INEQUALITIES INVOLVING BERNOULLI AND EULER NUMBERS

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Abstract. Classical Cauchy-Buniakowski-Schwartz's inequality and some identities involving Bernoulli and Euler polynomials are used to obtain some elementary numerical inequalities.

1. Introduction

Cauchy-Buniakowski-Schwartz's inequality ([1], [2], [4]) plays an important role in many branches of Pure and Applied Mathematics as it is well known. It is also a source of some interesting inequalities in the complex plane ([7]) or in Problem Solving ([5], [6]). In this note we use it and some numerical identities to obtain several inequalities involving Bernoulli polynomials and numbers and Bernoulli and Euler polynomials and Euler numbers ([3]).

2. Inequalities

In the sequel, using the complex version of Cauchy-Buniakowski-Schwartz inequality ([4]) and some numerical identities involving Bernoulli and Euler numbers ([3]) various elementary inequalities are obtained. We begins with ([7]):

Theorem 2.1. *Let $\alpha \in \mathbb{C}$ and $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \cdots, b_n$ be sequences of complex numbers, then holds the following inequality*

$$Re\left(\overline{\alpha}\sum_{k=0}^{n} a_k b_k\right) \leq \frac{1}{2}\left(\sum_{k=0}^{n} |a_k|^2 + |\alpha|^2 \sum_{k=0}^{n} |b_k|^2\right) \tag{2.1}$$

Note that the inequality claimed in the preceding statement generalizes the Cauchy-Buniakowski-Schwartz inequality.
Next we apply (2.1), and using Bernoulli and Euler polynomials and Bernoulli and Euler numbers, several numerical inequalities in the complex plane are given:

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Theorem 2.2. *If $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \cdots, b_n$ are sequences of complex numbers, then for all positive integers m, n holds*

$$\mathit{Re}\left(\sum_{k=0}^{n} a_k b_k\right) \le \frac{m+1}{2\left(B_{m+1}(n) - B_{m+1}\right)} \left(n \sum_{k=0}^{n} |a_k|^2 + \frac{B_{2m+1}(n)}{2m+1} \sum_{k=0}^{n} |b_k|^2\right) \tag{2.2}$$

where $B_p(n)$ and $B_p = B_p(0)$ are the Bernoulli polynomial of degree p and the pth Bernoulli's number respectively.

Proof. Setting $\alpha = k^m, 0 \le k \le n-1$ into (2.1), we get

$$k^m \operatorname{Re}\left(\sum_{k=0}^{n} a_k b_k\right) \le \frac{1}{2}\left(\sum_{k=0}^{n} |a_k|^2 + k^{2m} \sum_{k=0}^{n} |b_k|^2\right) \tag{2.3}$$

Adding up the inequalities in (2.3) and taking into account the well known identities

$$\sum_{k=0}^{n-1} k^m = \frac{1}{m+1}\left(B_{m+1}(n) - B_{m+1}(0)\right), \qquad \sum_{k=0}^{n-1} k^{2m} = \frac{1}{2m+1} B_{2m+1}(n)$$

yields

$$\frac{B_{m+1}(n) - B_{m+1}(0)}{m+1} \operatorname{Re}\left(\sum_{k=0}^{n} a_k b_k\right) \le \frac{1}{2}\left(n \sum_{k=0}^{n} |a_k|^2 + \frac{B_{2m+1}(n)}{2m+1} \sum_{k=0}^{n} |b_k|^2\right).$$

From the preceding inequality rearranging terms (2.2) immediately follows and the proof is complete. □

Corollary 2.3. *If $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \cdots, b_n$ are sequences of complex numbers, then the following inequality*

$$\mathit{Re}\left(\sum_{k=0}^{n} a_k b_k\right) \le \frac{m+1}{2\, B_{m+1}(n)} \left(n \sum_{k=0}^{n} |a_k|^2 + \frac{B_{2m+1}(n)}{2m+1} \sum_{k=0}^{n} |b_k|^2\right) \tag{2.4}$$

holds for every even positive integer number m.

Proof. The inequality claimed immediately follows from (2.2) and the fact that when $m \in \mathbb{N}$ is an even number then $B_{m+1} \equiv 0$. □

Theorem 2.4. *If $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \cdots, b_n$ are sequences of complex numbers, then for every positive integers m and n, holds*

$$\mathit{Re}\left(\sum_{k=0}^{n} a_k b_k\right) \le \frac{1}{E_m(n) - (-1)^n E_m(0)} \left(n \sum_{k=0}^{n} |a_k|^2 + \frac{B_{2m+1}(n)}{2m+1} \sum_{k=0}^{n} |b_k|^2\right) \tag{2.5}$$

where $B_p(n)$ and $E_p(n)$ are respectively the Bernoulli and Euler polynomials of degree p ([3]).

Proof. Setting $\alpha = (-1)^{n-k-1}\, k^m$, $0 \le k \le n-1$ into (2.1), we get for all $k \in \{0, 1, \cdots, n-1\}$,

$$(-1)^{n-k-1}\, k^m \operatorname{Re}\left(\sum_{k=0}^{n} a_k b_k\right) \le \frac{1}{2}\left(\sum_{k=0}^{n} |a_k|^2 + k^{2m} \sum_{k=0}^{n} |b_k|^2\right) \tag{2.6}$$

Adding up the inequalities in (2.6) and taking into account the well known identities

$$\sum_{k=0}^{n-1} (-1)^{n-k-1}\, k^m = \frac{E_m(n) - (-1)^n E_m(0)}{2} > 0, \qquad \sum_{k=0}^{n-1} k^{2m} = \frac{B_{2m+1}(n)}{2m+1},$$

yields

$$\frac{E_m(n) - (-1)^n E_m(0)}{2} \operatorname{Re}\left(\sum_{k=0}^{n} a_k b_k\right) \le \frac{1}{2}\left(n \sum_{k=0}^{n} |a_k|^2 + \frac{B_{2m+1}(n)}{2m+1} \sum_{k=0}^{n} |b_k|^2\right).$$

From the preceding inequality and rearranging terms (2.5) immediately follows and the proof is complete. □

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