THE NEW BOUNDS OF THE LOGARITHMIC MEAN

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ABSTRACT. In this paper, using the generalized power-type Heron mean and its dual form, we establish the new bounds of the logarithmic mean

$$G(a,b) \leqslant h_r(a,b;k_2) \leqslant L(a,b) \leqslant H_p(a,b;k_1) \leqslant M_q(a,b).$$

Furthermore, the constants $p = \frac{k_1}{k_1+2}, q = \frac{1}{3}$, and $r = \frac{k_2+1}{k_2-1}$ $(k_2 > 1)$ are the best possible.

1. Introduction

For positive numbers a, b, let

$$A = A(a,b) = \frac{a+b}{2}; \tag{1.1}$$

$$G = G(a,b) = \sqrt{ab}; \tag{1.2}$$

$$L = L(a,b) = \begin{cases} \dfrac{a-b}{\ln a - \ln b}, & a \neq b; \\ a, & a = b; \end{cases} \tag{1.3}$$

$$H = H(a,b) = \frac{a+\sqrt{ab}+b}{3}. \tag{1.4}$$

These are respectively called the arithmetic, geometric, logarithmic, and Heron means.

Let r be a real number, the r-order power mean (see [1]) is defined by

$$M_r = M_r(a,b) = \begin{cases} \left(\dfrac{a^r+b^r}{2}\right)^{\frac{1}{r}}, & r \neq 0; \\ \sqrt{ab}, & r = 0. \end{cases} \tag{1.5}$$

The well-known T.-P. Lin inequality (see also [1]) is stated as

$$G \leqslant L \leqslant M_{\frac{1}{3}}. \tag{1.6}$$

In 1993, the following interpolation inequalities are summarized and stated by J.-Ch. Kuang in [1]

$$G \leqslant L \leqslant M_{\frac{1}{3}} \leqslant M_{\frac{1}{2}} \leqslant H \leqslant M_{\frac{2}{3}} \leqslant A. \tag{1.7}$$

In [2], G. Jia and J.-D. Cao studied the power-type generalization of Heron mean

$$H_r = H_p(a,b) = \begin{cases} \left[\dfrac{a^r+(ab)^{r/2}+b^r}{3}\right]^{\frac{1}{r}}, & r \neq 0; \\ \sqrt{ab}, & r = 0; \end{cases} \tag{1.8}$$

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and obtained inequalities

$$L \leqslant H_p \leqslant M_q, \tag{1.9}$$

where $p \geqslant \frac{1}{2}, q \geqslant \frac{2}{3}p$. Furthermore, $p = \frac{1}{2}, q = \frac{1}{3}$ are the best constants.

In 2003, Zh.-G. Xiao and Zh.-H. Zhang [3] gave another generalization of Heron mean and its dual form respectively as follows

$$H(a,b;k) = \frac{1}{k+1}\sum_{i=0}^{k} a^{\frac{k-i}{k}} b^{\frac{i}{k}}, \tag{1.10}$$

and

$$h(a,b;k) = \frac{1}{k}\sum_{i=1}^{k} a^{\frac{k+1-i}{k+1}} b^{\frac{i}{k+1}}, \tag{1.11}$$

where k is a natural number. Authors proved that $H(a,b;k)$ is monotone decreasing function and $h(a,b;k)$ is monotone increasing function for k, and $\lim_{k\to+\infty} H(a,b;k) = \lim_{k\to+\infty} h(a,b;k) = L(a,b)$.

Combining (1.8), (1.10) and (1.11), in [5], two class of new means for two variables are defined. Let $a > 0, b > 0,$ k is a natural number, and r is a real number, then the generalized power-type Heron mean and its dual form are defined as follows

$$H_r(a,b;k) = \begin{cases} \left[\dfrac{1}{k+1}\displaystyle\sum_{i=0}^{k} a^{\frac{(k-i)r}{k}} b^{\frac{ir}{k}}\right]^{\frac{1}{r}}, & r \neq 0; \\ \sqrt{ab}, & r = 0; \end{cases} \tag{1.12}$$

and

$$h_r(a,b;k) = \begin{cases} \left[\dfrac{1}{k}\displaystyle\sum_{i=1}^{k} a^{\frac{(k+1-i)r}{k+1}} b^{\frac{ir}{k+1}}\right]^{\frac{1}{r}}, & r \neq 0; \\ \sqrt{ab}, & r = 0. \end{cases} \tag{1.13}$$

In the end [5], authors put the following:

Open problem: Prove that, if k_1, k_2 are two fixed natural number, and $p \geqslant \frac{k_1}{k_1+2}, q \geqslant \frac{(k_1+2)p}{3k_1}, 0 \leqslant r \leqslant \frac{k_2+1}{k_2-1}$, then the following inequalities for the new bounds of the logarithmic mean

$$G(a,b) \leqslant h_r(a,b;k_2) \leqslant L(a,b) \leqslant H_p(a,b;k_1) \leqslant M_q(a,b). \tag{1.14}$$

hold, and the constants $p = \frac{k_1}{k_1+2}, q = \frac{1}{3}$, and $r = \frac{k_2+1}{k_2-1}$ $(k_2 > 1)$ are the best possible.

In this paper, we solve the open problem above.

2. Lemmas

In order to prove the theorem of the next section, we require some lemmas in this section.

Lemma 2.1. *Let $a > 0, b > 0,$ k is a natural number, and r is a real number, then the generalized power-type Heron mean $H_r(a,b;k)$ and its dual form $h_r(a,b;k)$ can be written that*

$$H_r(a,b;k) = \begin{cases} \left[\dfrac{a^{\frac{(k+1)r}{k}} - b^{\frac{(k+1)r}{k}}}{(k+1)(a^{\frac{r}{k}} - b^{\frac{r}{k}})}\right]^{\frac{1}{r}}, & r \neq 0, a \neq b; \\ \sqrt{ab}, & r = 0, a \neq b; \\ a, & r \in R, a = b; \end{cases} \tag{2.1}$$

and

$$h_r(a,b;k)=\begin{cases}\left[\dfrac{a^{\frac{kr}{k+1}}-b^{\frac{kr}{k+1}}}{-k(a^{-\frac{r}{k+1}}-b^{-\frac{r}{k+1}})}\right]^{\frac{1}{r}}, & r\neq 0, a\neq b;\\ \sqrt{ab}, & r=0, a\neq b;\\ a, & r\in R, a=b;\end{cases}\tag{2.2}$$

Lemma 2.2. *If k is a fixed natural number, then $H_r(a,b;k)$ and $h_r(a,b;k)$ is monotone increasing function with r for fixed positive numbers a and b.*

Lemma 2.3. [3] *For any $r>0$, we have that $H_r(a,b;k)$ is monotonic decreasing function, and $h_r(a,b;k)$ is monotone increasing function with k.*

Lemma 2.4. *([4]) Let p,q,u,v be arbitrary with $p\neq q, u\neq v$, and*

$$E_{p,q}(a,b)=\begin{cases}\left[\dfrac{q}{p}\cdot\dfrac{a^p-b^p}{a^q-b^q}\right]^{1/(p-q)}, & pq(p-q)(a-b)\neq 0;\\ \left[\dfrac{1}{p}\cdot\dfrac{a^p-b^p}{\ln a-\ln b}\right]^{1/p}, & p(a-b)\neq 0, q=0;\\ \dfrac{1}{e}\left(\dfrac{a^a}{b^b}\right)^{\frac{1}{a-b}}, & p(a-b)\neq 0, p=q;\\ \sqrt{ab}, & (a-b)\neq 0, p=q=0;\\ a, & a=b.\end{cases}\tag{2.3}$$

Then the inequality

$$E_{p,q}(a,b)\geqslant E_{u,v}(a,b)\tag{2.4}$$

is satisfied for all $a,b>0, a\neq b$ if and only if

$$p+q\geqslant u+v,\tag{2.5}$$

and

$$e(p,q)\geqslant e(u,v),\tag{2.6}$$

where

$$e(x,y)=\begin{cases}(x-y)/\ln(x/y), & for\ xy>0, x\neq y;\\ 0, & for\ xy=0;\end{cases}\tag{2.7}$$

if either $0\leqslant\min\{p,q,u,v\}$ or $\max\{p,q,u,v\}\leqslant 0$; and

$$e(x,y)=(\mid x\mid-\mid y\mid)/(x-y),\ \ for\ x,y\in \mathrm{R}, x\neq y,\tag{2.8}$$

if either $\min\{p,q,u,v\}<0<\max\{p,q,u,v\}$.

Lemma 2.5. *If k is a natural number. Then*

$$(k+2)^{k(k+3)}\geqslant (k+1)^{(k+1)(k+2)},\tag{2.9}$$

or

$$\frac{k}{(k+2)\ln(k+1)}\geqslant\frac{k+1}{(k+3)\ln(k+2)}.\tag{2.10}$$

Proof. When $k = 1, 2$, we have $(1 + 2)^{1\cdot(1+3)} = 81 > 64 = (1 + 1)^{(1+1)(1+2)}$, and

$$(2 + 2)^{2\cdot(2+3)} = 1048576 > 531441 = (2 + 1)^{(2+1)(2+2)},$$

respectively. i.e. (2.9) or (2.10) holds.

If $k \geqslant 3$, then we have

$$\frac{k^3}{6} \geqslant \frac{k^2}{2}, \ \frac{k^4}{24} \geqslant k, \tag{2.11}$$

and

$$k(k + 3) - i \geqslant k(k + 1), 1 \leqslant i \leqslant 3. \tag{2.12}$$

Using the binomial theorem, we obtain

$$\begin{aligned} \left(1 + \frac{1}{k+1}\right)^{k(k+3)} &= 1 + \frac{k(k+3)}{k+1} + \frac{k(k+3)[k(k+3)-1]}{2(k+1)^2} \\ &+ \frac{k(k+3)[k(k+3)-1][k(k+3)-2]}{6(k+1)^3} \\ &+ \frac{k(k+3)[k(k+3)-1][k(k+3)-2][k(k+3)-3]}{24(k+1)^4} + \cdots \end{aligned} \tag{2.13}$$

From (2.11)-(2.13), we get

$$\begin{aligned} \left(1 + \frac{1}{k+1}\right)^{k(k+3)} &> 1 + k + \frac{k^2}{2} + \frac{k^3}{6} + \frac{k^4}{24} \\ &\geqslant 1 + k + \frac{k^2}{2} + \frac{k^2}{2} + k = 1 + 2k + k^2 = (k+1)^2 \end{aligned} \tag{2.14}$$

Rearranging (2.14) we immediately find (2.9) or (2.10). The proof of Lemma2.5 is completed. ∎

Lemma 2.6. *For fixed positive numbers a and b, $H_{\frac{k}{k+2}}(a, b; k)$ is monotonic decreasing function and $h_{\frac{k+1}{k-1}}(a, b; k)$ is monotone increasing function with k .*

Proof. Firstly, from Leema2.1, the proof of monotone decreasing for $H_{\frac{k}{k+2}}(a, b; k)$ is equivalent to the inequality

$$\left[\frac{a^{\frac{k+1}{k+2}} - b^{\frac{k+1}{k+2}}}{(k+1)(a^{\frac{1}{k+2}} - b^{\frac{1}{k+2}})}\right]^{\frac{k+2}{k}} \geqslant \left[\frac{a^{\frac{k+2}{k+3}} - b^{\frac{k+2}{k+3}}}{(k+2)(a^{\frac{1}{k+3}} - b^{\frac{1}{k+3}})}\right]^{\frac{k+3}{k+1}}, \tag{2.15}$$

where k is a natural number.

Setting $p_1 = \frac{k+1}{k+2}, q_1 = \frac{1}{k+2}, u_1 = \frac{k+2}{k+3}$, and $v_1 = \frac{1}{k+3}$, that (2.15) become

$$E_{p_1,q_1}(a, b) \geqslant E_{u_1,v_1}(a, b). \tag{2.16}$$

It is easy to see that

$$\min\{p_1, q_1, u_1, v_1\} = \frac{1}{k+3} > 0, \tag{2.17}$$

and

$$p_1 + q_1 = 1 = u_1 + v_1. \tag{2.18}$$

From Lemma2.5, we find that

$$e(p_1, q_1) = \frac{k}{(k+2)\ln(k+1)} \geqslant \frac{k+1}{(k+3)\ln(k+2)} = e(u_1, v_1), \tag{2.19}$$

where $e(x, y)$ is defined as (2.7) of Lemma2.4.

Using Lemma2.4, and combining expression (2.17)-(2.19), we can obtain (2.16), and immediately follow that expression (2.15) is true. Thus, the proof of monotone decreasing for $H_{\frac{k}{k+2}}(a,b;k)$ is completed.

We secondly prove that $h_{\frac{k+1}{k-1}}(a,b;k)$ is monotone increasing function with k, it is to prove the inequality

$$\left[\frac{a^{\frac{k}{k-1}}-b^{\frac{k}{k-1}}}{-k(a^{-\frac{1}{k-1}}-b^{-\frac{1}{k-1}})}\right]^{\frac{k-1}{k+1}} \leqslant \left[\frac{a^{\frac{k+1}{k}}-b^{\frac{k+1}{k}}}{-(k+1)(a^{-\frac{1}{k}}-b^{-\frac{1}{k}})}\right]^{\frac{k}{k+2}}, \tag{2.20}$$

holding, where k is a natural number.

Taking $p_2=\frac{k+1}{k}, q_2=-\frac{1}{k}, u_2=\frac{k}{k-1}$, and $v_2=-\frac{1}{k-1}$, that (2.20) is

$$E_{u_2,v_2}(a,b) \leqslant E_{p_2,q_2}(a,b). \tag{2.21}$$

We easily know that

$$\min\{p_2,q_2,u_2,v_2\} = -\frac{1}{k-1} < 0 < \frac{k}{k-1} = \max\{p_2,q_2,u_2,v_2\}, \tag{2.22}$$

$$p_2+q_2 = 1 = u_2+v_2, \tag{2.23}$$

and

$$e(p_2,q_2) = \frac{k}{k+2} \geqslant \frac{k-1}{k+1} = e(u_2,v_2), \tag{2.24}$$

where $e(x,y)$ is defined as (2.8) of Lemma2.4.

Using Lemma2.4, and combining expression (2.22)-(2.24), we similarity have (2.21).

Therefore, Theorem2.6 is proved. ∎

3. The New Bounds of the Logarithmic Mean

The following inequalities for the new bounds of the logarithmic mean are interesting.

Theorem 3.1. *Let k_1, k_2 are two fixed natural number, and $p \geqslant \frac{k_1}{k_1+2}, q \geqslant \frac{(k_1+2)p}{3k_1}, 0 \leqslant r \leqslant \frac{k_2+1}{k_2-1}$, we then have inequalities*

$$G(a,b) \leqslant h_r(a,b;k_2) \leqslant L(a,b) \leqslant H_p(a,b;k_1) \leqslant M_q(a,b). \tag{3.1}$$

Furthermore, $p=\frac{k_1}{k_1+2}, q=\frac{1}{3}$, and $r=\frac{k_2+1}{k_2-1}$ ($k_2>1$) are the best constants for (3.1). With equalities holding if and only if $a=b$.

Proof. We first prove, for $p=\frac{k_1}{k_1+2}, q=\frac{1}{3}$, and $r=\frac{k_2+1}{k_2-1}$ ($k_2>1$), that (3.1) is true.

Using Lemma2.6, we have

$$H_{\frac{i_1}{i_1+2}}(a,b;i_1) \leqslant H_{\frac{k_1}{k_1+2}}(a,b;k_1) \leqslant H_{\frac{j_1}{j_1+2}}(a,b;j_1), \tag{3.2}$$

and

$$h_{\frac{j_2+1}{j_2-1}}(a,b;j_2) \leqslant h_{\frac{k_2+1}{k_2-1}}(a,b;k_2) \leqslant h_{\frac{i_2+1}{i_2-1}}(a,b;i_2), \tag{3.3}$$

where i_t, j_t, k_t are three positive natural numbers which satisfy $j_t \leqslant k_t \leqslant i_t$ $(t=1,2)$.
Let $i_t \to \infty$, and $j_t=1,\ t=1,2$, from Proposition 1.1 of [5], those (3.2) and (3.3) are respectively

$$L(a,b) \leqslant H_{\frac{k_1}{k_1+2}}(a,b;k_1) \leqslant M_{\frac{1}{3}}(a,b). \tag{3.4}$$

and

$$G(a,b) \leqslant h_{\frac{k_2+1}{k_2-1}}(a,b;k_2) \leqslant L(a,b). \tag{3.5}$$

For $q \geqslant \frac{(k_1+2)p}{3k_1} > 0$, and altering $a \to a^{\frac{(k_1+2)p}{k_1}}, b \to b^{\frac{(k_1+2)p}{k_1}}$, from the right inequality of (3.4), and Lemma2.2, we get

$$H_p(a,b;k_1) \leqslant M_q(a,b). \tag{3.6}$$

Also, for $0 \leqslant r \leqslant \frac{k_2+1}{k_2-1}$, we similarity obtain

$$G(a,b) \leqslant h_r(a,b;k_2). \tag{3.7}$$

Using Lemma2.2, combining (3.4)-(3.7), we can conclude that

$$G(a,b) \leqslant h_r(a,b;k_2) \leqslant h_{\frac{k_2+1}{k_2-1}}(a,b;k_2) \leqslant L(a,b) \leqslant H_{\frac{k_1}{k_1+2}}(a,b;k_1) \leqslant H_p(a,b;k_1) \leqslant M_q(a,b), \tag{3.8}$$

where $p \geqslant \frac{k_1}{k_1+2}, q \geqslant \frac{(k_1+2)p}{3k_1}$ and $0 \leqslant r \leqslant \frac{k_2+1}{k_2-1}$.

Next, we prove that $p = \frac{k_1}{k_1+2}, q = \frac{1}{3}$, and $r = \frac{k_2+1}{k_2-1}$ $(k_2 > 1)$ are the best constants for (3.1). Assume the following inequalities have holden for any $x > 1$:

$$G(x,1) \leqslant h_r(x,1;k_2) \leqslant L(x,1) \leqslant H_p(x,1;k_1) \leqslant M_q(x,1). \tag{3.9}$$

There is no harm in supposing $1 < x \leqslant 2$ (In fact, if $n < x \leqslant n+1$, we can take $x = t+n$, where n is a positive integer). Setting $x = t+1$, applying Taylor's Theorem to the functions $G(x,1), L(x,1), H_p(x,1;k_1), h_r(x,1;k_2)$ and $M_q(x,1)$, we have

$$G(x,1) = G(t+1,1) = 1 + \frac{1}{2}t - \frac{1}{8}t^2 + \cdots, \tag{3.10}$$

$$L(x,1) = L(t+1,1) = 1 + \frac{1}{2}t - \frac{1}{12}t^2 + \cdots, \tag{3.11}$$

$$H_p(x,1;k_1) = H_p(t+1,1;k_1) = 1 + \frac{1}{2}t + \frac{(k_1+2)p - 3k_1}{24k_1}t^2 + \cdots, \tag{3.12}$$

$$h_r(x,1;k_2) = h_r(t+1,1;k_2) = 1 + \frac{1}{2}t + \frac{(k_2-1)r - 3(k_2+1)}{24(k_2+1)}t^2 + \cdots, \tag{3.13}$$

and

$$M_q(x,1) = M_q(t+1,1) = 1 + \frac{1}{2}t + \frac{q-1}{8}t^2 + \cdots. \tag{3.14}$$

With simple manipulations (3.10)-(3.14), together with (3.9), yield

$$-\frac{1}{8} \leqslant \frac{(k_2-1)r - 3(k_2+1)}{24(k_2+1)} \leqslant -\frac{1}{12} \leqslant \frac{(k_1+2)p - 3k_1}{24k_1} \leqslant \frac{q-1}{8}. \tag{3.15}$$

From (3.15), it immediately follows that

$$p \geqslant \frac{k_1}{k_1+2}, q \geqslant \frac{(k_1+2)p}{3k_1}, \text{ and } 0 \leqslant \mathrm{r} \leqslant \frac{\mathrm{k}_2+1}{\mathrm{k}_2-1}.$$

We then have, by virtue of Lemma2.2, that $p = \frac{k_1}{k_1+2}, q = \frac{1}{3}$, and $r = \frac{k_2+1}{k_2-1}$ $(k_2 > 1)$ are the best constants for (3.1).

The condition of these equalities holding is obvious. The proof of Theorem3.1 is completed. ∎

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