(t)^3}[(\psi'(t))^2+\psi''(t)].$$

In [2, (4.39)] it is proved that $[(\psi'(t))^2+\psi''(t)]>0$. Using this inequality we have $\theta''(e^{\psi(t)})<0$ for $t>0$. Proceeding as above we conclude that $\theta''(t)<0$ for all $t>0$.

To prove the theorem integrate both sides of (2.2) over $1\le u\le x$ to obtain

$$\int_1^x\psi(u+\theta(u))\,du=\int_1^x\log u du.$$

Making the change of variable $u=e^{\psi(t)}$ on the left hand side this becomes by (2.2)

$$\int_c^{x+\theta(x)}\psi(t)\ \psi'(t)\ e^{\psi(t)}dt=x\log x-x+1. \tag{2.6}$$

Since $\psi(t)\ge 0$ for all $t\ge c$, and $\psi'(t)e^{\psi(t)}<1$ by (2.5), this gives for $x\ge 0$

$$x\log x-x+1+\log\Gamma(c)\le\log\Gamma(x+\theta(x)).$$

Replace x by $e^{\psi(x)}$ and then employ (2.3) to get

$$\log\Gamma(c)+[\psi(x)e^{\psi(x)}-e^{\psi(x)}+1]\le\log\Gamma(x),$$

which implies the left-hand inequality of (2.1). Since θ' is decreasing we conclude from (2.4) that

$$\theta'(e^{\psi(t)})=\frac{1}{\psi'(t)\,e^{\psi(t)}}-1\le\theta'(e^{\psi(1)})=\frac{6e^\gamma}{\pi^2}-1$$

for all t ≥ 1. This implies that $\psi'(t)\,e^{\psi(t)}>e^{-\gamma}\pi^2/6$ for t ≥ 1. Hence, using (2.6) we arrive at after brief simplification

$$\log\Gamma(x+\theta(x))\le\frac{6e^\gamma}{\pi^2}[\,x\log x-x+1]+\log\Gamma(c).$$

Now replace x by $e^{\psi(x)}$ and then use (2.3) to get the right inequality in (2.1). This completes the proof of the theorem. $\square$

References

[1] M .Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1965.

[2] Horst Alzer, Sharp inequalities for digamma and polygamma functions, *Forum Math.*, 16, 2004, 181-221.

[3] Horst Alzer and S. Ruscheweyh, A subadditive property of the gamma function, *J. Math. Anal.*, 285, 2003, 564-577.

[4] Horst Alzer, On some inequalities for the gamma and psi functions, *Math. Comp.*, 66, No. 217, 1997, 373-389.

[5] Horst Alzer, Some gamma function inequalities, *Math. Comp.*, 60, No. 201, 1993, 337-346.

[6] G. D Anderson and S. L. Qui, A monotonicity property of the gamma function, *Proc. Amer. Math. Soc.*, 125, No. 11, 1997, 3355-3362.

[7] Chao-Ping Chen and Feng Chao-Ping, Monotonicity results for gamma function, *J. Inequal. Pure and Appl. Math.*, 4(2), Art. 44, 2003.

[8] W. E. Clark and M. E. H. Ismail, Inequalities involving gamma and psi function, *Analysis and Applications*, 1, No. 129, 2003, 129-140.

[9] A. Elbert and A. Laforgia, On some properties of the gamma function, *Proc. Amer. Math. Soc.*, 128, No. 9, 2000, 2667-2673.

[10] H. Vogt and J. Voigt, A monotonicity property of the Γ-function, *J. Inequal. Pure and Appl. Math.*, 3(5), Art. 73, 2002.

Department of Mathematics, Faculty of Art and Science, Yuzuncu Yil University, 65080, Van, Turkey

E-mail address: `necdet_batir@hotmail.com`