Some bounds for the logarithmic function

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Abstract

Bounds for the logarithmic function are studied. In particular, we establish bounds with rational functions as approximants. The study leads into the fascinating areas of Padé approximations ([2], [6]), continued fractions ([7], [11]) and orthogonal polynomials ([14], [4]) as well as the somewhat frightening jungle of special functions and associated identities ([5], [9]). Originally, the results aimed at establishing certain inequalities for Shannon entropy but are here discussed in their own right (the applications to entropy inequalities will be published elsewhere).

The exposition is informal, a kind of essay, with only occasional indications of proofs. The reader may take it as an invitation to further studies. Enough details are provided to enable the reader to verify all statements. To the expert in the fields pointed to there is little or nothing new.

Keywords Logarithmic inequalities, Padé approximation, continued fractions, Jacobi polynomials, Legendre polynomials.

1 Basic inequalities

Consider first the truly basic inequalities:

$$1 - \frac{1}{x} \leq \ ln\, x \leq x - 1 \,\text{for}\, x > 0\,. \tag{1}$$

Here and in all inequalities below it is understood that the inequalities shown are strict, except in easily recognizable cases of equality. One may prefer to write (1) in the form

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$$\frac{x}{1+x} \leq \ ln\,(1+x) \leq x \,\text{for}\, x > -1\,. \tag{2}$$

The bounds (1) and (2) involve rational functions of type $[1,1]$ (the left-hand-side) and of type $[1,0]$ (the right-hand-side). Here, the *type* refers to the degrees of numerator and denominator in the rational function concerned.

The simplicity, generality and usefulness of (1) and (2) is unbeatable. Nevertheless, there are many other interesting inequalities for the logarithmic function. For instance, if we separate positive and negative values of x, none of the following two inequalities is much more complex:

$$\frac{2x}{2+x} \leq \ ln\,(1+x) \leq \frac{x}{2}\cdot\frac{2+x}{1+x} \quad \text{for } 0 \leq x < \infty, \tag{3}$$

$$\frac{2x}{2+x} \geq \ ln\,(1+x) \geq \frac{x}{2}\cdot\frac{2+x}{1+x} \quad \text{for } -1 < x \leq 0 \tag{4}$$

–and these inequalities are sharper than (2). Also note that the first inequality of (3) is an improvement, for $x \geq 0$, over the first inequality of (2) with a rational function of the same type – whereas the second inequality of (2) cannot, of course, be improved in a similar way.

The inequalities (3) and (4) may be written as a single double inequality, e.g. by taking absolute values. Another possibility is to define the function λ by

$$\lambda(x) = \frac{ln\,(1+x)}{x} \tag{5}$$

(λ represents the slope of the chorde on the function $x \curvearrowright \ ln\,(1+x)$ which connects $(0,0)$ with $(x,\ ln\,(1+x)))$. By continuity, $\lambda(0) = 1$. Then

$$\frac{2}{2+x} \leq \lambda(x) \leq \frac{2+x}{2+2x} \,\text{for}\, x > -1\,. \tag{6}$$

For some possibilities to sharpen the bounds (3) and (4), see (22) and (23) further on. However, the form chosen is convenient for a special reason. Actually, (3) and (4) are equivalent as follows by writing $ln\,(1+x)$ in the form $-\ ln\,(1-\frac{x}{1+x})$. To further exploit this observation, we introduce a notion of *duality* between functions defined on $[0,\infty[$ and functions defined on $]-1,0]$. The *dual* of one such function, say ϕ, is defined by the relation

$$\phi^*(x) = -\phi\left(\frac{-x}{1+x}\right)\,. \tag{7}$$

Inequalities of the form $\phi(x) \leq \ ln\,(1+x) \leq \psi(x)$ for $x \in [0,\infty[$ (respectively, for $x \in]-1,0]$) then translate into inequalities $\phi^*(x) \geq \ ln\,(1+x) \geq \psi^*(x)$ for $x \in]-1,0]$ (respectively, for $x \in [0,\infty[$).

In (3) and (4) we have an instance with two *self-dual* functions in the sense that for the two analytic functions – say ϕ and ψ – given on the full interval $]-1,\infty[$ by, respectively the left-hand and the right-hand expression in (3) and (4), we find that

$$(\phi_{|[0,\infty[})^* = \phi_{|]-1,0]} \tag{8}$$

and similarly for ψ (here, the subscripts refer to restrictions of the domain of definition). As the functions we shall consider will be analytic in $]-1,\infty[$, we do not find it necessary to indicate the restrictions to the appropriate interval since the analytic form of the functions will be the same whether we consider the restriction to $[0,\infty[$ or to $]-1,0]$. For the functions above we may thus write simply $\phi^* = \phi$ and $\psi^* = \psi$.

Let us return to (2). Also there, duality is relevant. Indeed, writing (2) in the form $\phi(x) \leq \ln(1+x) \leq \psi(x)$, we find that $\phi^* = \psi$ (and $\psi^* = \phi$), i.e. (ϕ,ψ) is a *dual pair*. This explains why there is no restriction on x in (2), except the natural one, $x > -1$.

Duality takes a different – and somewhat simpler – form when having bounds for $\ln x$ in mind rather than bounds for $\ln(1+x)$. In order to save on notation, we use only one symbol, a "star" , for duality and then indicate by an overline "tilde" if the functions are intended as bounds for $\ln(x)$ rather than for $\ln(1+x)$. The duality definition for "tilde-functions" is then given by

$$\tilde{\phi}^*(x) = -\tilde{\phi}(x^{-1})\,. \tag{9}$$

As an example of a basic inequality of $\ln x$-type, we mention

$$|\ln x| \leq \frac{1}{2}|x - \frac{1}{x}| \text{ for } x > 0 \tag{10}$$

which is obtained from the right-hand inequalities of (3) and (4).

From (10), and after a further substitution $x := x^a$ with a a positive parameter, we obtain useful approximations of $\ln x$. In more detail, one finds that

$$\frac{1}{2a}(x^a - x^{-a}) \downarrow \ln x \text{ as } a \downarrow 0 \text{ for } 1 \leq x < \infty\,, \tag{11}$$

and that

$$\frac{1}{2a}(x^a - x^{-a}) \uparrow \ln x \text{ as } a \downarrow 0 \text{ for } 0 < x \leq 1\,. \tag{12}$$

These results – clearly duals of each other – can also be derived directly from the expansion

$$\frac{1}{2a}(x^a - x^{-a}) = \ln x + \sum_{n=1}^{\infty} \frac{a^{2n}}{(2n+1)!}(\ln x)^{2n+1}\,. \tag{13}$$

Consider, for instance, the case $a = \frac{1}{2}$. This leads to the inequality

$$\mathit{ln}\, x \le \frac{x-1}{\sqrt{x}} \text{ for } 1 \le x < \infty \tag{14}$$

with reversal of the inequality sign for $0 < x < 1$. Expressed in terms of the λ-function this shows that

$$\lambda(x) \le \frac{1}{\sqrt{1+x}} \text{ for } x > -1\,. \tag{15}$$

This useful inequality – which one could just as well had been led to by considering the product of the extreme terms in (6) – goes back at least to Karamata [8], cf. also Mitrinović [12, Section 3.6.15].

Let us end this section with a more special inequality which is, again, related to (6). It is our only result which involves two parameters (other such inequalities can be found in [12]). The inequality states that for $0 \le x \le 1$ and $0 \le y < \infty$,

$$(2-x)\lambda(y) - \frac{1-x}{1+y} \le \lambda(xy) \le x\lambda(y) + (1-x)\,. \tag{16}$$

This follows by a standard analysis of the inequalities which result when you keep y fixed (for the proof, consider the function equal to the difference between terms of the inequality you wish to prove, and differentiate twice). The indicated proof makes use of (6) (which is needed in order to discuss endpoint behaviour). One may also note that, for $0 \le x \le 1$, the second inequality in (6) follows by comparison of the two extreme terms in (16).

2 Rational approximants for $\mathit{ln}\,(1+x)$

We have demonstrated the usefulness of (6) and now turn to a more systematic study of similar bounds with more general rational functions. We choose to focus on bounds for $\mathit{ln}\,(1+x)$ for $x \ge 0$. We already have some examples of lower and upper bounds of $\mathit{ln}\,(1+x)$ and realize that it is natural to seek lower bounds of type $[n,n]$ and upper bounds of type $[n,n-1]$. We shall find such bounds for each $n \ge 1$.

To find criteria for the selection of good lower bounds, assume that $\phi(x) = \frac{F}{G}$ is such a bound with F and G polynomials, both of degree n, say. We insist that the bound is exact for $x = 0$, hence the constant term in F is 0. For the difference function $\delta(x) = \mathit{ln}\,(1+x) - \phi(x)$, we find that

$$\delta'(x) = \frac{G(x)^2 - (1+x)\Big(F'(x)G(x) - F(x)G'(x)\Big)}{(1+x)G(x)^2}\,. \tag{17}$$

For a loose consideration, let us neglect the fact that the denominator here varies with the choice of ϕ. Then we only need to assure that the numerator is small and non-negative. To choose among the possible bounds, note that the numerator is a polynomium of degree $2n$. We aim at a bound which is especially good for small values of x and realize that we should attempt to make lower terms vanish. We achieve this by insisting that *all* terms in the numerator vanish, except the leading term.

To be precise, we denote by ϕ_n the function of the form $\phi_n = \frac{F}{G}$ with F and G two polynomials of degree n such that $F(0) = 0$ and $G(0) > 0$ and such that the numerator in (17) is a positive constant times x^{2n}. Similarly, ψ_n is defined by $\psi_n = \frac{F}{G}$, where F is a polynomial of degree n with $F(0) = 0$ and G a polynomial of degree $n-1$ with $G(0) > 0$ and such that the numerator in (17) is a negative constant times x^{2n-1}.

It turns out that, for each $n \geq 1$, ϕ_n and ψ_n are uniquely determined by these requirements. We introduce the following standard representations:

$$\phi_n(x) = \frac{xP_{n-1}(x)}{Q_n(x)}\,,\ \psi_n(x) = \frac{xR_{n-1}(x)}{S_{n-1}(x)} \tag{18}$$

with $(P_n)_{n\geq 0}, (Q_n)_{n\geq 0}, (R_n)_{n\geq 0}$ and $(S_n)_{n\geq 0}$ four sets of polynomials of degrees as indicated by the subscript and such that

$$Q_{n,n} = 1\,,\ R_{n-1,n-1} = 1 \tag{19}$$

(with natural notation for the coefficients of the polynomials we consider). For convenience, we put $Q_0(x) \equiv 1$. By (19), the leading term in Q_n is x^n and the leading term in R_{n-1} is x^{n-1}, i.e. Q_n and R_{n-1} are *monic* polynomials. Observe that we normalize via the denominator in ϕ_n and via the numerator in ψ_n. With the chosen normalization of the representations for the ϕ- and ψ-functions, the P-, Q-, R- and S-polynomials are all uniquely determined. To save on notation, we shall normally write P_n (respectively Q_n, R_n, S_n and also ϕ_n and ψ_n) instead of $P_n(x)$ (respectively $Q_n(x), R_n(x)$ etc.). We often refer to the four sets of polynomials as the *PQRS-polynomials*.

To recapitulate, ϕ_n and ψ_n are defined by (18) and (19) and by the essential requirements

$$Q_n^2 - (1+x)\big((xP_{n-1})'Q_n - xP_{n-1}Q_n'\big) = x^{2n}\,, \tag{20}$$

$$(1+x)\big((xR_{n-1})'S_{n-1} - xR_{n-1}S_{n-1}'\big) - S_{n-1}^2 = S_{n-1,n-1}x^{2n-1}\,. \tag{21}$$

Clearly, ϕ_n is a lower- and ψ_n an upper bound of $\ln(1+x)$ for $x \in [0,\infty[$. This follows as the functions $x \curvearrowright \ln(1+x) - \phi_n(x)$ and $x \curvearrowright \psi_n(x) - \ln(1+x)$ both vanish at $x = 0$ and have positive derivatives in $]0,\infty[$.

The considerations leading to the above definitions express a key idea in the theory of *Padé approximation*, cf. Baker and Graves-Morris [2], a standard reference. In the terminology of that theory, ϕ_n is the $[n,n]$-*Padé approximant* and ψ_n the $[n,n-1]$-*Padé approximant* of $\ln(1+x)$. We shall refer to the ϕ_n's as simply the *lower approximants* (to $\ln(1+x)$) and to the ψ_n's as the *upper approximants.*

Instead of just referring the reader to the literature on Padé approximation, we base our exposition on experiments facilitated by modern computing tools. This will lead rather quickly to desired formulas and other insights. Full proofs of relations found experimentally may not always be so obvious. We shall include enough details to enable the reader to validate all statements in the usual rigorous mathematical style. The interested reader will find further results in the literature referred to.

3 Some experiments

In order to get a feel for the nature of the approximants defined in the previous section, it is natural to work out a number of examples. This can be done by equating coefficients of polynomials occuring in the defining relations (20) and (21). With hinsight, this can be done much more conveniently by recursion formulas developed later ((60)–(63)) or by simply asking MAPLE to work out the relevant Padé approximants. Anyhow, in one way or another we can get at the first few approximants and obtain a suitable table, cf. Table 1.

For instance, $\phi_1(x) \le \ln(1+x)$ for $x \ge 0$ which we recognize as the first inequality of (3). As $\phi_1^* = \phi_1$, the first inequality of (4) also follows. The determination of ψ_2 gives the inequality

$$\ln(1+x) \le \frac{x(6+x)}{2(3+2x)} \text{ for } x \ge 0\,, \tag{22}$$

a strengthening of the second inequality in (3) with a bound of the same type ($[2,1]$). And, if we dualize the inequality for ψ_2, we get $\ln(1+x) \ge \psi_2^*(x)$ for $-1 < x \le 0$, i.e.

$$\ln(1+x) \ge \frac{x(6+5x)}{2(1+x)(3+x)} \text{ for } -1 < x \le 0\,, \tag{23}$$

a streghtening of the second inequality of (4), though with a function of a different type ($[2,2]$ rather than $[2,1]$).

By (18) and (19), Table 1 reveals the identity of the first $PQRS$-polynomials, cf. Table 2.

n	ϕ_n	ψ_n
1	$\frac{2x}{2+x}$	x
2	$\frac{3x(2+x)}{6+6x+x^2}$	$\frac{x(6+x)}{2(3+2x)}$
3	$\frac{x(60+60x+11x^2)}{3(20+30x+12x^2+x^3)}$	$\frac{x(30+21x+x^2)}{3(10+12x+3x^2)}$
4	$\frac{5x(84+126x+52x^2+5x^3)}{6(70+140x+90x^2+20x^3+x^4)}$	$\frac{x(420+510x+140x^2+3x^3)}{12(35+60x+30x^2+4x^3)}$
5	$\frac{x(7560+15120x+9870x^2+2310x^3+137x^4)}{30(252+630x+560x^2+210x^3+30x^4+x^5)}$	$\frac{x(3780+6510x+3360x^2+505x^3+6x^4)}{30(126+280x+210x^2+60x^3+5x^4)}$
6	$\frac{7x(1320+3300x+2960x^2+1140x^3+174x^4+7x^5)}{10(924+2772x+3150x^2+1680x^3+420x^4+42x^5+x^6)}$	$\frac{x(13860+30870x+23520x^2+7035x^3+672x^4+5x^5)}{30(462+1260x+1260x^2+560x^3+105x^4+6x^5)}$

Table 1: Lower- and upper approximants to $ln\,(1+x)$

n	0	1	2	3
P_n	2	$6+3x$	$20+20x+\frac{11}{3}x^2$	$70+105x+\frac{130}{3}x^2+\frac{25}{6}x^3$
Q_n	1	$2+x$	$6+6x+x^2$	$20+30x+12x^2+x^3$
R_n	1	$6+x$	$30+21x+x^2$	$140+170x+\frac{140}{3}x^2+x^3$
S_n	1	$6+4x$	$30+36x+9x^2$	$140+240x+120x^2+16x^3$

Table 2: $PQRS$-polynomials of degrees $0, 1, 2$ and 3.

A graphical plot of the first four approximants is shown in Figure 1. After more extensive plotting – if the reader is not yet convinced – one will realize that the ϕ_n's increase with n and the ψ_n's decrease, both sequences with $ln\,(1+x)$ as limit function. In order to test this, we look at differences of the relevant functions. One finds that

$$\psi_n - \phi_n = \frac{x^{2n}}{Q_n S_{n-1}}\,, \tag{24}$$

$$\psi_{n+1} - \phi_n = \frac{x^{2n+1}}{Q_n S_n}\,, \tag{25}$$

$$\phi_{n+1} - \phi_n = \frac{2}{n+1}\frac{x^{2n+1}}{Q_{n+1} Q_n}\,, \tag{26}$$

$$\psi_n - \psi_{n+1} = (2n+1)\frac{x^{2n}}{S_{n-1} S_n}. \tag{27}$$

This may be obtained experimentally but also follows from the definitions of the approximants (except for the constants in (26) and (27) which depend on the later relations (29) and (30)).

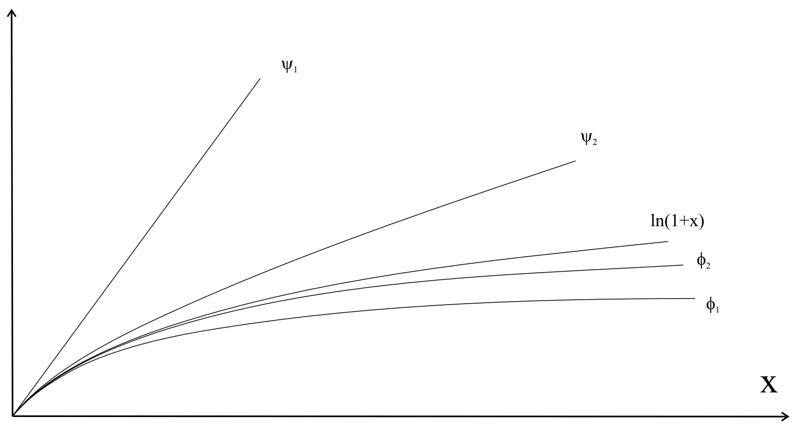

Fig.1. Upper and lower bounds of $ln\,(1+x)$ for $x \geq 0$.

In order to find good approximations of $ln\,(1+x)$ (for $x \geq 0$), we could proceed differently from the above approach which is tied to the defining relations (20) and (21). As a start, take x as a reasonable approximation of $ln\,(1+x)$. This is too much, thus in the next step we divide by 1 plus some non-negative function. Not to increase the complexity too much, take this function to be proportional to x, say cx. Then we get a type-$[1,1]$ approximation of $ln\,(1+x)$. If this is set to ϕ_1 – known to be a reasonable bound – we get $c = \frac{1}{2}$. Now, ϕ_1 is too small, hence in our next step we divide x by 2 plus a non-negative function. Choosing again a function proportional to x, we obtain a type-$[2,1]$ bound and it is reasonable to adjust this so that it coincides with ψ_2. We now have the bound

$$\psi_2(x) = \frac{x}{1 + \frac{x}{2+\frac{x}{3}}}$$

and may continue based on the identification of the first few good bounds given in Table 1.

The approach just described is of course well known and nothing but an attempt to represent $ln\,(1+x)$ by a *continued fraction.* The approach above does not lead in a unique way to (the beginnings of) a continued fraction. E.g., above we could choose to look, as we did, at x divided by 2 + something or we could have looked at $\frac{1}{2}x$ divided by 1 plus something. Experimenting with the possibilities, you soon see that a very simple structure emerges if you at each step divide by n plus something. Indeed, you then arrive at the beautiful representation

$$ln\,(1+x) = \frac{x}{1_+}\;\frac{1^2x}{2_+}\;\frac{1^2x}{3_+}\;\frac{2^2x}{4_+}\;\frac{2^2x}{5_+}\;\cdots\;{}_+\frac{n^2x}{2n_+}\;\frac{n^2x}{2n+1_+}\;\cdots\,. \tag{28}$$

This representation is not new. You find it in [3], [4], [2], [7] and [11]. In fact, the result is more than two hundred years old (!) and goes back to Lambert, cf. [10]. Regarding historical comments – to this and other aspects of the paper – we refer to [3] and to [7].

In a sense, the continued fraction (28) tells us all we need to know as it provides us with easy access to the approximants ϕ_n and ψ_n. We shall elaborate more on this in Section 5. For now, let us look at a few more "experiments" .

Some coefficients for the $PQRS$-polynomials are easy to guess from Tables 1 and 2, e.g., we see that

$$S_{n,n} = (n+1)^2 \,. \tag{29}$$

As another example, we realize that $P_{n,n} - P_{n-1,n-1} = \frac{2}{n+1}$, hence

$$P_{n,n} = 2\sum_{\nu=1}^{n+1} \frac{1}{\nu} \,. \tag{30}$$

One may also go hunting for relationships among the coefficients by consulting the "On-Line Encyclopedia of Integer Sequences" , cf. Sloane [13]. For instance, enquiring about the sequence $6, 30, 90, 210$, the feed-back from this source will reveal the fact that

$$Q_{n,n-2} = \frac{1}{4}[n+2]_4 \,.^1$$

Many other relations for the $PQRS$-coefficients can be discovered in this way. Instead of pursuing this line of investigation, we shall have a look at the approximants after a transformation so that they approximate $\ln x$ rather than $\ln(1+x)$. In other words, we are asking about the "tilde-functions"

$$\tilde{\phi}_n(x) = \phi_n(x-1) \ ; \ \tilde{\psi}_n(x) = \psi_n(x-1) \,. \tag{31}$$

This leads to the functions in Table 3.

We end this section by an investigation of the zeroes of the $PQRS$-polynomials. The result – found by numerical computation using MAPLE – is reported in Table 4. We realize that the *interlacing property* holds for each set of polynomials, e.g., between adjacent zeroes of P_n you find a zero of P_{n-1}. And also some "mixed interlacing" takes place, as we shall return to at the very end of the paper.

[1] The notation $[a]_k$ is used for the *descending factorial* $a(a-1)\cdots(a-k+1)$.

n	$\tilde{\phi}_n$	$\tilde{\psi}_n$
1	$\frac{2(x-1)}{1+x}$	$x-1$
2	$\frac{3(x-1)(1+x)}{1+4x+x^2}$	$\frac{(x-1)(x+5)}{2(1+2x)}$
3	$\frac{(x-1)(11+38x+11x^2)}{3(1+9x+9x^2+x^3)}$	$\frac{(x-1)(10+19x+x^2)}{3(1+6x+3x^2)}$
4	$\frac{5(x-1)(5+37x+37x^2+5x^3)}{6(1+16x+36x^2+16x^3+x^4)}$	$\frac{(x-1)(47+239x+131x^2+3x^3)}{12(1+12x+18x^2+4x^3)}$
5	$\frac{(x-1)(137+1762x+3762x^2+1762x^3+137x^4)}{30(1+25x+100x^2+100x^3+25x^4+x^5)}$	$\frac{(x-1)(131+1281x+1881x^2+481x^3+6x^4}{30(1+20x+60x^2+40x^3+5x^4)}$
6	$\frac{7(x-1)(7+139x+514x^2+514x^3+139x^4+7x^5)}{10(1+36x+225x^2+400x^3+225x^4+36x^5+x^6)}$	$\frac{(x-1)(142+2272x+6397x^2+4397x^3+647x^4+5x^5)}{30(1+30x+150x^2+200x^3+75x^4+6x^5)}$

Table 3: Lower- and upper approximants to $ln\,(x)$

P_1	-2.0000				
P_2	-4.1356	-1.3189			
P_3	-7.2397	-2.0000	-1.1603		
P_4	-11.3204	-2.9243	-1.5197	-1.0969	
P_5	-16.3907	-4.0764	-2.0000	-1.3251	-1.0650
Q_1	-2.0000				
Q_2	-4.7321	-1.2679			
Q_3	-8.8730	-2.0000	-1.1270		
Q_4	-14.4026	-3.0302	-1.4926	-1.0746	
Q_5	-21.3174	-4.3334	-2.0000	-1.3000	-1.0492
R_1	-6.0000				
R_2	-19.4582	-1.5418			
R_3	-42.7683	-2.6743	-1.2240		
R_4	-77.0830	-4.2481	-1.7114	-1.1242	
R_5	-123.2945	-6.2526	-2.3648	-1.4089	-1.0792
S_1	-1.5000				
S_2	-2.8165	-1.1835			
S_3	-4.7094	-1.6934	-1.0992		
S_4	-7.1551	-2.4015	-1.3828	-1.0606	
S_5	-10.1487	-3.2837	-1.7793	-1.2469	-1.0415

Table 4: Zeroes of the $PQRS$-polynomials.

4 Some facts

In this section it is assumed that any occurring x is non-negative unless stated otherwise explicitly.

The experiments of the previous section lead to important facts about the ϕ_n's and ψ_n's. Let us start by looking at the quality of these functions as bounds for $\ln(1+x)$.

Apparently, all coefficients in the $PQRS$-polynomials are positive. From (24) - (27) it therefore follows that

$$\phi_1(x) \le \phi_2(x) \le \cdots \le \ln(1+x) \le \cdots \le \psi_2(x) \le \psi_1(x) \,. \tag{32}$$

Furthermore, by (25), recalling also (19) and (29), we see that

$$\psi_{n+1}(x) - \phi_n(x) \le \frac{x}{(n+1)^2} \,. \tag{33}$$

Though this bound is quite loose, it is sharp enough to imply that, for each $x \ge 0$, the sequences $\phi_n(x)$ and $\psi_n(x)$ both converge to $\ln(1+x)$ as $n \to \infty$. We also note that

$$\ln(1+x) - \phi_n(x) \le \frac{x^{2n+1}}{Q_n(x)S_n(x)} \,, \tag{34}$$

$$\psi_n(x) - \ln(1+x) \le \frac{x^{2n}}{Q_n(x)S_{n-1}(x)} \,. \tag{35}$$

These bounds are less explicit but typically much sharper than (33).

The behaviour of $\phi_n(x)$ and $\psi_n(x)$ for large x is given by the relations

$$\phi_n(x) \approx 2\sum_{\nu=1}^{n} \frac{1}{\nu} \,; \;\; \psi_n(x) \approx \frac{x}{n^2} \tag{36}$$

in the sense that the ratios involved converge to 1 for $x \to \infty$. These relations follow from (29) and (30).

We also note that the functions ϕ_n and ψ_n are increasing. As for ϕ_n this follows from (20) (see also (17)) which tells us that

$$\phi_n'(x) = \frac{Q_n(x)^2 - x^{2n}}{(1+x)Q_n(x)^2} \ge 0 \,. \tag{37}$$

The approximants may be characterized in a natural way which is quite different from the definitions via (20) and (21). Indeed, for each $n \ge 1$, ϕ_n is the unique type-$[n,n]$ rational function with $\phi_n(0) = 0$ and $\phi_n(x) \le \ln(1+x)$

for $x \geq 0$ which dominates any other such function *locally*, i.e., for any such function f, there exists $\varepsilon > 0$ such that $\phi_n(x) \geq f(x)$ for $0 \leq x < \varepsilon$. In fact, for this conclusion we may assume about type-$[n, n]$ functions f considered only that $f \leq \psi_{n+1}$, locally, instead of $f(x) \leq \ln(1+x)$ for $x \geq 0$. A similar characterization holds for ψ_n.

Note that we have to work locally with inequalities in the characterizations just pointed out (consider, for example, the function given by $f(x) = \frac{3x}{4+x}$; then $f(x) \leq \ln(1+x)$ for $x \geq 0$ – as $f \leq \phi_2$ – but $f \leq \phi_1$ does not hold globally for $x \geq 0$, only locally). The proofs of the characterizations rest on the facts that if a type-$[n, n]$-function is not identical to ϕ_n, then there is a $\nu \leq 2n$ such that $f^{(\nu)}(0) \neq \phi_n^{(\nu)}(0)$ and if a type-$[n, n-1]$ function g is not identical to ψ_n, then there is a $\nu \leq 2n-1$ such that $g^{(\nu)}(0) \neq \psi_n^{(\nu)}(0)$.

The determination of the $PQRS$-polynomials in closed form is not that obvious based on our experiments. Most strikingly, perhaps, is that if we define $\tilde{Q}_n$ by

$$\tilde{Q}_n(x) = Q_n(x-1)\,,$$

then

$$\tilde{Q}_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 x^k\,,$$

cf. Table 3. As $Q_n(x) = \tilde{Q}_n(1+x)$, this leads to the formula

$$Q_{n,k} = \binom{n}{k} \sum_{\nu=0}^{n-k} \binom{n-k}{\nu} \binom{n}{n-k-\nu}\,.$$

The sum here can be evaluated by using the *binomial Chu-Vandermonde formula*:

$$\sum_{j=0}^{N} \binom{N}{j} [x]_j [y]_{N-j} = [x+y]_N\,,$$

or, more directly, by the equivalent *Chu-Vandermonde convolution* formula:

$$\sum_{j=0}^{N} \binom{x}{j} \binom{y}{N-j} = \binom{x+y}{N}\,.$$

For these identities, see [1]. What we find is that

$$Q_{n,k} = \binom{n}{k} \binom{2n-k}{n-k} = \binom{2n-k}{k, n-k, n-k}\,. \tag{38}$$

Thus, the coefficients in Q_n have been identified as certain trinomial coefficients.

Having determined the $Q_{n,k}$'s we may use the defining relation (20) to determine the $P_{n-1,k}$'s. With some effort this leads to the formula

$$P_{n-1,k} = \sum_{\nu=0}^{k} \frac{(-1)^\nu}{\nu+1} Q_{n,k-\nu}\,. \tag{39}$$

Combined with (38) this gives a reasonable determination of the coefficients of the P-polynomials, e.g. it shows that $P_{n-1,0} = Q_{n,0} = \binom{2n}{n}$. However, the formulas can sometimes be simplified (consider the case $k = n-1$ when $P_{n-1,n-1} = 2\sum_1^n \frac{1}{\nu}$ according to (30)).

A striking feature of the $\tilde{\phi}_n$-functions is that they are self-dual. This is seen from Table 3. Indeed, writing

$$\tilde{\phi}_n(x) = \frac{(x-1)\tilde{P}_{n-1}(x)}{\tilde{Q}_n(x)}$$

with $\tilde{P}_{n-1}(x) = P_{n-1}(x-1)$ and $\tilde{Q}_n(x) = Q_n(x-1)$, we realize that these polynomials are *self-reflected* in the sense that

$$x^{n-1}\tilde{P}_{n-1}(\frac{1}{x}) = \tilde{P}_{n-1}(x)\,,$$
$$x^n\tilde{Q}_n(\frac{1}{x}) = \tilde{Q}_n(x)\,.$$

By simple calculation, this implies that the $\tilde{\phi}_n$'s are self-dual. Hence also the ϕ_n's are self-dual: $\phi_n^* = \phi_n$. By duality, this implies that ϕ_n is a best type-$[n,n]$ upper bound of $\ln(1+x)$ for $-1 < x \le 0$ (with "best" being understood in much the same way as discussed for the bounds found for $x \ge 0$).

The functions ψ_n are not self-dual. The duals ψ_n^* are even of a different type, viz. of type $[n,n]$. These functions then, are the best type-$[n,n]$ lower bounds of $\ln(1+x)$ for $-1 < x \le 0$.

5 Exploiting Lamberts continued fraction

It is natural to seek procedures which allow recursive determination of the $PQRS$-polynomials. This can be achieved based on the defining relations and the derived relations (24)–(27). However, this will, typically, involve products of polynomials and simpler procedures are desirable. Here Lamberts expansion (28) provides the proper tool.

Following usual terminology of continued fraction theory, let A_n and B_n denote the *approximants* related to (28). These are polynomials associated in the natural manner to the corresponding finite continued fractions, e.g.

$$\frac{A_1}{B_1} = \frac{x}{1}\,, \ \frac{A_2}{B_2} = \frac{x}{1 + \dfrac{1^2 x}{2}}\,, \ \frac{A_3}{B_3} = \frac{x}{1 + \dfrac{1^2 x}{2 + \dfrac{1^2 x}{3}}}\,.$$

The first polynomials are

$$A_0 = 0\,, \ A_1 = x\,, \ B_0 = 1\,, \ B_1 = 1 \tag{40}$$

and, exploiting the special form of (28), the following recursive relations hold for $n \geq 1$:

$$A_{2n} = 2nA_{2n-1} + n^2 x A_{2n-2}\,, \tag{41}$$

$$A_{2n+1} = (2n+1)A_{2n} + n^2 x A_{2n-1}\,, \tag{42}$$

$$B_{2n} = 2nB_{2n-1} + n^2 x B_{2n-2}\,, \tag{43}$$

$$B_{2n+1} = (2n+1)B_{2n} + n^2 x B_{2n-1}\,. \tag{44}$$

We realize that, for $n \geq 1$,

$$\phi_n = \frac{A_{2n}}{B_{2n}}\,, \ \psi_n = \frac{A_{2n-1}}{B_{2n-1}}\,. \tag{45}$$

The polynomials A_{2n-1}, A_{2n}, B_{2n}, and B_{2n+1} are all of degree n and for basic coefficients we find

$$A_{n,0} = 0\,, \ A_{n,1} = B_{n,0} = n!\,, \tag{46}$$

$$A_{2n+1,n+1} = B_{2n,n} = \big(n!\big)^2\,. \tag{47}$$

Together with (45), (47) tells us that

$$xP_{n-1} = \frac{A_{2n}}{(n!)^2}\,, \ Q_n = \frac{B_{2n}}{(n!)^2}\,, \tag{48}$$

$$xR_{n-1} = \frac{A_{2n-1}}{((n-1)!)^2}\,, \ S_{n-1} = \frac{B_{2n-1}}{((n-1)!)^2}\,. \tag{49}$$

Now, Q_n is known, cf. (38), hence also B_{2n} can be found. And then, from (43), we can determine B_{2n-1}, hence, by (49), also S_{n-1}. This leads to the formula

$$S_{n-1} = n \sum_{\nu=0}^{n-1} \binom{2n-\nu-1}{\nu, n-\nu, n-\nu-1} x^\nu \,. \tag{50}$$

Similarly, a formula for R_{n-1} can be found based on (39), (41) and (49). Going through the details you find that

$$R_{n-1} = n \sum_{\nu=0}^{n-1} \sum_{\mu=1}^{\nu+1} (-1)^{\mu-1} \binom{2n-\nu+\mu-2}{\nu-\mu+1, n-\nu+\mu-1, n-\nu+\mu-2} \mu^{-1} x^\nu \,. \tag{51}$$

The expressions (38), (39), (50) and (51) thus provide formulas for the $PQRS$-polynomials in closed form with very satisfactory expressions for the denominator polynomials (Q, S) and somewhat less satisfactory formulas for the numerator polynomials (P, R).

For practical calculations, however, the recursive relations (41)–(44) are more expedient and simple to programme. By (48) and (49) these relations may be written directly in terms of the $PQRS$-polynomials:

$$P_{n-1} = \frac{2}{n} R_{n-1} + x P_{n-2} \,, \tag{52}$$

$$R_n = (2n+1) P_{n-1} + x R_{n-1} \,, \tag{53}$$

$$Q_n = \frac{2}{n} S_{n-1} + x Q_{n-1} \,, \tag{54}$$

$$S_n = (2n+1) Q_n + x S_{n-1} \,. \tag{55}$$

Here, $n \geq 1$ and for (52) we have put $P_{-1} = 0$.

The special structure of these recurrence relations may be emphasized by introducing the notation ΔP_n for the polynomial

$$(\Delta P_n)(x) = P_n(x) - x P_{n-1}(x)$$

and similarly for ΔQ_n etc. Then,

$$\Delta P_{n-1} = \frac{2}{n} R_{n-1} \,; \; \Delta R_n = (2n+1) P_{n-1} \,, \tag{56}$$

$$\Delta Q_n = \frac{2}{n} S_{n-1} \,; \; \Delta S_n = (2n+1) Q_n \,. \tag{57}$$

Here, $n \geq 1$ (with $\Delta P_0 = 2$).

Another form of (52)–(55) is obtained for the *reflected polynomials*. Following [6], the reflected polynomial of a polynomial A is the polynomial denoted $\overline{A}$ which is given by

$$\overline{A}(x) = x^N A(x^{-1})$$

with N the degree of A. We find:

$$\overline{P}_{n-1} - \overline{P}_{n-2} = \frac{2}{n}\overline{R}_{n-1}\,;\ \overline{R}_n - \overline{R}_{n-1} = (2n+1)x\overline{P}_{n-1}\,, \tag{58}$$

$$\overline{Q}_n - \overline{Q}_{n-1} = \frac{2}{n}x\overline{S}_{n-1}\,;\ \overline{S}_n - \overline{S}_{n-1} = (2n+1)\overline{Q}_n\,. \tag{59}$$

6 Relations to orthogonal polynomials

The recurrence relations (52)–(55) are ideal for recursive computation of the $PQRS$-polynomials. In (52) and (53), numerator polynomials are "entangled" , and in (54) and (55) so are denominator polynomials. It is, however easy to "unentangle" the formulas so as to only involve one set of polynomials at a time. In this way you find, for $n \geq 2$:

$$(n+1)P_n - (2n+1)(x+2)P_{n-1} + nx^2P_{n-2} = 0\,, \tag{60}$$

$$nQ_n - (2n-1)(x+2)Q_{n-1} + (n-1)x^2Q_{n-2} = 0\,, \tag{61}$$

$$(2n-1)R_n - \big(4nx + \frac{2}{n}(4n^2-1)\big)R_{n-1} + (2n+1)x^2R_{n-2} = 0\,, \tag{62}$$

$$(2n-1)S_n - \big(4nx + \frac{2}{n}(4n^2-1)\big)S_{n-1} + (2n+1)x^2S_{n-2} = 0\,. \tag{63}$$

In passing we note that it is easy to prove from (60) and (61) that the ϕ_n's are self-dual.

It is well known that orthogonal polynomials satisfy three-term recurrence relations. With the above formulas in mind it therefore lies nearby to investigate the possible relations of the $PQRS$-polynomials to orthogonal polynomials. However, the recurrence relations (60)-(63) are not in standard form as known from the theory of orthogonal polynomials because of the appearence of x in the last term. This points to the desirability of rewriting the formulas by introducing a suitable transformation.

Such a transformation may be suggested by looking at the zeroes of the $PQRS$-polynomials, recalling that these are dense in the support of any underlying measure. Here, the information provided by Table 4 is useful. It shows that the zeroes lie in $]-\infty,-1[$ and also, that the "interlacing" behaviour which is characteristic for orthogonal polynomials applies to our polynomials.

n	$\mathbb{P}_n$	$\mathbb{Q}_n$	$\mathbb{R}_n$	$\mathbb{S}_n$
0	2	1	1	1
1	$3x$	x	$3x-2$	$3x+1$

Table 5: $\mathbb{PQRS}$-polynomials of degrees 0 and 1.

Possibly after some experimentation, it is found that a transformation which places the zeroes in the interval $]-1,1[$ have a simplifying effect. This may be achieved by first making a transformation to the reflected polynomials, which brings the zeroes into $]-1,0[$, and then consider the natural linear transformation into $]-1,1[$. We therefore define new polynomials, indicated by the letters $\mathbb{P},\mathbb{Q},\mathbb{R},\mathbb{S}$. For the $\mathbb{P}$-polynomials, definition and notation is as follows:

$$\mathbb{P}_n(x) = \overline{P}_n(\frac{1}{2}(x-1)) = 2^{-n}(x-1)^n P_n\Big(\frac{2}{x-1}\Big)\,.$$

The other polynomials are defined similarly. We find:

$$\mathbb{P}_n - \mathbb{P}_{n-1} = \frac{2}{n+1}\mathbb{R}_n\,, \tag{64}$$

$$\mathbb{Q}_n - \mathbb{Q}_{n-1} = \frac{x-1}{n}\mathbb{S}_{n-1}\,, \tag{65}$$

$$\mathbb{R}_n - \mathbb{R}_{n-1} = (n+\frac{1}{2})(x-1)\mathbb{P}_{n-1}\,, \tag{66}$$

$$\mathbb{S}_n - \mathbb{S}_{n-1} = (2n+1)\mathbb{Q}_n\,, \tag{67}$$

and the three-term recurrence relations become

$$(n+1)\mathbb{P}_n - (2n+1)x\mathbb{P}_{n-1} + n\mathbb{P}_{n-2} = 0\,, \tag{68}$$

$$n\mathbb{Q}_n - (2n-1)x\mathbb{Q}_{n-1} + (n-1)\mathbb{Q}_{n-2} = 0\,, \tag{69}$$

$$n(2n-1)\mathbb{R}_n - \big((4n^2-1)x+1\big)\mathbb{R}_{n-1} + n(2n+1)\mathbb{R}_{n-2} = 0\,, \tag{70}$$

$$n(2n-1)\mathbb{S}_n - \big((4n^2-1)x+1\big)\mathbb{S}_{n-1} + n(2n+1)\mathbb{S}_{n-2} = 0\,. \tag{71}$$

Note the disappearence of any x's in the last terms. We may apply (68)–(71) with $n \geq 2$ in connection with the start polynomials given in Table 5.

We shall relate the $\mathbb{PQRS}$-polynomials to the classical *Jacobi polynomials*, here denoted $P_n^{\alpha,\beta}$, which are associated with the measures on $[-1,1]$ with densities $(1-x)^\alpha(1+x)^\beta$. From standard sources, e.g. [14], we see that these polynomials may be determined from the recurrence relations

$$P_n^{\alpha,\beta} - \frac{(2n-1+\alpha+\beta)\big((2n+\alpha+\beta)(2n+\alpha+\beta-2)x+(\alpha^2-\beta^2)\big)}{2n(n+\alpha+\beta)(2n-2+\alpha+\beta)}P_{n-1}^{\alpha,\beta} + \frac{(n+\alpha-1)(n+\beta-1)(2n+\alpha+\beta)}{n(n+\alpha+\beta)(2n-2+\alpha+\beta)}P_{n-2}^{\alpha,\beta} = 0 \tag{72}$$

for $n \geq 2$ in conjunction with the first polynomials which are given by

$$P_0^{\alpha,\beta} = 1\,;\ P_1^{\alpha,\beta} = \frac{1}{2}\big((\alpha+\beta+2)x+(\alpha-\beta)\big)\,. \tag{73}$$

The Jacobi polynomials have a simple expression in closed form:

$$P_n^{\alpha,\beta} = \frac{1}{2^n}\sum_{\nu=0}^{n}\binom{n+\alpha}{\nu}\binom{n+\beta}{n-\nu}(x-1)^{n-\nu}(x+1)^{\nu}\,. \tag{74}$$

We realize that

$$\mathbb{Q}_n = P_n^{0,0}\,, \tag{75}$$

i.e. the $\mathbb{Q}_n$'s are nothing but the classical *Legendre polynomials*. By (65) the $\mathbb{S}$-polynomials are closely related to these polynomials. However, apart from a constant factor, they may also be identified directly as orthogonal polynomials. Indeed, by Table 5, (71), (72) and (73) it is easy to check that

$$\mathbb{S}_n = (n+1)P_n^{1,0}\,. \tag{76}$$

The $\mathbb{P}$- and $\mathbb{R}$-polynomials are harder to identify. We appeal to [4] for a systematic table of orthogonal polynomials. The table works with monic polynomials. Let the monic polynomials corresponding to the $\mathbb{P}_n$'s be denoted $[\mathbb{P}_n]$, i.e.

$$[\mathbb{P}_n] = \frac{2^{n-1}}{\binom{2n+1}{n}}\mathbb{P}_n\,. \tag{77}$$

Then you find the recursion formula

$$[\mathbb{P}_n] - x[\mathbb{P}_{n-1}] + \frac{n^2}{4n^2-1}[\mathbb{P}_{n-2}] = 0\,. \tag{78}$$

By a look-up in [4] (case 5, page 219), you realize that the $[\mathbb{P}_n]$'s are orthogonal polynomials associated with the somewhat bizarre measure on $]-1,1[$ with density function

$$\frac{4}{\pi^2 + \left(\, ln\, \frac{1+x}{1-x}\right)^2}\,.$$

We have not identified the $\mathbb{R}$-polynomials in a similar manner. Possibly, it is more reasonable to look at the polynomials $\mathbb{S}_n - \mathbb{R}_n$ when searching for properties related to orthogonal polynomials.

Lastly we note two interlacing properties with mixed polynomials. For this, we may as well return to the original $PQRS$-polynomials. In fact, from the defining relation (20) and an investigation of the sign of P_{n-1} at zeroes of Q_n it follows that between any two neighbouring zeroes of Q_n there is a zero of P_{n-1} (and of course only one zero). And from (21) a similar investigation shows that between any two neighbouring zeroes of S_{n-1} there is a zero of R_{n-1}. This accounts for $n-2$ zeroes of R_{n-1}. As is easily seen, the last zero of R_{n-1} is smaller than the smallest zero of S_{n-1}.

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