

# ON AN INEQUALITY OF DIANANDA, II

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ABSTRACT. We extend the result in part I ( Int. J. Math. Math. Sci., **2003**(2003), 2061-2068; MR1990724 (2004f:26027) ) of certain inequalities among the generalized power means.

## 1. INTRODUCTION

Let  $P_{n,r}(\mathbf{x})$  be the generalized weighted means:  $P_{n,r}(\mathbf{x}) = (\sum_{i=1}^n q_i x_i^r)^{\frac{1}{r}}$ , where  $P_{n,0}(\mathbf{x})$  denotes the limit of  $P_{n,r}(\mathbf{x})$  as  $r \rightarrow 0^+$ ,  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  and  $q_i > 0 (1 \leq i \leq n)$  are positive real numbers with  $\sum_{i=1}^n q_i = 1$ . In this paper, we let  $q = \min q_i$  and always assume  $n \geq 2, 0 \leq x_1 < x_2 < \dots < x_n$ .

We define  $A_n(\mathbf{x}) = P_{n,1}(\mathbf{x}), G_n(\mathbf{x}) = P_{n,0}(\mathbf{x}), H_n(\mathbf{x}) = P_{n,-1}(\mathbf{x})$  and we shall write  $P_{n,r}$  for  $P_{n,r}(\mathbf{x}), A_n$  for  $A_n(\mathbf{x})$  and similarly for other means when there is no risk of confusion.

For mutually distinct numbers  $r, s, t$  and any real number  $\alpha, \beta$ , we define

$$\Delta_{r,s,t,\alpha,\beta} = \left| \frac{P_{n,r}^\alpha - P_{n,t}^\alpha}{P_{n,r}^\beta - P_{n,s}^\beta} \right|,$$

where we interpret  $P_{n,r}^0 - P_{n,s}^0$  as  $\ln P_{n,r} - \ln P_{n,s}$ . When  $\alpha = \beta$ , we define  $\Delta_{r,s,t,\alpha}$  to be  $\Delta_{r,s,t,\alpha,\alpha}$ . We also define  $\Delta_{r,s,t}$  to be  $\Delta_{r,s,t,1}$ .

Bounds for  $\Delta_{r,s,t,\alpha,\beta}$  have been studied by many mathematicians. For the case  $\alpha \neq \beta$ , we refer the reader to the articles [2, 7, 10] for the detailed discussions. In the case  $\alpha = \beta$  and  $r > s > t$ , we seek the following bound

$$(1.1) \quad f_{r,s,t,\alpha}(q) \geq \Delta_{r,s,t,\alpha} \geq g_{r,s,t,\alpha}(q),$$

where  $f_{r,s,t,\alpha}(q)$  is a decreasing function of  $q$  and  $g_{r,s,t,\alpha}(q)$  is an increasing function of  $q$ .

For  $r = 1, s = 0, \alpha = 0, t = -1$  in (1.1), we can take  $f_{1,0,t,0}(q) = 1/q, g_{1,0,t,0}(q) = 1/(1-q)$ , when  $q_i = 1/n, 1 \leq i \leq n$ , this is the well-known Sierpiński's inequality[12](see [5] for a refinement of this). If we further require  $t, \alpha > 0$ , then consideration of the case  $n = 2, x_1 \rightarrow 0, x_2 = 1$  leads to the choice  $f_{r,s,t,\alpha} = C_{r,s,t}((1-q)^\alpha), g_{r,s,t,\alpha} = C_{r,s,t}(q^\alpha)$ , where

$$C_{r,s,t}(x) = \frac{1 - x^{1/t-1/r}}{1 - x^{1/s-1/r}}, t > 0; C_{r,s,0}(x) = \frac{1}{1 - x^{1/s-1/r}}.$$

We will show in Lemma 2.1 that  $C_{r,s,t}(x)$  is an increasing function of  $x (0 < x < 1)$  so the above choice for  $f, g$  is plausible. From now on, we will assume  $f, g$  to be so chosen.

Note when  $t > 0$ , the limiting case  $\alpha \rightarrow 0$  in (1.1) leads to Liapunov's inequality(see [8, p. 27]):

$$(1.2) \quad \Delta_{r,s,t,0} = \frac{\ln P_{n,r} - \ln P_{n,t}}{\ln P_{n,r} - \ln P_{n,s}} \leq \frac{s(r-t)}{t(r-s)} =: C(r, s, t).$$

From this(or by letting  $q \rightarrow 0$  when  $\alpha = 1$ ), one easily deduces the following result of H.Hsu[9](see also [1]):  $\Delta_{r,s,t} \leq C(r, s, t), r > s > t > 0$ .

The consideration of  $n = 2, x_2 \rightarrow x_1$  shows that the two inequalities in (1.1) can't hold simultaneously and from now on by saying (1.1) holds for  $r > s > t \geq 0, \alpha > 0$ , we mean the left-hand side

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inequality in (1.1) holds when  $C_{r,s,t}((1-q)^\alpha) \geq (r-t)/(r-s)$  and the right-hand side inequality in (1.1) holds when  $(r-t)/(r-s) \geq C_{r,s,t}(q^\alpha)$ .

More generally, for any set  $\{a, b, c\}$  with  $a, b, c$  mutually distinct and non-negative, we let  $r = \max\{a, b, c\}$ ,  $t = \min\{a, b, c\}$ ,  $s = \{a, b, c\}/\{r, t\}$ . By saying (1.1) holds for the set  $\{a, b, c\}$ ,  $\alpha > 0$  we mean (1.1) holds for  $r > s > t \geq 0, \alpha > 0$ .

In the case  $\alpha = 1$ , a result of P. Diananda([3], [4])(see also [1],[11]) shows (1.1) holds for  $\{1, 1/2, 0\}$  and recently, the author[6] has shown that (1.1) holds for  $\{r, 1, 0\}$  and  $\{r, 1, 1/2\}$ , where  $r \in (0, \infty)$ . It is the goal of this paper to further extend the result in [6].

## 2. LEMMAS

**Lemma 2.1.** *For  $0 < x < 1, 0 \leq t < s < r, C_{r,s,t}(x)$  is a strictly increasing function of  $x$ . In particular, for  $0 < q \leq 1/2, C_{r,s,t}(1-q) \geq C_{r,s,t}(q)$ .*

*Proof.* We may assume  $t > 0$ . Note  $C_{r,s,t}(x) = C_{1,t/r,s/r}(x^{1/r})$ , thus it suffices to prove the lemma for  $C_{1,r,s}$  with  $1 > r > s > 0$ . By the mean value theorem,

$$\frac{1/s - 1}{1/r - 1} \cdot \frac{1 - x^{1/r-1}}{1 - x^{1/s-1}} = \eta^{1/r-1/s} < x^{1/r-1/s}$$

for some  $x < \eta < 1$  and this implies  $C'_{1,r,s}(x) > 0$  which completes the proof.  $\square$

**Lemma 2.2.** *For  $1/2 < r < 1, C_{1,r,1-r}(1/2) > r/(1-r)$ .*

*Proof.* By setting  $x = r/(1-r) > 1$ , it suffices to show  $f(x) > 0$  where  $f(x) = 1 - 2^{-x} - x(1 - 2^{-1/x})$ . Now  $f''(x) = (\ln 2)^2 2^{-x} x^{-3} (2^{x-1/x} - x^3)$  and let  $g(x) = (x - 1/x) \ln 2 - 3 \ln x$ . Note  $g'(x)$  has one root in  $(1, \infty)$  and  $g(1) = 0$ , it follows that  $g(x)$  hence  $f''(x)$  has only one root  $x_0$  in  $(1, \infty)$ . Note when  $f''(x) > 0$  for  $x > x_0$ , this together with the observation that  $f(1) = 0, f'(1) = \ln 2 - \frac{1}{2} > 0, \lim_{x \rightarrow \infty} f(x) = 1 - \ln 2 > 0$  shows  $f(x) > 0$  for  $x > 1$ .  $\square$

**Lemma 2.3.** *Let  $0 < q \leq 1/2$ . For  $0 < s < r < 1, r + s \geq 1, C_{1,r,s}(1-q) > (1-s)/(1-r)$ . For  $0 \leq s < 1 < r, C_{r,1,s}(1-q) > (r-s)/(r-1)$  and for  $1 < s < r, C_{r,s,1}(1-q) > (r-1)/(r-s)$ .*

*Proof.* We shall give a proof for the case  $1 > r > s > 0, r + s \geq 1$  here and the proofs for the other cases are similar. We note first that in this case  $1/2 < r < 1$ . By Lemma 2.1, it suffices to prove  $C_{1,r,s}(1/2) > (1-s)/(1-r)$ . Consider

$$f(s) = (1-r)(1 - (\frac{1}{2})^{1/s-1}) - (1-s)(1 - (\frac{1}{2})^{1/r-1}).$$

We have  $f(r) = 0$  and Lemma 2.2 implies  $f(1-r) > 0$ . Now  $f'(r) = 2^{1-1/r} g(1/r)$  where  $g(x) = -\ln 2(x^2 - x) + 2^{x-1} - 1$  with  $1 < x < 2$ . One checks easily  $g(1) = g'(1) = 0, g''(x) < 0$  which implies  $g(x) < 0$ . Hence  $f'(r) < 0$ , this combined with the observation that

$$f''(s) = (1-r) \ln 2 (\frac{1}{2})^{1/s-1} (2s - \ln 2) / s^4$$

has at most one root and  $f''(r) > 0, f(1-r) > 0, f(r) = 0$  implies  $f(s) > 0$  for  $1-r \leq s < r$ .  $\square$

## 3. THE MAIN THEOREMS

**Theorem 3.1.** *Let  $\alpha = 1$ . For the set  $\{1, r, s\}$ , with  $1, r, s$  mutually distinct and  $r > s \geq 0, r + s \geq 1$ , the left-hand side inequality of (1.1) holds. The equality holds if and only if  $n = 2, x_1 = 0, q_1 = q$ .*

*Proof.* The case  $s = 0$  was treated in [6], so we may assume  $s > 0$  here. We shall give a proof for the case  $1 > r > s > 0$  here and the proofs for the other cases are similar. Define

$$D_n(\mathbf{x}) = A_n - P_{n,r} - C(1-q)(A_n - P_{n,s}), C(x) = \frac{1 - x^{1/r-1}}{1 - x^{1/s-1}}.$$

By Lemma 2.3, we need to show  $D_n \geq 0$  and we have

$$(3.1) \quad \frac{1}{q_n} \frac{\partial D_n}{\partial x_n} = 1 - P_{n,r}^{1-r} x_n^{r-1} - C(1-q)(1 - P_{n,s}^{1-s} x_n^{s-1}).$$

By a change of variables:  $\frac{x_i}{x_n} \rightarrow x_i, 1 \leq i \leq n$ , we may assume  $0 \leq x_1 < x_2 < \dots < x_n = 1$  in (3.1) and rewrite it as

$$(3.2) \quad g_n(x_1, \dots, x_{n-1}) := 1 - P_{n,r}^{1-r} - C(1-q)(1 - P_{n,s}^{1-s}).$$

We want to show  $g_n \geq 0$ . Let  $\mathbf{a} = (a_1, \dots, a_{n-1}) \in [0, 1]^{n-1}$  be the point in which the absolute minimum of  $g_n$  is reached. We may assume  $a_1 \leq a_2 \leq \dots \leq a_{n-1}$ . If  $a_i = a_{i+1}$  for some  $1 \leq i \leq n-2$  or  $a_{n-1} = 1$ , by combing  $a_i$  with  $a_{i+1}$  and  $q_i$  with  $q_{i+1}$  or  $a_{n-1}$  with 1 and  $q_{n-1}$  with  $q_n$ , it follows from Lemma 2.1 that we can reduce the determination of the absolute minimum of  $g_n$  to that of  $g_{n-1}$  with different weights. Thus without loss of generality, we may assume  $a_1 < a_2 < \dots < a_{n-1} < 1$ .

If  $\mathbf{a}$  is a boundary point of  $[0, 1]^{n-1}$ , then  $a_1 = 0$ , and we can regard  $g_n$  as a function of  $a_2, \dots, a_{n-1}$ , then we obtain

$$\nabla g_n(a_2, \dots, a_{n-1}) = 0.$$

Otherwise  $a_1 > 0$ ,  $\mathbf{a}$  is an interior point of  $[0, 1]^{n-1}$  and

$$\nabla g_n(a_1, \dots, a_{n-1}) = 0.$$

In either case  $a_2, \dots, a_{n-1}$  solve the equation

$$(r-1)P_{n,r}^{1-2r} x^{r-1} + C(1-q)(1-s)P_{n,s}^{1-2s} x^{s-1} = 0.$$

The above equation has at most one root (regarding  $P_{n,r}, P_{n,s}$  as constants), so we only need to show  $g_n \geq 0$  for the case  $n = 3$  with  $0 = a_1 < a_2 = x < a_3 = 1$  in (3.2). In this case we regard  $g_3$  as a function of  $x$  and we get

$$\frac{1}{q_2} g_3'(x) = P_{3,r}^{1-2r} x^{r-1} h(x),$$

where

$$h(x) = r - 1 + (1-s)C(1-q)(q_2 x^{s/2} + q_3 x^{-s/2})^{(1-2s)/s} (q_2 x^{r/2} + q_3 x^{-r/2})^{(2r-1)/r}.$$

If  $q_2 = 0$  (note  $q_3 > 0$ ), then

$$h(x) = r - 1 + (1-s)C(1-q)q_3^{1/s-1/r} x^{s-r}.$$

One easily checks that in this case  $h(x)$  has exactly one root in  $(0, 1)$ . Now assume  $q_2 > 0$ , then

$$h'(x) = (1-s)C(1-q)P_{3,s}^{1-3s} P_{3,r}^{r-1} x^{-\frac{r+s+2}{2}} p(x),$$

where

$$p(x) = (r-s)(q_2^2 x^{r+s} - q_3^2) + (r+s-1)q_2 q_3 (x^r - x^s).$$

Now

$$p'(x) = x^{s-1}((r^2 - s^2)q_2^2 x^r + (r+s-1)q_2 q_3 (rx^{r-s} - s)) := x^{s-1} q(x).$$

If  $r+s \geq 1$  then  $q'(x) > 0$  which implies there can be at most one root for  $p'(x) = 0$ . Since  $p(0) < 0$  and  $\lim_{x \rightarrow \infty} p(x) = +\infty$ , we conclude that  $p(x)$  hence  $h'(x)$  has at most one root. Since  $h(1) < 0$  by Lemma 2.3 and  $\lim_{x \rightarrow 0^+} h(x) = +\infty$ , this implies  $h(x)$  has exactly one root in  $(0, 1)$ .

Thus  $g_3'(x)$  has only one root  $x_0$  in  $(0, 1)$ . Since  $g_3'(1) < 0$ ,  $g_3(x)$  takes its maximum value at  $x_0$ . Thus  $g_3(x) \geq \min\{g_3(0), g_3(1)\} = 0$ .

Thus we have shown  $g_n \geq 0$ , hence  $\frac{\partial D_n}{\partial x_n} \geq 0$  with equality holding if and only if  $n = 1$  or  $n = 2, x_1 = 0, q_1 = q$ . By letting  $x_n$  tend to  $x_{n-1}$ , we have  $D_n \geq D_{n-1}$  (with weights  $q_1, \dots, q_{n-2}, q_{n-1} + q_n$ ). Since  $C$  is an increasing function of  $q$ , it follows by induction that  $D_n > D_{n-1} > \dots > D_2 = 0$  when  $x_1 = 0, q_1 = q$  in  $D_2$ . Else  $D_n > D_{n-1} > \dots > D_1 = 0$ . Since we assume  $n \geq 2$  in this paper, this completes the proof.  $\square$

The relations between (1.1) and (1.2) seems to suggest that if the left-hand side inequality of (1.1) holds for  $r > s > t \geq 0, \alpha > 0$ , then the left-hand side inequality of (1.1) also holds for  $r > s > t \geq 0, k\alpha$  with  $k < 1$  and if the right-hand side inequality of (1.1) holds for  $r > s > t \geq 0, \alpha > 0$ , then the right-hand side inequality of (1.1) also holds for  $r > s > t \geq 0, k\alpha$  with  $k > 1$ . We don't know the answer in general but for a special case, we have the following:

**Theorem 3.2.** *Let  $r > s > 0$ , if the right-hand side inequality of (1.1) holds for  $\{r, s, 0\}, \alpha > 0$ , then it also holds for  $\{r, s, 0\}, k\alpha$  with  $k > 1$ . If the left-hand side inequality of (1.1) holds for  $\{r, s, 0\}, \alpha > 0$ , then it also holds for  $\{r, s, 0\}, k\alpha$  with  $0 < k < 1$ .*

*Proof.* We will only prove the first assertion here and the second can be proved similarly. By the assumption, we have

$$P_{n,r}^\alpha - G_n^\alpha \geq \frac{1}{1 - (q^\alpha)^{\frac{1}{s} - \frac{1}{r}}} (P_{n,r}^\alpha - P_{n,s}^\alpha).$$

We write the above as

$$(3.3) \quad P_{n,s}^\alpha \geq (q^\alpha)^{\frac{1}{s} - \frac{1}{r}} P_{n,r}^\alpha + (1 - (q^\alpha)^{\frac{1}{s} - \frac{1}{r}}) G_n^\alpha.$$

We now need to show for  $k > 1$ ,

$$P_{n,s}^{k\alpha} \geq (q^{k\alpha})^{\frac{1}{s} - \frac{1}{r}} P_{n,r}^{k\alpha} + (1 - (q^{k\alpha})^{\frac{1}{s} - \frac{1}{r}}) G_n^{k\alpha}.$$

Note by (3.3), via setting  $w = (q^{k\alpha})^{1/s-1/r}, x = G_n/P_{n,r}$ , it suffices to show

$$f(x) =: (w + (1-w)x^k)^{1/k} - w^{1/k} - (1-w^{1/k})x \leq 0,$$

for  $0 \leq w, x \leq 1$ . Note

$$f'(x) = (1-w)(wx^{-k} + (1-w))^{1/k-1} - (1-w^{1/k}),$$

thus  $f'(x)$  can have at most one root in  $(0, 1)$ , note also  $f(0) = f(1) = 0$  and  $f'(1) > 0$ , we then conclude  $f(x) \leq 0$  for  $0 \leq x \leq 1$  and this completes the proof.  $\square$

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