SOME MONOTONICITY QUESTIONS FOR GEOMETRICALLY CONVEX FUNCTIONS

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ABSTRACT. In this paper, by using some properties of geometrically convex functions, some monotone functions and sequences are constructed. In the final, an open problem is posed.

1. INTRODUCTION

Throughout the paper we assume $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$, and R^n be the n-dimensional Euclidean Space, $R_+^n = \{(x_1, x_2, \cdots, x_n), x_i > 0, i = 1, 2, \cdots, n\}$, and $\alpha x = (\alpha x_1, \alpha x_2, \cdots, \alpha x_n), e^x = (e^{x_1}, e^{x_2}, \cdots, e^{x_n}), x^\alpha = (x_1^\alpha, x_2^\alpha, \cdots, x_n^\alpha), \ln x = (\ln x_1, \ln x_2, \cdots, \ln x_n), x \cdot y = (x_1 y_1, x_2 y_2, \cdots, x_n y_n)$, where $\alpha \in R$, and $x = (x_1, x_2, \cdots, x_n) \in R^n$, $y = (y_1, y_2, \cdots, y_n) \in R^n$.

For the convex function, the following Definition 1.1 and Hadamard's inequality (1.2) are well-known:

Definition 1.1. *Let $\Gamma \subseteq R^n$ be a convex set, $f : \Gamma \to (-\infty, +\infty)$is a continuous function. For any $x \in \Gamma, y \in \Gamma$, and $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$, then f is called a convex function on Γ, if*

$$f(\alpha x + \beta y) \leqslant \alpha f(x) + \beta f(y). \tag{1.1}$$

And f is called a concave function on Γ, if inequality (1.1) is inverse.

Theorem 1.1. ([1]) *If $f : [a,b] \to R$ be a convex function, then*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a} \int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}. \tag{1.2}$$

In [3] and [5], a special property for the convex function is respectively studied by S. S. Dragomir, P. Agarwal and K. Hu. One of results is the following theorem constructing monotone functions:

Theorem 1.2. *Let f be a continuous convex function in interval $[a,b]$. For any $x \in [a,b]$, if*

$$F(x) = \int_a^x f(t)dt - (x-a) f\left(\frac{x+a}{2}\right), \tag{1.3}$$

and

$$G(x) = (x-a)\frac{f(a)+f(x)}{2} - \int_a^x f(t)dt, \quad (x \in [a,b]), \tag{1.4}$$

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then the functions F and G both are increasing in $[a,b]$, and the following refinements of inequalities (1.2) *hold*

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^x f(x)dx-\frac{x-a}{b-a}f\left(\frac{a+x}{2}\right)+f\left(\frac{a+b}{2}\right)$$
$$\leqslant\frac{1}{b-a}\int_a^b f(x)dx\leqslant\frac{x-a}{b-a}\cdot\frac{f(a)+f(x)}{2}+\frac{1}{b-a}\int_x^b f(x)dx\leqslant\frac{f(a)+f(b)}{2}. \tag{1.5}$$

In [2] and [8, 9, 10], the authors obtained some relative definitions of geometrically convex function and geometrically convex set:

Definition 1.2. ([2, 8, 9]) *Let $f: I\subseteq(0,+\infty)\to(0,+\infty)$ be a continuous function, then f is called a geometrically convex function on I, if there exists $n\geqslant 2$, such that one of the following three inequalities holds for any $x_1,x_2,\cdots,x_n\in I$ and $\lambda_1,\lambda_2,\cdots,\lambda_n>0$ with $\lambda_1+\lambda_2+\cdots+\lambda_n=1$.*

$$f(\sqrt{x_1x_2})\leqslant\sqrt{f(x_1)f(x_2)}, \tag{1.6}$$

$$f\left(\sqrt[n]{\prod\nolimits_{i=1}^n x_i}\right)\leqslant\sqrt[n]{\prod\nolimits_{i=1}^n f\left(x_i\right)}, \tag{1.7}$$

$$f\left(\prod\nolimits_{i=1}^n x_i^{\lambda_i}\right)\leqslant\prod\nolimits_{i=1}^n f^{\lambda_i}\left(x_i\right), \tag{1.8}$$

and f is called a geometrically concave function on I if one of three inequalities (1.6)-(1.8) *is inverse.*

Remark 1.1. *The above three inequalities* (1.6)-(1.8) *are equivalent to each other (see* [2, p. 13-17]*).*

Definition 1.3. ([2]) *$H\subseteq R_+^n$ is called a geometrically convex set if $x^\alpha y^\beta\in H$ for any $x,y\in H$.*

Definition 1.4. ([2, 9]) *Let $H\subseteq R_+^n$ is a geometrically convex set, $f: H\to(0.+\infty)$ is a continuous function, f is called a geometrically convex function if $f(x^\alpha y^\beta)\leqslant f^\alpha(x)f^\beta(y)$ for any $x,y\in H$, f is called a geometrically concave function if the above inequality is inverse, where $\alpha,\beta\in[0,1]$ with $\alpha+\beta=1$.*

Definition 1.5. ([2, 9, 10]) *Let $x=(x_1,x_2,\cdots,x_n)\in R_+^n, y=(y_1,y_2,\cdots,y_n)\in R_+^n, (x_{[1]},x_{[2]},\cdots,x_{[n]})$ and $(y_{[1]},y_{[2]},\cdots,y_{[n]})$ are the decreasing queue of $(x_1,x_2,\cdots,x_n)$ and $(y_1,y_2,\cdots,y_n)$ respectively. We say $(x_1,x_2,\cdots,x_n)$ logarithm control $(y_1,y_2,\cdots,y_n)$, denotes $\ln x\succ\ln y$ if*

$$\begin{cases}\prod\nolimits_{i=1}^k x_{[i]}\geq\prod\nolimits_{i=1}^k y_{[i]}, k=1,2,\cdots,n-1,\\ x_1x_2\cdots x_n=y_1y_2\cdots y_n.\end{cases} \tag{1.9}$$

We can easily find $(\ln x_1,\ln x_2,\cdots,\ln x_n)\succ(\ln y_1,\ln y_2,\cdots,\ln y_n)$.

In this paper, by using some properties of geometrically convex functions, we shall construct some monotone functions and sequences.

2. Lemmas

To prove below theorems, the following lemmas are necessary.

Lemma 2.1. ([2]) *(1) If $f: H\subseteq R_+^n\to R_+^n$ is a geometrically convex (concave) function. Let $\ln H=\{\ln x|x\in H\}$, and $g(x)=\ln f(e^x)$ on $\ln H$. Then g is a convex (concave) function.*
(2) If g is a convex (concave) function on $\Gamma(\subseteq R_n)$. Let $e^\Gamma=\{e^x|x\in\Gamma\}$, and $f(x)=e^{g(\ln x)}$ on e^Γ. Then f is a geometrically convex (concave) function.

Lemma 2.2. ([2]) *Let $H\subseteq(0,+\infty)$, $H^n=\{(x_1,x_2,\cdots,x_n)\,|x_i\in H, 1\leqslant i\leqslant n\}$, if $f: H\to(0,+\infty)$ is a geometrically convex function, $g(x)=f(x_1)+f(x_2)+\cdots+f(x_n)$ and $h(x)=f(x_1)f(x_2)\cdots f(x_n)$, then g and h are geometrically convex functions on H^n.*

Lemma 2.3. ([2]) *Let $x, y \in H \subseteq R_+^n$, H is a symmetric convex set, and $\ln x \succ \ln y$, then $f(x) \leqslant (\geqslant) f(y)$ if f is a geometrically convex (concave) function on H.*

Lemma 2.4. ([2]) *If $f : [a,b] \to R_+$ is a geometrically convex function, then*

$$f(\frac{1}{e} b^{\frac{b}{b-a}} a^{\frac{a}{a-b}}) \leqslant \frac{1}{b-a} \int_a^b f(x)dx.$$

Lemma 2.5. ([4]) *Let f is a convex function on H, and $x_i \in H, i = 1,2,3,4$ with $x_1 < x_2 < x_4$, $x_1 < x_3 < x_4$. If $\alpha = \frac{x_4 - x_3}{x_2 - x_1}$, then we have*

$$\alpha \left(f(x_2) - f(x_1)\right) \leqslant f(x_4) - f(x_3).$$

Lemma 2.6. *Let $f : [a,b] \to R_+$ is a geometrically convex function, $x_i \in [a,b], i = 1,2,3,4$ and $x_1 < x_2 < x_4$,$x_1 < x_3 < x_4$, if α satisfies $\left(\frac{x_2}{x_1}\right)^\alpha = \frac{x_4}{x_3}$, then*

$$\frac{f^\alpha(x_2)}{f^\alpha(x_1)} \leqslant \frac{f(x_4)}{f(x_3)}. \tag{2.1}$$

Proof. Setting $F(x) = \ln f(e^x)$, from Lemma 2.1, we have that F is a convex function on $(\ln a, \ln b)$. For $\left(\frac{x_2}{x_1}\right)^\alpha = \frac{x_4}{x_3}$, we get $\alpha\left(\ln x_2 - \ln x_1\right) = \ln x_4 - \ln x_3$. Combining Lemma 2.5, these are deduced that

$$\alpha \cdot \left(\ln f\left(e^{\ln x_2}\right) - \ln f\left(e^{\ln x_1}\right)\right) \leqslant \ln f(e^{\ln x_4}) - \ln f(e^{\ln x_3}),$$

which implies the inequality (2.1). The proof of Lemma 2.6 is completed. ∎

Lemma 2.7. ([2])([10])*Let $(a,b) \subset (0,+\infty)$, $f : (a,b) \to (0,+\infty)$, and f is two order derivable. Then $x[f(x)f"(x) - (f'(x))^2] + f(x)f'(x) \geqslant (\leqslant) 0$, if and only if f is a geometrically convex (concave) function.*

Lemma 2.8. *Let $\lambda > 1$,$n \in N_+$, then $(n+1)(n+2)$ dimensional vector*

$$X = (\underbrace{\lambda, \cdots, \lambda}_{n+2}, \underbrace{\lambda^{\frac{n-1}{n}}, \cdots, \lambda^{\frac{n-1}{n}}}_{n+2}, \underbrace{\lambda^{\frac{n-2}{n}}, \cdots, \lambda^{\frac{n-2}{n}}}_{n+2}, \cdots, \underbrace{1, \cdots, 1}_{n+2})$$

logarithm control

$$Y = (\underbrace{\lambda, \cdots, \lambda}_{n+1}, \underbrace{\lambda^{\frac{n}{n+1}}, \cdots, \lambda^{\frac{n}{n+1}}}_{n+1}, \underbrace{\lambda^{\frac{n-1}{n+1}}, \cdots, \lambda^{\frac{n-1}{n+1}}}_{n+1}, \cdots, \underbrace{1, \cdots, 1}_{n+1}).$$

Proof. In vector X, multiplying all numbers is

$$\lambda^{n+2} \cdot \lambda^{\frac{(n-1)(n+2)}{n}} \cdot \cdots \cdot \lambda^{\frac{1}{n}(n+2)} = \lambda^{\frac{(n+1)(n+2)}{2}}.$$

In vector Y, multiplying all numbers is also $\lambda^{\frac{(n+1)(n+2)}{2}}$, and take $k \in N, 1 \leqslant k \leqslant (n+1)(n+2) - 1$ arbitrarily, use c, d to stand for integral part of $\frac{k}{n+2}$ and $\frac{k}{n+1}$ respectively, then the product of front k numbers of vector X is

$$\begin{aligned} &\lambda^{n+2} \cdot \lambda^{\frac{(n-1)(n+2)}{n}} \cdot \cdots \cdot \lambda^{\frac{n-c+1}{n}(n+2)} \cdot \lambda^{(k-c(n+2)) \cdot \frac{n-c}{n}} \\ =&\lambda^{\frac{n+2}{n}(n+(n-1)+\cdots+(n-c+1)) + (k-c(n+2))\frac{n-c}{n}} \\ =&\lambda^{\frac{(n+2)c^2 + (n-2k+2)c + 2kn}{2n}}. \end{aligned} \tag{2.2}$$

The product of front k numbers of vector Y is

$$\begin{aligned} &\lambda^{n+1} \cdot \lambda^{\frac{n}{n+1}(n+1)} \cdot \cdots \cdot \lambda^{\frac{n-d+2}{n+1}(n+1)} \cdot \lambda^{(k-d(n+1)) \cdot \frac{n-d+1}{n+1}} \\ =&\lambda^{\frac{(1+n)d^2 + (1+n-2k)d + 2kn + 2k}{2(n+1)}}. \end{aligned} \tag{2.3}$$

Thus according to the definition of logarithmic control, only to testify

$$\lambda^{\frac{(n+2)c^2+(n-2k+2)c+2kn}{2n}} \geqslant \lambda^{\frac{(1+n)d^2+(1+n-2k)d+2kn+2k}{2(n+1)}}$$

$$\Leftrightarrow (n+1)(n+2)c^2+(n+1)(n-2k+2)c \geqslant n(1+n)d^2+n(1+n-2k)d,$$

meanwhile

$$\frac{k}{n+1}-\frac{k}{n+2}=\frac{k}{(n+1)(n+2)} \leqslant \frac{(n+1)(n+2)-1}{(n+1)(n+2)} < 1,$$

so $d-1 \leqslant c \leqslant d$, if $c=d$, it is true. If $c=d-1$, it is equivalent to

$$(n+1)(n+2)(d-1)^2+(n+1)(n-2k+2)(d-1) \geqslant n(1+n)d^2+n(1+n-2k)d \tag{2.4}$$

$$\Leftrightarrow (d-\frac{k}{n+1})(d-n-1) \geqslant 0, \tag{2.5}$$

because

$$d \leqslant \frac{k}{n+1} \leqslant \frac{(n+1)(n+2)-1}{n+1} = n+2-\frac{1}{n+1},$$

so $d \leqslant \frac{k}{n+1}$ and $d \leqslant n+1$,$(d-\frac{k}{n+1})(d-n-1) \geqslant 0$, thence Lemma 2.8 is true. ∎

3. Construction of Some Monotone Functions

Theorem 3.1. *Let $0<a<b$, if f is a geometrically convex function on $[a,b]$, and $g(x)=f(x)+f(a)-2f(\sqrt{xa})$. Then g is a increasing function on $[a,b]$.*

Proof. For any $z_1,z_2 \in [a,b], z_1<z_2$, next we shall prove that

$$f(z_2)+f(a)-2f(\sqrt{z_2a}) \geqslant f(z_1)+f(a)-2f(\sqrt{z_1a}),$$

$$f(z_2)+2f(\sqrt{z_1a}) \geqslant f(z_1)+2f(\sqrt{z_2a}). \tag{3.1}$$

It is easy to obtain that $f(x_1)+f(x_2)+f(x_3)$ is a geometrically convex function on $[a,b]\times[a,b]\times[a,b]$ and

$$z_2 \geqslant \sqrt{z_2a}, \quad z_2\cdot\sqrt{z_1a} \geqslant \sqrt{z_2a}\cdot\sqrt{z_2a}, \quad z_2\sqrt{z_1a}\sqrt{z_1a}=\sqrt{z_2a}\cdot\sqrt{z_2a}\cdot z_1.$$

If $\sqrt{z_2a} \leqslant z_1$ or not, $(z_2,\sqrt{z_1a},\sqrt{z_1a})$ logarithm control $(z_1,\sqrt{z_2a},\ \sqrt{z_2a})$, and we know that (3.1) holds by Lemma 2.3, g is a increasing function on $[a,b]$. ∎

Theorem 3.2. *Let $0<a<b$, if f is a geometrically convex function on $[a,b]$, and $g(x)=\int_a^x f(t)dt-(x-a)f\left(\frac{1}{e}x^{\frac{x}{x-a}}a^{\frac{a}{a-x}}\right)$. Then g is increasing on $[a,b]$.*

Proof. For any $z_1,z_2 \in [a,b]$, $z_1<z_2$, next we shall prove $g(z_1) \leqslant g(z_2)$, or

$$\int_a^{z_2} f(t)dt-(z_2-a)f\left(\frac{1}{e}z_2^{\frac{z_2}{z_2-a}}a^{\frac{a}{a-z_2}}\right) \geqslant \int_a^{z_1} f(t)dt-(z_1-a)f\left(\frac{1}{e}z_1^{\frac{z_1}{z_1-a}}a^{\frac{a}{a-z_1}}\right),$$

$$\int_{z_1}^{z_2} f(t)dt \geqslant (z_2-a)f\left(\frac{1}{e}z_2^{\frac{z_2}{z_2-a}}a^{\frac{a}{a-z_2}}\right)-(z_1-a)f\left(\frac{1}{e}z_1^{\frac{z_1}{z_1-a}}a^{\frac{a}{a-z_1}}\right).$$

We need only to prove

$$(z_2-z_1)f(\frac{1}{e}z_2^{\frac{z_2}{z_2-z_1}}z_1^{\frac{z_1}{z_1-z_2}}) \geqslant (z_2-a)f\left(\frac{1}{e}z_2^{\frac{z_2}{z_2-a}}a^{\frac{a}{a-z_2}}\right)-(z_1-a)f\left(\frac{1}{e}z_1^{\frac{z_1}{z_1-a}}a^{\frac{a}{a-z_1}}\right),$$

By Lemma 2.4, we need only to prove

$$\frac{z_2-z_1}{z_2-a}f(\frac{1}{e}z_2^{\frac{z_2}{z_2-z_1}}z_1^{\frac{z_1}{z_1-z_2}})+\frac{z_1-a}{z_2-z_1}f\left(\frac{1}{e}z_1^{\frac{z_1}{z_1-a}}a^{\frac{a}{a-z_1}}\right) \geqslant f\left(\frac{1}{e}z_2^{\frac{z_2}{z_2-a}}a^{\frac{a}{a-z_2}}\right).$$

By Young inequality, to prove the above inequality we need only to prove

$$\left(f\left(\frac{1}{e}z_2^{\frac{z_2}{z_2-z_1}}z_1^{\frac{z_1}{z_1-z_2}}\right)\right)^{\frac{z_2-z_1}{z_2-a}}\left(f\left(\frac{1}{e}z_1^{\frac{z_1}{z_1-a}}a^{\frac{a}{a-z_1}}\right)\right)^{\frac{z_1-a}{z_2-a}}\geqslant f\left(\frac{1}{e}z_2^{\frac{z_2}{z_2-a}}a^{\frac{a}{a-z_2}}\right).$$

By in Definition 1.2 with $n=2$, it is easy to obtain the above inequality by simple calculation. ∎

For the case in Definition 1.2 with $n=2$, we have the following Theorem 3.3.

Theorem 3.3. *Let $0<a<b$, if f is a geometrically convex function on $[a,b]$, and $g(x)=\frac{f^{\alpha}(x)f^{\beta}(a)}{f(x^{\alpha}a^{\beta})}$. Then g is a increasing function on $[a,b]$.*

Proof. For any $z_1,z_2\in[a,b]$, $z_1<z_2$, next we shell prove

$$\frac{f^{\alpha}(z_2)f^{\beta}(a)}{f\left(z_2^{\alpha}a^{\beta}\right)}\geqslant\frac{f^{\alpha}(z_1)f^{\beta}(a)}{f\left(z_1^{\alpha}a^{\beta}\right)}$$

$$\Leftrightarrow\frac{f\left(z_2\right)}{f\left(z_1\right)}\geqslant\left(\frac{f(z_2^{\alpha}a^{\beta})}{f\left(z_1^{\alpha}a^{\beta}\right)}\right)^{\frac{1}{\alpha}},\tag{3.2}$$

if $z_1=a$, then (3.2) is obvious by the definition of geometrically convex function. If $z_1\neq a$, then $z_1^{\alpha}a^{\beta}<z_2^{\alpha}a^{\beta}<z_2$, $z_1^{\alpha}a^{\beta}<z_1<z_2$ and $\frac{z_2}{z_1}=\left(\frac{z_2^{\alpha}a^{\beta}}{z_1^{\alpha}a^{\beta}}\right)^{\frac{1}{\alpha}}$, (3.2) is also holds by Lemma 2.6. Hence g is increasing on $[a,b]$. ∎

Theorem 3.4. *Let $0<a<b$, if f is a increasing geometrically convex function, and $g(x)=f^{\alpha}(x)f^{\beta}(a)-f(x^{\alpha}a^{\beta})$. Then g is increasing on $[a,b]$.*

Proof. For any $z_1,z_2\in[a,b]$, $z_1<z_2$, next we shall prove

$$f^{\alpha}(z_2)f^{\beta}(a)-f(z_2^{\alpha}a^{\beta})\geqslant f^{\alpha}(z_1)f^{\beta}(a)-f(z_1^{\alpha}a^{\beta}),$$

by (3.2) only to prove

$$\frac{f^{\alpha}\left(z_1\right)f\left(z_2^{\alpha}a^{\beta}\right)}{f\left(z_1^{\alpha}a^{\beta}\right)}f^{\beta}(a)-f(z_2^{\alpha}a^{\beta})\geqslant f^{\alpha}(z_1)f^{\beta}(a)-f(z_1^{\alpha}a^{\beta})$$

$$\Leftrightarrow f^{\alpha}\left(z_1\right)f\left(z_2^{\alpha}a^{\beta}\right)f^{\beta}\left(a\right)-f(z_2^{\alpha}a^{\beta})f\left(z_1^{\alpha}a^{\beta}\right)\geqslant f^{\alpha}(z_1)f^{\beta}(a)f\left(z_1^{\alpha}a^{\beta}\right)-f^{2}(z_1^{\alpha}a^{\beta})$$

$$\Leftrightarrow\left(f^{\alpha}\left(z_1\right)f^{\beta}\left(a\right)-f\left(z_1^{\alpha}a^{\beta}\right)\right)\cdot\left(f\left(z_2^{\alpha}a^{\beta}\right)-f\left(z_1^{\alpha}a^{\beta}\right)\right)\geqslant 0.\tag{3.3}$$

Since $f^{\alpha}\left(z_1\right)f^{\beta}\left(a\right)\geqslant f\left(z_1^{\alpha}a^{\beta}\right)$ by the definition of geometrically convex function, $z_2^{\alpha}a^{\beta}>z_1^{\alpha}a^{\beta}$ and f is increasing, hence (3.3) holds. ∎

Theorem 3.5. *Let $0<a<b$, if f is a increasing geometrically convex function, and $g(x)=f(x)f^{\frac{\beta}{\alpha}}(a)-f^{\frac{1}{\alpha}}(x^{\alpha}a^{\beta})$. Then g is increasing on $[a,b]$.*

Theorem 3.6. *Let $0<a<b$, if f is a geometrically convex function on $[a,b]$,and $g(x)=\frac{f(x)f(a)}{f(\sqrt{ax})}-f\left(\sqrt{ax}\right)$. Then g is increasing function on $[a,b]$.*

The proof of Theorem 3.5 and Theorem 3.6 are similar to the proof of Theorem 3.4.

The following counter example show that the condition "f is increasing" in Theorem 3.4-Theorem 3.6 can not be removed.

Let $f(x)=e^{-\sqrt{\ln x}}$, $x\in[1,10^8]$, we can prove that f is a geometrically convex function easily by Lemma 2.7, but

$$f^{\frac{1}{2}}(x)f^{\frac{1}{2}}(1)-f(\sqrt{1\cdot x})=f^{\frac{1}{2}}(x)-f(\sqrt{x}),$$

$$f(x)f(1)-f^2(\sqrt{1\cdot x})=f(x)-f^2(\sqrt{x}),$$

and

$$\frac{f(x)f(1)}{f(\sqrt{1\cdot x})}-f\left(\sqrt{1\cdot x}\right)=\frac{f(x)}{f(\sqrt{x})}-f\left(\sqrt{x}\right)$$

are not increasing in $[1,10^8]$.

4. Construction of Some Monotone sequences

Theorem 4.1. *Let $a>0$, f is a geometrically convex function on $[a,b]$, for $x_i\in[a,b], i=1,2,\cdots$, and $F(n)=\sum_{i=1}^n f(x_i)-nf\left(\sqrt[n]{\prod_{i=1}^n x_i}\right)$, then ordered series of numbers $\{F(n)\}_{n=1}^{+\infty}$ is a monotonous increasing sequence.*

Proof. $F(n+1)-F(n)=f(x_{n+1})-(n+1)f\left(\sqrt[n+1]{\prod_{i=1}^{n+1}x_i}\right)+nf\left(\sqrt[n]{\prod_{i=1}^n x_i}\right)$, then

$$F(n+1)-F(n)\geqslant 0$$

$$\Leftrightarrow \frac{1}{n+1}f(x_{n+1})+\frac{n}{n+1}f\left(\sqrt[n]{\prod_{i=1}^{n}x_i}\right)\geqslant f\left(\sqrt[n+1]{\prod_{i=1}^{n+1}x_i}\right).$$

By Young inequality and the definition of geometrically convex functions,

$$\frac{1}{n+1}f(x_{n+1})+\frac{n}{n+1}f\left(\sqrt[n]{\prod_{i=1}^{n}x_i}\right)\geqslant f^{\frac{1}{n+1}}(x_{n+1})\cdot f^{\frac{n}{n+1}}\left(\sqrt[n]{\prod_{i=1}^{n}x_i}\right) \tag{4.1}$$

$$\geqslant f\left(x_{n+1}^{\frac{1}{n+1}}\cdot\left(\sqrt[n]{\prod_{i=1}^{n}x_i}\right)^{\frac{n}{n+1}}\right)=f\left(\sqrt[n+1]{\prod_{i=1}^{n+1}x_i}\right).$$

Thence Theorem 4.1 is true. ∎

Theorem 4.2. *Let $a>0$, f is a geometrically convex function on $[a,b]$, and*

$$F(n)=\sqrt[n+1]{\prod_{i=0}^{n}f\left(\sqrt[n]{a^{n-i}b^i}\right)},$$

then $\{F(n)\}_{n=1}^{+\infty}$ is a monotonous decreasing sequences, and convergence to $\exp\left(\frac{1}{\ln b-\ln a}\int_{\ln a}^{\ln b}\ln f(e^x)dx\right)$.

Proof. Let $\lambda=\frac{b}{a}>1$ in Lemma 2.8, then

$$\left(\underbrace{\frac{b}{a},\cdots,\frac{b}{a}}_{n+2},\underbrace{\left(\frac{b}{a}\right)^{\frac{n-1}{n}},\cdots,\left(\frac{b}{a}\right)^{\frac{n-1}{n}}}_{n+2},\underbrace{\left(\frac{b}{a}\right)^{\frac{n-2}{n}},\cdots,\left(\frac{b}{a}\right)^{\frac{n-2}{n}}}_{n+2},\cdots,\underbrace{1,\cdots,1}_{n+2}\right)$$

logarithm control

$$\left(\underbrace{\frac{b}{a},\cdots,\frac{b}{a}}_{n+1},\underbrace{\left(\frac{b}{a}\right)^{\frac{n}{n+1}},\cdots,\left(\frac{b}{a}\right)^{\frac{n}{n+1}}}_{n+1},\underbrace{\left(\frac{b}{a}\right)^{\frac{n-1}{n+1}},\cdots,\left(\frac{b}{a}\right)^{\frac{n-1}{n+1}}}_{n+1},\cdots,\underbrace{1,\cdots,1}_{n+1}\right).$$

So

$$\left(\underbrace{b,\cdots,b}_{n+2},\underbrace{b^{\frac{n-1}{n}}a^{\frac{1}{n}},\cdots,b^{\frac{n-1}{n}}a^{\frac{1}{n}}}_{n+2},\underbrace{b^{\frac{n-2}{n}}a^{\frac{2}{n}},\cdots,b^{\frac{n-2}{n}}a^{\frac{2}{n}}}_{n+2},\cdots,\underbrace{a,\cdots,a}_{n+2}\right)$$

logarithm control

$$\left(\underbrace{b,\cdots,b}_{n+1},\underbrace{b^{\frac{n}{n+1}}a^{\frac{1}{n+1}},\cdots,b^{\frac{n}{n+1}}a^{\frac{1}{n+1}}}_{n+1},\underbrace{b^{\frac{n-1}{n+1}}a^{\frac{2}{n+1}},\cdots,b^{\frac{n-1}{n+1}}a^{\frac{2}{n+1}}}_{n+1},\cdots,\underbrace{a,\cdots,a}_{n+1}\right).$$

Function $\prod_{i=1}^{(n+1)(n+2)} f(x_i)$ is a geometrically convex function on $[a,b]^{(n+1)(n+2)}$ by Lemma 2.2, so that

$$\left(\prod{}_{i=0}^{n} f\left(\sqrt[n]{a^{n-i}b^i}\right)\right)^{n+2} \geqslant \left(\prod{}_{i=0}^{n+1} f\left(\sqrt[n+1]{a^{n+1-i}b^i}\right)\right)^{n+1}$$

by Lemma 2.3, implies

$$\left(\frac{F(n+1)}{F(n)}\right)^{(n+1)(n+2)} = \left(\frac{\left(\prod_{i=0}^{n+1} f(\sqrt[n+1]{a^{n+1-i}b^i})\right)^{n+1}}{\left(\prod_{i=0}^{n} f(\sqrt[n]{a^{n-i}b^i})\right)^{n+2}}\right) \leqslant 1,$$

$$F(n+1) \leqslant F(n),$$

so $\{F(n)\}_{n=1}^{+\infty}$ is monotonous decreasing sequences by degrees, and

$$\ln F(n) = \frac{1}{n+1}\sum{}_{i=0}^{n} \ln\left(f\left(\exp\left(\frac{n-i}{n}\ln a + \frac{i}{n}\ln b\right)\right)\right)$$

$$= \frac{1}{\ln b - \ln a}\cdot\frac{\ln b - \ln a}{n+1}\cdot\sum{}_{i=0}^{n} \ln\left(f\left(\exp\left(\frac{n-i}{n}\ln a + \frac{i}{n}\ln b\right)\right)\right).$$

If $n \to +\infty$, then

$$\ln F(n) \to \frac{1}{\ln b - \ln a}\int_{\ln a}^{\ln b} \ln f(e^x)dx,$$

$$F(n) \to \exp\left(\frac{1}{\ln b - \ln a}\int_{\ln a}^{\ln b} \ln f(e^x)dx\right).$$

∎

Theorem 4.3. *Let $a > 0 f$ is a geometrically convex function on $[a,b]$, and*

$$F(n) = \frac{1}{n+1}\sum{}_{i=0}^{n} f\left(\sqrt[n]{a^{n-i}b^i}\right),$$

then $\{F(n)\}_{n=1}^{+\infty}$ is a monotonous decreasing sequences, and convergence to $\frac{1}{\ln b - \ln a}\int_{\ln a}^{\ln b} f(e^x)dx$.

Proof. Function $\sum_{i=1}^{(n+1)(n+2)} f(x_i)$ is a geometrically convex function on $[a,b]^{(n+1)(n+2)}$ by Lemma 2.2, so that

$$(n+2)\sum{}_{i=0}^{n} f\left(\sqrt[n]{a^{n-i}b^i}\right) \geqslant (n+1)\sum{}_{i=0}^{n+1} f\left(\sqrt[n+1]{a^{n+1-i}b^i}\right)$$

by Lemma 2.3, implies

$$\frac{1}{n+1}\sum{}_{i=0}^{n} f\left(\sqrt[n]{a^{n-i}b^i}\right) \geqslant \frac{1}{n+2}\sum{}_{i=0}^{n+1} f\left(\sqrt[n+1]{a^{n+1-i}b^i}\right),$$

$$F(n) \geqslant F(n+1).$$

And if $n \to +\infty$, then

$$F(n) = \frac{1}{n+1}\sum{}_{i=0}^{n} \ln\left(f\left(\exp\left(\frac{n-i}{n}\ln a + \frac{i}{n}\ln b\right)\right)\right) \to \frac{1}{\ln b - \ln a}\int_{\ln a}^{\ln b} \ln f(e^x)dx.$$

∎

Corollary 4.1. *(Rado's inequality) Let* $x_i > 0, i = 1, 2, \cdots$ *, and* $A_n(x) = \frac{1}{n}\sum_{i=1}^n x_i, G_n(x) = \sqrt[n]{\prod_{i=1}^n x_i}, R_n(x) = n\left[A_n(x) - G_n(x)\right]$*, then*

$$R_{n-1}(x) \leqslant R_n(x).$$

Proof. In Theorem 4.1, let $a = \min\{x_1, x_2, \cdots, x_n\}, b = \max\{x_1, x_2, \cdots, x_n\}$, and $f(x) = x, x \in [a, b]$, then Corollary 4.1 is true. ∎

Corollary 4.2. *Let* $0 < a < b, n \geqslant 2$*, then*

$$\frac{a+b}{2} \geqslant \frac{a+\sqrt{ab}+b}{3} \geqslant \cdots \geqslant \frac{a + a^{\frac{n-1}{n}}b^{\frac{1}{n}} + \cdots + a^{\frac{1}{n}}b^{\frac{n-1}{n}} + b}{n+1} \geqslant \cdots \geqslant \frac{b-a}{\ln b - \ln a}.$$

Proof. In Theorem 4.3, let $f(x) = x, x \in [a, b]$, then Corollary 4.2 is true. ∎

5. An Open Problems

In the final, we pose the following open problem according to some results in [2].

Problem 5.1. *Let* $0 < a < b$*,* f *is a geometrically convex function on* $[a, b]$*, and*

$$g_1(x) = \frac{xf(x) - af(a)}{\ln(xf(x)) - \ln(af(a))} \cdot \ln\frac{x}{a} - \int_a^x f(t)dt, \tag{5.1}$$

$$g_2(x) = \left(\frac{x-a}{\ln x - \ln a} - a\right)f(a) + \left(x - \frac{x-a}{\ln x - \ln a}\right)f(x) - \int_a^x f(t)dt, \tag{5.2}$$

$$g_3(x) = \sqrt{xa}(\ln x - \ln a)f(\sqrt{ax}) - \int_a^x f(t)dt. \tag{5.3}$$

We conjecture that all the $g_i^{,}s$ *are increasing on* $[a, b]$.

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