SOME MITROVIC TYPE TRIGONOMETRIC INEQUALITIES

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Abstract. In this short note, we give some parameter trigonometric inequalities.

1. Introduction

In 1967, Z.Mitrovic [1] obtained the following inequality for the parameter form of the triangle:

Theorem 1.1. *If λ is a real number, then in every triangle ABC, we have*

$$\cos A + \lambda(\cos B + \cos C) \leqslant 1 + \frac{\lambda^2}{2} \tag{1.1}$$

with equality holding if and only if $0 < \lambda < 2$, and $B = C = \frac{\pi}{2} - \arccos\frac{\lambda}{2}$.

Inequality (1.1) is called Mitrovic's inequality. In this short note, we give some new results on Mitrovic type inequality for the triangle.

2. Some Results for the Sine and Cosine

In this part, we will give some Mitrovic type inequalities for the sine and cosine on the triangle.

Theorem 2.1. *If λ is a real number, then in every triangle ABC, we have*

$$\cos 2A + \lambda(\sin 2B + \sin 2C) \leqslant 1 + \frac{\lambda^2}{2} \tag{2.1}$$

with equality holding if and only if $0 \leqslant \lambda \leqslant 2$, and $B = C = \frac{\pi}{2} - \frac{1}{2}\arcsin\frac{\lambda}{2}$.

Proof. Utilizing the facts that

$$\sin 2B + \sin 2C = 2\sin(B + C)\cos(B - C) = 2\sin A\cos(B - C),$$

and

$$\cos 2A = 1 + 2\cos^2 A,$$

we obtain

$$\begin{aligned}
\cos 2A + \lambda(\sin 2B + \sin 2C) &= \cos 2A + 2\lambda\sin A\cos(B - C)\\
&\leqslant \cos 2A + 2|\lambda|\sin A\\
&= -2\left(\sin A - \frac{|\lambda|}{2}\right)^2 + 1 + \frac{\lambda^2}{2}\\
&\leqslant 1 + \frac{\lambda^2}{2}
\end{aligned}$$

with equality holding if and only if $B = C, |\lambda| = \lambda$, and $\sin A = \frac{|\lambda|}{2}$, these are $0 \leqslant \lambda \leqslant 2$, and $B = C = \frac{\pi}{2} - \frac{1}{2}\arcsin\frac{\lambda}{2}$. ∎

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Corollary 2.1. *If λ is a real number, then in every triangle ABC, we have*

$$\cos A + \lambda(\sin B + \sin C) \leqslant 1 + \frac{\lambda^2}{2} \tag{2.2}$$

with equality holding if and only if $0 \leqslant \lambda \leqslant 2$, and $B = C = \arcsin \frac{\lambda}{2}$.

Corollary 2.2. *If λ is a real number, then in every triangle ABC, we have*

$$\cos 2A + \sqrt{3}(\sin 2B + \sin 2C) \leqslant \frac{5}{2} \tag{2.3}$$

with equality holding if and only if the triangle ABC is the equilateral one or $B = C = \frac{\pi}{6}$.

Theorem 2.2. *If λ is a real number, then in every triangle ABC, we have*

$$\cos A + \lambda(\sin 2B + \sin 2C) \leqslant \sqrt{1 + 4\lambda^2} \tag{2.4}$$

with equality holding if and only if $0 < \lambda$, and $B = C = \frac{\pi}{2} - \frac{1}{2}\arccos \frac{1}{\sqrt{1+4\lambda^2}}$.

Proof. By using the facts that

$$\sin 2B + \sin 2C = 2\sin(B + C)\cos(B - C) = 2\sin A \cos(B - C),$$

and Cauchy inequality, we obtain

$$\begin{aligned}
\cos A + \lambda(\sin 2B + \sin 2C) &= \cos A + 2\lambda \sin A \cos(B - C) \\
&\leqslant \cos A + 2|\lambda| \sin A \\
&\leqslant \sqrt{1 + 4\lambda^2}
\end{aligned}$$

with equality holding if and only if $B = C$ and $\frac{1}{\cos A} = \frac{2|\lambda|}{\sin A}$, these are $0 < \lambda$, and $B = C = \frac{\pi}{2} - \frac{1}{2}\arccos \frac{1}{\sqrt{1+4\lambda^2}}$. The proof of inequality (2.4) is completed. ∎

Corollary 2.3. *If λ is a real number, then in every triangle ABC, we have*

$$\sin \frac{A}{2} + \lambda(\sin B + \sin C) \leqslant \sqrt{1 + 4\lambda^2} \tag{2.5}$$

with equality holding if and only if $0 < \lambda$, and $B = C = \arccos \frac{1}{\sqrt{1+4\lambda^2}}$.

The proof of the following theorems and corollaries will be left to the readers.

Theorem 2.3. *If λ is a real number, then in every triangle ABC, we have*

$$\sin A + \lambda(\cos 2B + \cos 2C) \leqslant \sqrt{1 + 4\lambda^2} \tag{2.6}$$

with equality holding if and only if $0 < \lambda$, and $B = C = \frac{1}{2}\arccos \frac{1}{\sqrt{1+4\lambda^2}}$ or $0 \geqslant \lambda$, and $B = C = \frac{\pi}{2} - \frac{1}{2}\arccos \frac{1}{\sqrt{1+4\lambda^2}}$.

Corollary 2.4. *In every triangle ABC, and real number λ, we have*

$$\cos \frac{A}{2} + \lambda(\cos B + \cos C) \leqslant \sqrt{1 + 4\lambda^2} \tag{2.7}$$

with equality holding if and only if $0 < \lambda$, and $B = C = \arccos \frac{1}{\sqrt{1+4\lambda^2}}$.

Theorem 2.4. *If λ is a real number, then in every triangle ABC, we have*

$$\sin^2 A + \lambda(\sin^2 B + \sin^2 C) \leqslant 1 + \lambda + \frac{\lambda^2}{4} \tag{2.8}$$

with equality holding if and only if $0 < \lambda < 2$, and $B = C = \frac{\pi}{2} - \frac{1}{2}\arccos \frac{\lambda}{2}$.

Corollary 2.5. *If λ is a real number, then in every triangle ABC, we have*

$$\sin^2 A + \lambda(\sin B \sin C) \leqslant 1 + \frac{\lambda}{2} + \frac{\lambda^2}{16} \tag{2.9}$$

with equality holding if and only if $0 \leqslant \lambda < 4$, and $B = C = \frac{\pi}{2} - \frac{1}{2}\arccos\frac{\lambda}{4}$.

Remark 2.1. *When $\lambda = 1$, inequality (2.9) become Berkolajko's inequality [2]:*

$$\sin^2 A + \sin B \sin C \leqslant \frac{25}{16} \tag{2.10}$$

Corollary 2.6. *If λ is a real number, then in every triangle ABC, we have*

$$\cos^2 A + \lambda(\cos^2 B + \cos^2 C) \geqslant \lambda - \frac{\lambda^2}{4} \tag{2.11}$$

with equality holding if and only if $0 \leqslant \lambda < 2$, and $B = C = \frac{\pi}{2} - \frac{1}{2}\arccos\frac{\lambda}{2}$.

Corollary 2.7. *If λ is a real number, then in every triangle ABC, we have*

$$\cos^2\frac{A}{2} + \lambda(\cos^2\frac{B}{2} + \cos^2\frac{C}{2}) \geqslant \lambda - \frac{\lambda^2}{4} \tag{2.12}$$

with equality holding if and only if $0 < \lambda < 2$, and $B = C = \arccos\frac{\lambda}{2}$.

Theorem 2.5. *If λ is a real number, then in every triangle ABC, we have*

$$\sin^2 A + \lambda(\cos^2 B + \cos^2 C) \leqslant 1 + \lambda + \frac{\lambda^2}{4} \tag{2.13}$$

with equality holding if and only if $0 < \lambda < 2$, and $B = C = \frac{1}{2}\arccos\frac{\lambda}{2}$.

Corollary 2.8. *If λ is a real number, then in every triangle ABC, we have*

$$\sin^2 A + \lambda(\cos B \cos C) \leqslant 1 + \frac{\lambda}{2} + \frac{\lambda^2}{16} \tag{2.14}$$

with equality holding if and only if $0 \leqslant \lambda < 4$, and $B = C = \frac{1}{2}\arccos\frac{\lambda}{4}$.

Corollary 2.9. *If λ is a real number, then in every triangle ABC, we have*

$$\cos^2 A + \lambda(\sin^2 B + \sin^2 C) \geqslant \lambda - \frac{\lambda^2}{4} \tag{2.15}$$

with equality holding if and only if $0 < \lambda < 2$, and $B = C = \frac{1}{2}\arccos\frac{\lambda}{2}$.

Theorem 2.6. *If λ is a real number, then in every triangle ABC, we have*

$$\sin A + \lambda(\sin B + \sin C) \leqslant \frac{1}{8}(\lambda\sqrt{\lambda^2+8} - \lambda^2 + 4)\sqrt{2\lambda\sqrt{\lambda^2+8} + 2\lambda^2 + 4} \tag{2.16}$$

with equality holding if and only if $0 < \lambda$, and $B = C = \arccos\frac{\lambda\sqrt{\lambda^2+8}-\lambda^2}{4}$.

3. The Inequalities for the Tangent and Cotangent

Theorem 3.1. *Let $\lambda > 0$, then in every triangle ABC, we have*

$$\tan\frac{A}{2} + \lambda(\tan B + \tan C) \geqslant 2\sqrt{2\lambda} \tag{3.1}$$

with equality holding if and only if $B = C = \arctan\sqrt{2\lambda}$.

Proof. From the fact that

$$\tan B + \tan C = \frac{2\sin A}{\cos(B-C) - \cos A} \geqslant \frac{2\sin A}{1-\cos A} = 2\cot\frac{A}{2},$$

we get

$$\tan\frac{A}{2} + \lambda(\tan B + \tan C) \geqslant \tan\frac{A}{2} + 2\lambda\cot\frac{A}{2} \geqslant 2\sqrt{2\lambda},$$

with equality holding if and only if $B = C$, and $\blacksquare$

By the same way, we obtain

Theorem 3.2. *Let $\lambda > 0$, then in every triangle ABC, we have*

$$\cot\frac{A}{2} + \lambda(\cot B + \cot C) \geqslant 2\sqrt{2\lambda} \tag{3.2}$$

with equality holding if and only if $B = C = \arctan\sqrt{2\lambda}$.

4. Some Weighted Inequalities

Wolstenholme's inequality (4.1) [1] is a well-known weighted inequality for the triangle:

Theorem 4.1. *Let x, y, z are three real numbers, then in every triangle ABC, we have*

$$2yz\cos A + 2zx\cos B + 2xy\cos C \leqslant x^2 + y^2 + z^2 \tag{4.1}$$

with equality holding if and only if $x : y : z = \sin A : \sin B : \sin C$.

Theorem 4.2. *Let x, y, z are three real numbers for $xyz > 0$, and $u, v, w > 0$, then in every triangle we have the inequality*

$$x\sin A + y\sin B + z\sin C \leq \frac{1}{2}\left(\frac{yz}{x}u + \frac{zx}{y}v + \frac{xy}{z}w\right)\sqrt{\frac{u+v+w}{uvw}} \tag{4.2}$$

with both equalities holding if and only if $x\cos A = y\cos B = z\cos C$ and $u\cot A = v\cot B = w\cot C$.

Proof. Let $x = x_2x_3, y = x_3x_1$, and $z = x_1x_2$, then we have

$$\begin{aligned} x\sin A + y\sin B + z\sin C = {} & \frac{x_2x_3\cos(\pi - A - \theta_1)}{\sin\theta_1} + \frac{x_3x_1\cos(\pi - B - \theta_2)}{\sin\theta_2} \\ & + \frac{x_2x_3\cos(\pi - C - \theta_3)}{\sin\theta_3} + x_2x_3\cot\theta_1\cos A + x_3x_1\cot\theta_2\cos B + x_1x_2\cot\theta_3\cos C, \end{aligned} \tag{4.3}$$

where $\theta_1, \theta_2, \theta_3 > 0$ for $\theta_1 + \theta_2 + \theta_3 = \pi$.
Utilizing the fact that

$$\tan\theta_1 + \tan\theta_2 + \tan\theta_3 = \tan\theta_1\tan\theta_2\tan\theta_3, \tag{4.4}$$

we can set

$$\tan\theta_1 = \lambda\sqrt{\frac{\lambda+\mu+\nu}{\lambda\mu\nu}}, \tan\theta_2 = \mu\sqrt{\frac{\lambda+\mu+\nu}{\lambda\mu\nu}}, \tan\theta_3 = \nu\sqrt{\frac{\lambda+\mu+\nu}{\lambda\mu\nu}} \tag{4.5}$$

From Theorem4.1, we easily obtain

$$\begin{aligned} & \frac{x_2x_3\cos(\pi - A - \theta_1)}{\sin\theta_1} + \frac{x_3x_1\cos(\pi - B - \theta_2)}{\sin\theta_2} + \frac{x_2x_3\cos(\pi - C - \theta_3)}{\sin\theta_3} \\ & \leqslant \frac{1}{2}\left[(x_2^2 + x_3^2)\cot\theta_1 + (x_3^2 + x_1^2)\cot\theta_2 + (x_1^2 + x_2^2)\cot\theta_3\right], \end{aligned} \tag{4.6}$$

and

$$\begin{aligned}(4.7)\qquad & x_2x_3\cot\theta_1\cos A + x_3x_1\cot\theta_2\cos B + x_1x_2\cot\theta_3\cos C\\ & \leqslant \frac{1}{2}\cot\theta_1\cot\theta_2\cot\theta_3(x_1^2\tan^2\theta_1 + x_2^2\tan^2\theta + x_3^2\tan^2\theta).\end{aligned}$$

From (4.4), we find also that

$$\begin{aligned}(4.8)\qquad & \frac{1}{2}\left[(x_2^2+x_3^2)\cot\theta_1 + (x_3^2+x_1^2)\cot\theta_2 + (x_1^2+x_2^2)\cot\theta_3\right]\\ & + \frac{1}{2}\cot\theta_1\cot\theta_2\cot\theta_3(x_1^2\tan^2\theta_1 + x_2^2\tan^2\theta + x_3^2\tan^2\theta)\\ & = \frac{1}{2}(x_1^2\tan\theta_1 + x_2^2\tan\theta_2 + x_3\tan\theta_3).\end{aligned}$$

Combining $x = x_2x_3, y = x_3x_1, z = x_1x_2$, (4.3) and (4.5)-(4.8), we have the inequality (4.2). The proof of Theorem4.8 is completed. ∎

The inequality (4.2) is obtained by X.-Zh. Yang in [4]. There following theorems are the special cases of Theorem4.8.

Theorem 4.3. *(Oppenheim* [1]*) Let x, y, z are three real numbers, then in every triangle ABC, we have*

$$(4.9)\qquad yz\sin A + zx\sin B + xy\sin C \leqslant \frac{1}{2\sqrt{3}}(x+y+z)^2$$

with equality holding if and only if $x = y = z$ and triangle ABC is the equilateral one.

Theorem 4.4. *(Vasic* [1]*) Let x, y, z are three real numbers for $xyz > 0$, then in every triangle ABC, we have*

$$(4.10)\qquad x\sin A + y\sin B + z\sin C \leqslant \frac{\sqrt{3}}{2}\left(\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}\right)$$

with equality holding if and only if $x = y = z$ and triangle ABC is the equilateral one.

Theorem 4.5. *(Klamkin* [1]*) Let $x, y, z > 0$, then in every triangle ABC, we have*

$$(4.11)\qquad x\sin A + y\sin B + z\sin C \leqslant \frac{1}{2}(xy+yz+zx)\sqrt{\frac{x+y+z}{xyz}}$$

with equality holding if and only if $x = y = z$ and triangle ABC is the equilateral one.

Theorem 4.6. *(*[3]*) Let $x, y, z > 0$, and in every triangle we have the inequality*

$$(4.12)\qquad \sqrt{\frac{x}{y+z}}\sin A + \sqrt{\frac{y}{z+x}}\sin B + \sqrt{\frac{z}{x+y}}\sin C \le \sqrt{\frac{(x+y+z)^3}{(x+y)(y+z)(z+x)}}$$

with both equalities holding if and only if $x : y : z = \tan A : \tan B : \tan C$ or

$$\frac{\sin^2 A}{x(y+z)} = \frac{\sin^2 B}{y(z+x)} = \frac{\sin^2 C}{z(x+y)}.$$

Theorem 4.7. *(*[4]*) Let x, y, z are three real numbers, and $u, v, w > 0$, then in every triangle we have the inequality*

$$(4.13)\qquad yz\sin A + zx\sin B + xy\sin C \le \frac{1}{2}\left(\frac{x^2}{u} + \frac{y^2}{v} + \frac{z^2}{w}\right)\sqrt{vw+wu+uv}$$

with both equalities holding if and only if $x : \cos A = y : \cos B = z : \cos C$ and $u : \cot A = v : \cot B = w : \cot C$.

Theorem 4.8. *([3]) If $k, u, v, w > 0$, and*

$$\frac{1}{u^2+k} + \frac{1}{v^2+k} + \frac{1}{w^2+k} = \frac{2}{k} \tag{4.14}$$

in every triangle, we have the inequality

$$u\sin A + v\sin B + w\sin C \le \frac{1}{k}\sqrt{(u^2+k)(v^2+k)(w^2+k)} \tag{4.15}$$

with equality holding if and only if

$$\frac{u^2+k}{u}\sin A = \frac{v^2+k}{v}\sin B = \frac{w^2+k}{w}\sin C$$

or

$$u\cos A = v\cos B = w\cos C.$$

Theorem 4.9. *([3]) Let x, y, z are three real numbers, if $xyz > 0$, then in every triangle ABC, we have*

$$x\cos A + y\cos B + z\cos C \leqslant \frac{1}{2}\left(\frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z}\right), \tag{4.16}$$

and the reverse inequality holds if $xyz < 0$. With equality holding if and only if $\frac{1}{x} : \frac{1}{y} : \frac{1}{z} = \sin A : \sin B : \sin C$.

From Theorem4.1, we easily obtain the following corollary:

Corollary 4.1. *Let x, y, z are three real numbers, then in every triangle ABC, we have*

$$2yz\sin\frac{A}{2} + 2zx\sin\frac{B}{2} + 2xy\sin\frac{C}{2} \leqslant x^2 + y^2 + z^2 \tag{4.17}$$

with equality holding if and only if $x : y : z = \cos\frac{A}{2} : \cos\frac{B}{2} : \cos\frac{C}{2}$.

The proof of the following two inequalities will be left to the readers.

Theorem 4.10. *Let $x, y, z > 0$, then in every triangle ABC, we have*

$$(y+z)\cot A + (z+x)\cot B + (x+y)\cot C \geqslant 2\sqrt{yz+zx+xy}, \tag{4.18}$$

with equality holding if and only if $x : y : z = \cot A : \cot B : \cot C$.

Theorem 4.11. *Let $x, y, z > 0$, then in every triangle ABC, we have*

$$x\sin^2 A + y\sin^2 B + z\sin^2 C \leqslant \frac{(yz+zx+xy)^2}{4xyz}, \tag{4.19}$$

with equality holding if and only if $x\sin 2A = y\sin 2B = z\sin 2C$.

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