

# ON INTEGRAL REPRESENTATION OF FIRST KIND BESSEL FUNCTION

BISERKA DRAŠČIĆ, TIBOR K. POGÁNY

ABSTRACT. A first kind Fredholm integral equation with nondegenerate kernel is given, which particular solution is the Bessel function of first kind. This equation is solved by means of Mellin transform pair.

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## 1. INTRODUCTION

Concerning the sources of special functions, the most exhaustive collection of 396 formulæ involving first kind Bessel functions the authors find on the widely known website [3]. The main purpose of this short note is to give a new, definite integral definition of Bessel function of the first kind (such that is not contained on the above mentioned website). This goal we realize by getting a Fredholm type integral equation of first kind with degenerate kernel. Particular solution of this equation is the desired Bessel  $J$ . To solve the integral equation we use the Mellin integral transform technique.

The most frequently used notations we need are  $J_\nu(x)$  - the Bessel function of the first kind;

$$\mathcal{L}_x f := \int_0^\infty e^{-xv} f(v) dv, \quad x > 0$$

denotes the real parameter ordinary Laplace transform of certain  $f$ , while

$$\mathcal{M}_p(g) := \int_0^\infty x^{p-1} g(x) dx, \quad \mathcal{M}_x^{-1}(g) := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-p} \mathcal{M}_p(g) dp,$$

is the Mellin transform pair of  $g$ . Here the real  $c$  belongs to the so-called *fundamental strip*  $\langle \cdot, \cdot \rangle$  of the inverse Mellin transform  $\mathcal{M}^{-1}$ , see [8].

We will say that functions  $f, g$  are *orthogonal a.e.* with respect to the ordinary Lebesgue measure on the positive halfline when  $\int_0^\infty f(x)g(x)dx$  vanishes, writing this by  $f \perp g$ .

Let us introduce the series

$$S_\mu(r, \alpha) = \sum_{n=0}^{\infty} \frac{2n^{\alpha/2}}{(n^\alpha + r^2)^{\mu+1}}, \quad \alpha, \mu, r > 0, \quad (1)$$

which has fundamental (but passive!) role in this note and such that is the subject of interest of many authors in the last time. This series we call *generalized Mathieu series* in the sequel following Qi, see [6],[7].

Finally,  $[t]$ ,  $\{t\}$  stay for the integer and the fractional part of real  $t$ ; *if and only if* is abbreviated into **iff** and  $\square$  will denote the end of proofs.

## 2. INTEGRAL EQUATION FOR $J_\nu(x)$

The generalized Mathieu series  $S_\mu(r, \alpha)$  possesses several closed form representations involving definite integrals. The most recent ones are given by Cerone & Lenard and by Pogány, consult [1], [5]. (We have to remark at this point that Tomovski deduced a similar result. Namely, instead of Euler-McLaurin summation formula used by Pogány in [5] he apply the trapesoidal rule to derive the integral expression for  $S_\mu(r, \alpha)$ , [9]). So, the heart of the matter are the mentioned integral expressions of the same subject, of the generalized Mathieu series  $S_\mu(r, \alpha)$ .

**Theorem 1.** *The first kind Fredholm type convolutional integral equation*

$$\int_0^\infty x^{\nu+1} \psi(rx) \mathcal{L}_x[v^{2/\alpha}] dx = g_\nu(r, \alpha) \quad (2)$$

*possesses particular solution  $\psi(x) = J_\nu(x) + h(x)$ ,  $h \perp x^{\nu+1} \mathcal{L}_x \mathbb{I}_{[1, \infty)}(v)[v^{2/\alpha}]$  iff*

$$g_\nu(r, \alpha) = \frac{(2r)^\nu \Gamma(\nu + 5/2)}{\sqrt{\pi}(\alpha + 2)} \int_1^\infty \frac{4[t^{1/\alpha}]^{\alpha/2+1} + \alpha(\alpha + 2) \int_1^{[t^{1/\alpha}]} u^{\alpha/2-1} \{u\} du}{(r^2 + t)^{\nu+5/2}} dt. \quad (3)$$

*Proof.* In [1] Cerone and Lenard show that

$$S_\mu(r, \alpha) = \frac{\sqrt{\pi}}{(2r)^{\mu-1/2} \Gamma(\mu + 1)} \int_0^\infty x^{\mu+1/2} J_{\mu-1/2}(rx) \left( \sum_{n=1}^{\infty} e^{-n^{\alpha/2} x} \right) dx. \quad (4)$$

On the other hand the standard Dirichlet-series

$$\mathcal{D}_\alpha(x) = \sum_{n=1}^{\infty} e^{-n^{\alpha/2} x}$$

possesses well-known integral representation of the form

$$\mathcal{D}_\alpha(x) = x \int_0^\infty e^{-xv} A(v) dv, \quad (5)$$

with the counting function

$$A(v) = \sum_{n: n^{\alpha/2} \leq v} 1 = [v^{2/\alpha}],$$

compare [4, 5]. So, by reducing (4), we clearly get

$$S_\mu(r, \alpha) = \frac{\sqrt{\pi}}{(2r)^{\mu-1/2}\Gamma(\mu+1)} \int_0^\infty x^{\mu+1/2} J_{\mu-1/2}(rx) \int_1^\infty e^{-xv} [v^{2/\alpha}] dv dx. \quad (6)$$

By obvious reasons, taking the Laplace transform notation for  $[v^{2/\alpha}]$  the relation (6) becomes

$$S_\mu(r, \alpha) = \frac{\sqrt{\pi}}{(2r)^{\mu-1/2}\Gamma(\mu+1)} \int_0^\infty x^{\mu+1/2} J_{\mu-1/2}(rx) \mathcal{L}_x[v^{2/\alpha}] dx. \quad (7)$$

Having in mind the second authors integral expression result [5, Eq.(13)] reads as follows

$$S_\mu(r, \alpha) = \frac{\mu+1}{\alpha+2} \int_1^\infty \frac{4[t^{1/\alpha}]^{\alpha/2+1} + \alpha(\alpha+2) \int_1^{[t^{1/\alpha}]} u^{\alpha/2-1} \{u\} du}{(r^2+t)^{\mu+2}} dt \quad (8)$$

by comparing (7) and (8) and taking the basic identity  $z\Gamma(z) = \Gamma(z+1)$  with  $z = \mu+1$ , we deduce

$$\begin{aligned} & \int_0^\infty x^{\mu+1/2} J_{\mu-1/2}(rx) \mathcal{L}_x[v^{2/\alpha}] dx \\ &= \frac{(2r)^{\mu-1/2}\Gamma(\mu+2)}{\sqrt{\pi}(\alpha+2)} \int_0^\infty \frac{4[t^{1/\alpha}]^{\alpha/2+1} + \alpha(\alpha+2) \int_0^{[t^{1/\alpha}]} u^{\alpha/2-1} \{u\} du}{(r^2+t)^{\mu+2}} dt \\ &= \frac{(2r)^{\mu-1/2}\Gamma(\mu+2)}{\sqrt{\pi}(\alpha+2)} \int_1^\infty \frac{4[t^{1/\alpha}]^{\alpha/2+1} + \alpha(\alpha+2) \int_1^{[t^{1/\alpha}]} u^{\alpha/2-1} \{u\} du}{(r^2+t)^{\mu+2}} dt. \end{aligned} \quad (9)$$

where the second relation holds because the integration domain reduces from  $\mathbb{R}_+$  to  $[1, \infty)$  by  $[u] = 0$ ,  $0 \leq u < 1$ .

Endly, writing  $\nu = \mu - 1/2$  in the last equation the right hand expression in (9) becomes  $g_\nu(r, \alpha)$  defined by (3). The derived first kind Fredholm convolution has a solution given in Theorem 1.  $\square$

**Example 1.** We give a simple construction of function  $h$  mentioned in the Theorem 1. In this goal consider the nonnegative random variable  $\xi_{\nu, \alpha}$  defined on fixed probability space  $(\Omega, \mathfrak{F}, \mathbb{P})$  having probability density function

$$f_{\nu, \alpha}(x) = \begin{cases} \frac{x^{\nu+1} \mathcal{L}_x[v^{2/\alpha}]}{\Gamma(\nu+2) \mathcal{M}_{-\nu-1}([v^{2/\alpha}])} & x > 0, \\ 0 & x \leq 0. \end{cases} \quad (10)$$

It is interesting that  $f_{\nu, 2}(x)$  reduces to

$$f_{\nu, 2}(x) = \begin{cases} \frac{(\nu+1)x^\nu}{\Gamma(\nu+2)\zeta(\nu+1)(e^x-1)} & x > 0, \\ 0 & x \leq 0, \end{cases}$$

where  $\zeta(\cdot)$  denotes the Riemann zeta function, and by obvious reasons  $\nu \geq 1$ .

As the probability is monotonous increasing with respect to the increasing sequences of random events, there exists the unique median  $x_{0.5}$ , i.e. the solution of the equation

$$\mathbb{P}\{\xi_{\nu,\alpha} \leq x_{0.5}\} = \int_0^{x_{0.5}} f_{\nu,\alpha}(x) dx = \frac{1}{2}.$$

Now the desired function

$$h(x) = \begin{cases} 1 & x \in [0, x_{0.5}), \\ -1 & x \geq x_{0.5}. \end{cases}$$

is the solution of the homogeneous variant of the equation (2).  $\square$

### 3. $J_\nu(x)$ GIVEN VIA MELLIN TRANSFORM

It is obvious that the equation (2) is invariant with respect to  $\alpha$ , so without any loss of generality we can put  $\alpha = 2$ . Now (9) from the proof of the Theorem becomes

$$\int_0^\infty \frac{x^{\nu+1}}{e^x - 1} J_\nu(rx) dx = \frac{\Gamma(\nu + \frac{5}{2})(2r)^\nu}{\sqrt{\pi}} \int_1^\infty \frac{[\sqrt{t}]([\sqrt{t}] + 1)}{(r^2 + t)^{\nu+5/2}} dt, \quad (11)$$

see [1],[5, Eq.(16)]. By applying Mellin transform for the convolution as given in [2], from (11) we get the following equation

$$\Phi(s) = \frac{2^\nu \Gamma(\nu + \frac{5}{2})}{\sqrt{\pi} \Gamma(\nu - s + 2) \zeta(\nu - s + 2)} \mathcal{M}_s \left( r^\nu \int_1^\infty \frac{[\sqrt{t}]([\sqrt{t}] + 1)}{(r^2 + t)^{\nu+5/2}} dt \right) \quad (12)$$

where  $\Phi(s)$  is the Mellin transform of the Bessel function  $J_\nu(x)$ . To express the Bessel function explicitly we now apply the inverse Mellin transform on (12), and we get the following identity:

$$J_\nu(x) = \frac{2^{\nu-1} \Gamma(\nu + \frac{5}{2})}{\pi^{3/2} i} \int_{c-i\infty}^{c+i\infty} \frac{\mathcal{M}_s \left( r^\nu \int_1^\infty \frac{[\sqrt{t}]([\sqrt{t}] + 1)}{(r^2 + t)^{\nu+5/2}} dt \right)}{\Gamma(\nu - s + 2) \zeta(\nu - s + 2) x^s} ds, \quad (13)$$

where  $c$  in the integral bounds is from the fundamental strip [8] of the function

$$\phi(r) = r^\nu \int_1^\infty \frac{[\sqrt{t}]([\sqrt{t}] + 1)}{(r^2 + t)^{\nu+5/2}} dt. \quad (14)$$

Now, it can be easily shown that the fundamental strip for  $\phi(r)$  contains  $\langle -\nu, \nu + 1 \rangle$ , so we can put  $c = \frac{1}{2}$ , say. Indeed, after some short calculation we have

$$\phi(r) \leq \begin{cases} \frac{(4\nu + 3)r^\nu}{(\nu + 1)(2\nu + 1)} & r \rightarrow 0, \\ \frac{(4\nu + 3)r^\nu}{(\nu + 1)(2\nu + 1)(1 + r^2)^{\nu+1/2}} & r \rightarrow \infty. \end{cases}$$

From this we deduce that the fundamental strip does contain  $\langle -\nu, \nu + 1 \rangle$  since

$$\phi(r) = \begin{cases} \mathcal{O}(r^\nu) & r \rightarrow 0, \\ \mathcal{O}(r^{-\nu-1/2}) & r \rightarrow \infty. \end{cases}$$

Therefore we can clearly choose the value  $c = \frac{1}{2}$  for the Bromwich - Wagner type integration contour in deriving the inverse Mellin transform. Therefore (13) becomes

$$J_\nu(x) = \frac{2^{\nu-1}\Gamma(\nu + \frac{5}{2})}{\pi^{3/2}i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\mathcal{M}_s \left( r^\nu \int_1^\infty \frac{[\sqrt{t}]([\sqrt{t}]+1)}{(r^2+t)^{\nu+5/2}} dt \right)}{\Gamma(\nu - s + 2)\zeta(\nu - s + 2)x^s} ds \quad (15)$$

which is the desired particular solution of (11) and the integral representation of the Bessel function of the first kind.

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DEPARTMENT OF SCIENCES, FACULTY OF MARITIME STUDIES, UNIVERSITY OF RIJEKA, 51000 RIJEKA, STUDENTSKA 2, CROATIA

*E-mail address:* bdrascic@pfri.hr, poganj@pfri.hr