RIEMANN HYPOTHESIS IN SPECIAL CASES

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Abstract. In this note, we show that the Riemann Hypothesis is true in some special cases.

1. Introduction

The Riemann zeta-function is defined for $Re(s) > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

and extended by analytic continuation to the complex plan with one singularity at $s = 1$; in fact a simple pole with residues 1. The Riemann hypothesis [1] states that the non-real zeros of the Riemann zeta-function all lie on the line $Re(s) = \frac{1}{2}$. Now, let $\sigma(n)$ denote the sum of positive divisors of n; in 2002 Lagarias [3] showed that Riemann hypothesis holds if and only if

$$\sigma(n) \leq H_n + e^{H_n} \ln H_n, \tag{1}$$

for every $\mathbb{N}$, where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.
In this note we show that the inequality (1) holds, when n is a power of a prime number and for some sufficiently large square free values of n; by square free integer we mean one that in its factoring to primes, the power of factors all are equal to 1.

2. Main Results

Let $\mathbb{P}$ be the set of all primes and $H_n = \sum_{k=1}^{n} 1/k$. It is easy to see that

$$H_n > \ln n \qquad (n \in \mathbb{N}). \tag{2}$$

Theorem 1. *The inequality (1) holds for all $n \in \mathbb{P}$.*

Proof. Suppose $p \in \mathbb{P}$ and $p \geq 17$. since $17 > e^e$, we have $p \ln\ln p > p$ and $\ln p > 1$. Thus, $\ln p + p \ln\ln p > p + 1 = \sigma(p)$ and combining this with (2) yields result for $p \geq 17$. For $p < 17$, we obtain the result by a simple calculation. ☐

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Theorem 2. *The inequality (1) holds for all $n = p^a$, in which $p \in \mathbb{P}$ and $a \in \mathbb{N}$.*

Proof. We know that

$$\sigma(p^a) = \sum_{t=0}^{a} p^t = \frac{p^{a+1}-1}{p-1} < 2p^a, \tag{3}$$

and by (2) we have

$$H_{p^a} > \ln p^a = a \ln p.$$

So,

$$H_{p^a} + e^{H_{p^a}} \ln H_{p^a} > a \ln p + p^a \ln\ln p^a. \tag{4}$$

For $p^a \geq 1619 > e^{(e^2)}$, we have $\ln\ln p^a > 2$ and $a \ln p > 0$, so

$$p^a(\ln\ln p - 2) + a \ln p > 0,$$

combining this inequality with (3) and (4) yields (1) for $n = p^a \geq 1619$. For $p^a \leq 1618$, if $a = 1$ then (1) holds by previous theorem. The other possible cases are: $(a = 2, p = 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37), (a = 3, p = 2, 3, 5, 7, 11), (a = 4, p = 2, 3, 5), (a = 5, 6, p = 2, 3)$ and $(a = 7, 8, 9, 10, p = 2)$, which in all of them, (1) follow by a simple calculation. □

Theorem 3. *The inequality (1) holds for some sufficiently large square free values of n.*

Proof. Suppose $n = p_1 p_2 \cdots p_k$ in which $p_i \in \mathbb{P}$ and $2 \leq p_1 < p_2 < \cdots < p_k$. Since $\sigma(n) = (p_1+1)(p_2+1)\cdots(p_k+1)$ and

$$\frac{\sigma(n)}{n} = (1+\frac{1}{p_1})(1+\frac{1}{p_2})\cdots(1+\frac{1}{p_k}) < (1+\frac{1}{2})(1+\frac{1}{3})\cdots(1+\frac{1}{k}) = \frac{k+2}{2},$$

we obtain

$$\sigma(n) < (1+\frac{k}{2})n.$$

Now, for $n > e^{(e^{1+\frac{k}{2}})}$ we yield $\ln\ln n > 1+\frac{k}{2}$ and $n \ln\ln n > (1+\frac{k}{2})n > \sigma(n)$. Combining this with relation (2) yields (1) for $n > e^{(e^{1+\frac{k}{2}})}$ and n square free with k distinct prime factors. □

Note 1. In the theorem 3, $n = p_1 p_2 \cdots p_k > k! > \Gamma(k)$ and so,

$$k < \Gamma^{-1}(n).$$

Corollary 1. *The inequality (1) holds for all $n = pq$, in which $p, q \in \mathbb{P}$ and $2 \leq p < q$.*

Proof. For $n > e^{(e^2)}$ or $n \geq 1619$, use Theorem 3, and for $n \leq 1618$ check it by a computer. □

Corollary 2. *For proving (1) for $n = pqr$, we should check it for $n \leq 195339$ and the other cases yield by Theorem 3.*

Note 2. We guess that if we consider the ABC-conjecture [4](or [5]), then we can yield the inequality (1) at least for all sufficiently large square free integers and since the density of them is $\frac{6}{\pi^2}$ [2], we may yield that the probability that the Riemann hypothesis be true is more that 60%.

References

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