New Perturbed Proximal Point Algorithms for Set-valued Quasi Variational Inclusions *

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Abstract. In this paper, by using some new and innovative techniques, some perturbed iterative algorithms for solving generalized set-valued variational inclusions are suggested and analyzed. Since the generalized set-valued variational inclusions include many variational inclusions , variational inequalities and set-valued operator equation studied by others in recent years, the results obtained in this paper continue to hold for them and represent a significant refinement and improvement of the previously known results in this area.

Keywords: Set-valued quasi variational inclusion;Proximal point algorithms; Convergence

1.Introductions

In recent years, variational inequalities have been extended and generalized in different directions using novel and innovative techniques both for their own sake and for applications. A useful and important generalization of variational inequalities is generalized set-valued variational inclusions in Hilbert space H:Let $A : D(A) \to 2^H$ be an maximal monotone mapping, $\Omega \subset H$ be a closed and convex set, find $u \in H, g(u) \in \Omega, w \in Tu, y \in Vu$ such that

$$0 \in N(w, y) + A(g(u)). \tag{1.1}$$

This problem was introduced and studied by Noor[2,3]. In [2,3], Noor gave some algorithms and convergence analysis under the assumption that N is strongly monotone with the respect to the first argument. Inspired and motivated by the result in Noor [2,3] and He B. S.[4,5], the purpose of this paper is to suggest iterative methods for solving the generalized set-valued variational inequalities under the assumption that the operator underlying is monotone with the respect to the first and second arguments. It is well known that variational inclusion of monotone mappings is more difficult to solve than those of strongly monotone mappings. So a new and innovative technique is used in this paper. The results presented in this paper generalize, improve and unify the corresponding results of S. S. Chang [1], Noor [2,3,4], He [5,6],

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Huang [7,8], zeng [9], and kazmi [10].

2.Preliminary

Let H be a real Hilbert space whose inner product and norm are denoted by $< \cdot, \cdot >$ and $\| \cdot \|$, respectively. Let $CB(H)$ be a family of all nonempty closed and bounded subsets of H. Let $T, V : H \to CB(H)$ be the set-valued operators, $g : H \to H$ be a single-valued operator and $A(\cdot, \cdot) : H \times H \to H$ be a maximal monotone operator.

Definition 2.1 For all $u_1, u_2 \in H$, the operator $N(\cdot, \cdot)$ is said $g-$monotone and Lipschitz continuous with respect to the first argument, if there exist constants $\alpha > 0, \beta > 0$ such that

$$< N(w_1, \cdot) - N(w_2, \cdot), g(u_1) - g(u_2) > \geq 0, \forall w_1 \in Tu_1, w_2 \in Tu_2,$$

$$\|N(u_1, \cdot) - N(u_2, \cdot)\| \leq \beta \|u_1 - u_2\|.$$

In a similar way, we can define the $g-$monotonicity and Lipschitz continuity of the operator $N(\cdot, \cdot)$ with respect to the second argument.

Definition 2.2 Let $T : E \to CB(E)$ be a set-valued mapping and $H(\cdot, \cdot)$is a Hausdorff metric in $CB(E)$, T is said to be ξ-Lipschitz continuous , if for any $x, y \in E$,

$$H(Tx, Ty) \leq \xi \|x - y\|,$$

where $\xi > 0$ is a constant.

To prove the main result, we need the following lemmas.

Lemma 2.1[11] Let E be a complete metric space, $T : E \to CB(E)$ be a set-valued mapping. Then for any given $\varepsilon > 0$ and any given $x, y \in E, u \in Tx$, there exists $v \in Ty$ such that

$$d(u, v) \leq (1 + \varepsilon) H(Tx, Ty).$$

In the section 3, we often used the following inequality of projection operator.

$$\|P_\Omega(x') - x\| \leq \|x' - x\|, \forall x' \in H, x \in \Omega. \tag{2.1}$$

3. Main result

In this section, we suggest a new perturbed proximal point algorithm for finding approximate solution of the problem (1.1). Then we show that the sequence of approximate solution strongly converges to the exact solution of the problem (1.1).

we first suggest some algorithms for the problem (1.1).

Algorithm 3.1 Given $u_0 \in H, g(u_0) \in \Omega$, compute u_k by the following schemes:

$$g(u_k) + e_k \in g(\bar{u}_k) + \beta_k(N(\bar{w}_k, \bar{y}_k) + A(g(\bar{u}_k))), \tag{3.1}$$

$$\|e_k\| \leq \eta_k \|g(u_k) - g(\bar{u}_k), \|$$

$$
\begin{aligned}
g(u_{k+1}) &= P_\Omega[g(\bar{u}_k) - e_k],\\
\|w_k - w_{k+1}\| &\le (1 + \frac{1}{n+1})H(Tu_k, Tu_{k+1}),\\
\|y_k - y_{k+1}\| &\le (1 + \frac{1}{n+1})H(Vu_k, Vu_{k+1}),\\
\|\bar{w}_k - \bar{w}_{k+1}\| &\le (1 + \frac{1}{n+1})H(T\bar{u}_k, T\bar{u}_{k+1}),\\
\|\bar{y}_k - \bar{y}_{k+1}\| &\le (1 + \frac{1}{n+1})H(V\bar{u}_k, V\bar{u}_{k+1}),
\end{aligned}
$$

where $Sup_{k\ge 0}\eta_k = \nu < 1$ and β_k is given.
As the extension of Algorithm 3.1, we have the following.

Algorithm 3.2 Given $u_0 \in H, g(u_0) \in \Omega$, compute u_k by the following schemes:

$$
\begin{aligned}
g(u_k) + e_k &\in g(\bar{u}_k) + \beta_k(N(\bar{w}_k, \bar{y}_k) + A(g(\bar{u}_k))),\\
\|e_k\| &\le \eta_k\|g(u_k) - g(\bar{u}_k)\|,\\
d_k &= g(u_k) - g(\bar{u}_k) + e_k,\\
g(u_{k+1}) &= P_\Omega[g(u_k) - \alpha_k d_k], \qquad (3.2)\\
\|w_k - w_{k+1}\| &\le (1 + \frac{1}{n+1})H(Tu_k, Tu_{k+1}),\\
\|y_k - y_{k+1}\| &\le (1 + \frac{1}{n+1})H(Vu_k, Vu_{k+1}),\\
\|\bar{w}_k - \bar{w}_{k+1}\| &\le (1 + \frac{1}{n+1})H(T\bar{u}_k, T\bar{u}_{k+1}),\\
\|\bar{y}_k - \bar{y}_{k+1}\| &\le (1 + \frac{1}{n+1})H(V\bar{u}_k, V\bar{u}_{k+1}),
\end{aligned}
$$

where $Sup_{k\ge 0}\eta_k = \nu < 1$, β_k is given and

$$
\alpha_k = \gamma_k\alpha_k^*, \alpha_k^* = \frac{< g(u_k) - g(\bar{u}_k), d_k >}{\|d_k\|^2}, \gamma_k \in (0, 2).
$$

Remark 3.1 Since

$$
\begin{aligned}
&< g(u_k) - g(\bar{u}_k), d_k >\\
&=< g(u_k) - g(\bar{u}_k), g(u_k) - g(\bar{u}_k) + e_k >\\
&= \|g(u_k) - g(\bar{u}_k)\|^2 + < g(u_k) - g(\bar{u}_k), e_k >\\
&> \frac{1}{2}\|g(u_k) - g(\bar{u}_k)\|^2 + < g(u_k) - g(\bar{u}_k), e_k > + \frac{1}{2}\|e_k\|^2 = \frac{1}{2}\|d_k\|^2,
\end{aligned}
$$

$\alpha_k^* > \frac{1}{2}$. If $\gamma_k = \frac{1}{\alpha_k^*} \in (0, 2)$, then $\alpha_k \equiv 1$, this implies that Algorithm 3.1 is the special case of Algorithm 3.2.

In the following, we will give the convergence analysis for Algorithm 3.1. For this purpose, we need the following Lemmas.

Lemma 3.1 Let $N(\cdot,\cdot): H \times H \to H$ be g-monotone with respect to the two arguments, set-valued operator $A: H \to 2^H$ be maximal monotone, and $T, V: H \to CB(H)$ be two set-valued operators. Then for given $u_k \in H$, $\beta_k > 0$, and $u^* \in H, w^* \in Tu^*, y^* \in Vu^*$, which is a solution of the variational inclusion (1.1), we have

$$\begin{aligned} &< g(u_k) - g(u^*), g(u_k) - g(\bar{u}_k) + e_k > \\ &\geq < g(u_k) - g(\bar{u}_k), g(u_k) - g(\bar{u}_k) + e_k > \end{aligned} \tag{3.3}$$

and

$$\|g(\bar{u}_k) - e_k - g(u^*)\|^2 \leq \|g(u_k) - g(u^*)\|^2 - (\|g(u_k) - g(\bar{u}_k)\|^2 - \|e_k\|^2). \tag{3.4}$$

Proof. From (3.1), we have

$$\frac{1}{\beta_k}(g(u_k) - g(\bar{u}_k) + e_k) \in N(\bar{w}_k, \bar{y}_k) + A(g(\bar{u}_k)).$$

Since $0 \in N(w^*, y^*) + A(g(u^*))$, A is maximal monotone and N is g-monotone with respect to two arguments, we have

$$< \frac{1}{\beta_k}(g(u_k) - g(\bar{u}_k) + e_k) - 0, g(\bar{u}_k) - g(u^*) >\geq 0.$$

From $\beta_k \geq 0$, we have

$$< g(u_k) - g(\bar{u}_k) + e_k, g(\bar{u}_k) - g(u^*) >\geq 0,$$

so the inequality (3.3) is obtained immediately. Thus,

$$\begin{aligned} &\|g(\bar{u}_k) - e_k - g(u^*)\|^2 \\ &= \|g(u_k) - g(u^*) - (g(u_k) - g(\bar{u}_k) + e_k)\|^2 \\ &= \|g(u_k) - g(u^*)\|^2 - 2 < g(u_k) - g(u^*), g(u_k) - g(\bar{u}_k) + e_k > + \|g(u_k) - g(\bar{u}_k) + e_k\|^2 \\ &\leq \|g(u_k) - g(u^*)\|^2 - 2 < g(u_k) - g(\bar{u}_k), g(u_k) - g(\bar{u}_k) + e_k > + \|g(u_k) - g(\bar{u}_k) + e_k\|^2 \\ &= \|g(u_k) - g(u^*)\|^2 - (\|g(u_k) - g(\bar{u}_k)\|^2 - \|e_k\|^2). \end{aligned}$$

The required results.

Lemma 3.2 For any given $u_k \in H, g(u_k) \in \Omega$, let $\bar{u}_k$ and e_k satisfy the variational inclusion (3.2), and N, T, V, A, g be as Lemma 3.1, u_{k+1} is obtained from Algorithm 3.1, we have

$$\|g(u_{k+1}) - g(u^*)\|^2 \leq \|g(u_k) - g(u^*)\|^2 - (1 - \eta_k^2)\|g(u_k) - g(\bar{u}_k)\|^2, \tag{3.5}$$

where $u^* \in H$ is a solution of the variational inclusion (2.1).
Proof.Since $u^* \in H, g(u^*) \in \Omega$ and $g(u_{k+1}) = P_\Omega(g(\bar{u}_k) - e_k)$, take $x = g(u^*)$ and $x' = g(\bar{u}_k) - e_k$ in (2.1), we obtain

$$\|g(u_{k+1}) - g(u^*)\| \leq \|g(\bar{u}_k) - e_k - g(u^*)\|. \tag{3.6}$$

From (3.4), (3.6)and $\|e_k\| \leq \eta_k \|g(u_k) - g(\bar{u}_k)\|$, we have

$$\|g(u_{k+1}) - g(u^*)\|^2$$
$$\leq \|g(u_k) - g(u^*)\|^2 - (\|g(u_k) - g(\bar{u}_k)\|^2 - \|e_k\|^2)$$
$$\leq \|g(u_k) - g(u^*)\|^2 - (1 - \eta_k^2)\|g(u_k) - g(\bar{u}_k)\|^2.$$

The required results.

From $Sup_{k\geq 0}\eta_k = \nu < 1$ and (3.5),

$$\|g(u_{k+1}) - g(u^*)\|^2 \leq \|g(u_k) - g(u^*)\|^2 - (1 - \nu^2)\|g(u_k) - g(\bar{u}_k)\|^2.$$

So $\{g(u_k)\}$ is bounded , and

$$\lim_{k\to\infty} \|g(u_k) - g(\bar{u}_k)\| = 0,$$

thus $\{g(\bar{u}_k)\}$ is also bounded.

Now we give the convergence analysis for the Algorithm 3.1.

Theorem 3.1 Let $N(\cdot, \cdot) : H \times H \to H$ be g-monotone and Lipschitz continuous with respect to the two arguments, set-valued operator $A : H \to 2^H$ be maximal monotone, and $T, V : H \to CB(H)$ be H-Lipschitz continuous with constants $\mu > 0$ and $\xi > 0$, g is inverse and g^{-1} is Lipschitz continuous, and H is finite dimensional. Then the sequence $\{u_k\}$ generated by Algorithm 3.1 converges strongly to a solution of generalized set-valued variational inclusion (1.1). Proof. From Lemma 3.2, $\{g(\bar{u}_k)\}$ is bounded, thus $\{g(\bar{u}_k)\}$ has a cluster point $g(u^\infty)$. So there exists a subsequence $\{g(\bar{u}_{k_j})\}$,which converges to $\{g(u^\infty)\}$. Let

$$x_k = \frac{1}{\beta_k}(g(u_k) - g(\bar{u}_k) + e_k),$$

then

$$x_{k_j} \in N(\bar{w}_{k_j}, \bar{y}_{k_j}) + A(g(\bar{u}_{k_j})). \tag{3.7}$$

From $\lim_{k\to\infty} \|g(u_k) - g(\bar{u}_k)\| = 0$ and $e_k \to 0$,we have

$$\lim_{j\to\infty} x_{k_j} = \lim_{j\to\infty} \frac{1}{\beta_{k_j}}(g(u_{k_j}) - g(\bar{u}_{k_j}) + e_{k_j}) = 0 \tag{3.8}$$

From the Lipschitz continuity of g^{-1}, and $g(\bar{u}_{k_j}) \to g(u^\infty)$, we have

$$\bar{u}_{k_j} \to u^\infty, (j \to \infty).$$

From Lemma 2.1 and the H-Lipschitz continuity of T, we have for any $\bar{w}_{k_j} \in T(\bar{u}_{k_j})$, there exists $\bar{w}'_{k_j} \in Tu^\infty$ such that

$$\|\bar{w}_{k_j} - \bar{w}'_{k_j}\| \leq 2H(T\bar{u}_{k_j}, Tu^\infty)$$

$$\leq 2\mu\|\bar{u}_{k_j} - u^\infty\| \to 0.(j \to \infty)$$

Since $Tu^\infty \in CB(E)$,there exists $w^\infty \in Tu^\infty$ such that

$$\bar{w}'_{k_j} \to w^\infty.$$

Thus

$$\bar{w}_{k_j} \to w^\infty. \tag{3.9}$$

In a similar way, we have there exists $y^\infty \in Tu^\infty$, such that

$$\bar{y}_{k_j} \to y^\infty. \tag{3.10}$$

Let $j \to \infty$, from (3.7)-(3.10), we have

$$0 \in N(w^\infty, y^\infty) + A(g(u^\infty)).$$

So $u^\infty \in H, w^\infty \in Tu^\infty, y^\infty \in Vu^\infty$ is the solution of variational inclusion (1.1). From (3.5), we have

$$\|g(u_{k+1}) - g(u^\infty)\|^2 \leq \|g(u_k) - g(u^\infty)\|^2, \forall k \geq 0.$$

Since $g(\bar{u}_{k_j}) \to g(u^\infty), g(u_k) - g(\bar{u}_k) \to 0$, for any $\varepsilon > 0$, $\exists L > 0$, such that

$$\|g(\bar{u}_{k_l}) - g(u^\infty)\| \leq \varepsilon/2, \|g(\bar{u}_{k_l}) - g(u_{k_l})\| \leq \varepsilon/2. (l > L)$$

So, for any $k > k_l$, from (3.9) and (3.10),

$$\|g(u_k) - g(u^\infty)\| \leq \|g(u_{k_l}) - g(u^\infty)\| \leq \|g(u_{k_l}) - g(\bar{u}_{k_l})\| + \|g(\bar{u}_{k_l}) - g(u^\infty)\| < \varepsilon$$

i. e., $\{g(u_k)\}$ strongly converges to $g(u^\infty)$.
From the Lipschitz continuity of g^{-1}, we have $\{u_k\}$ strongly converges to u^∞.

Remark 3.2 Theorem 3.1 improve and extend the corresponding results in Noor [2,3,4] and He [5,6].

Theorem 3.2 Let the sequences $\{u_k\}$, $\{\bar{u}_k\}$ and $\{e_k\}$ satisfy variational inclusion (3.2), $\{\eta_k\}$ is the positive sequence, then the sequence $\{u_k\}$ generated by Algorithm 3.2 satisfy

$$\|g(u_{k+1} - g(u^*)\|^2 \leq \|g(u_k) - g(u^*)\|^2$$
$$-\frac{1}{2}\gamma_k(2-\gamma_k)(1-\eta_k)\|g(u_k) - g(\bar{u}_k)\|^2.$$

Proof. Let $u^* \in H, w^* \in Tu^*, y^* \in Vu^*$ be the solution of variational inclusion (2.1). Since $g(u^*) \in \Omega$ and

$$g(u_{k+1}) = P_\Omega[g(u_k) - \alpha_k(g(u_k) - g(\bar{u}_k) + e_k)],$$

$$\|g(u_{k+1} - g(u^*)\| \leq \|g(u_k) - g(u^*) - \alpha_k(g(u_k) - g(\bar{u}_k) + e_k)\|.$$

From (3.3) and $\alpha_k^* = \frac{<g(u_k)-g(\bar{u}_k), d_k>}{\|d_k\|^2}$, we have

$$\|g(u_{k+1} - g(u^*)\|^2 \leq \|g(u_k) - g(u^*) - \alpha_k(g(u_k) - g(\bar{u}_k) + e_k)\|^2$$

$$\leq \|g(u_k) - g(u^*)\|^2 - 2\alpha_k < g(u_k) - g(\bar{u}_k), g(u_k) - g(\bar{u}_k) + e_k >$$

$$+\alpha_k^2\|g(u_k) - g(\bar{u}_k) + e_k\|^2$$
$$= \|g(u_k) - g(u^*)\|^2 - \gamma_k(2-\gamma_k)\alpha_k^* < g(u_k) - g(\bar{u}_k), d_k > . \tag{3.12}$$

Notice

$$< g(u_k) - g(\bar{u}_k), d_k > \geq \|g(u_k) - g(\bar{u}_k)\|^2 - \|g(u_k) - g(\bar{u}_k)\|\|e_k\|$$
$$\geq (1-\eta_k)\|g(u_k) - g(\bar{u}_k)\|^2. \tag{3.13}$$

From (3.12), (3.13) and $\alpha_k^* > 1/2$, (3.11) is obtained immediately. The required results.

Remark 3.3 By using Theorem 3.2, one can easily obtain the convergence analysis of Algorithm 3.2.

References

[1] S. S. Chang, Set-valued variational inclusions in Banach spaces, J. of Mathematical Analysis and Applications, 248 (2000) : 438-454.

[2] M. A. Noor, Generalized set-valued variational inclusions and resolvent equations. J. of Mathematical Analysis and Applications, 228(1998): 206-220.

[3] M. A. Noor, K. I. Noor, and Th. M. Rassias, Set-valued resolvent equations and mixed variational inequalities, J. of Mathematical Analysis and Applications, 220(1998): 741-759.

[4] M. A. Noor, Multivalued quasi variational inclusions and implicit resolvent equations, Nonlinear Analysis 48 (2002):159-174.

[5] Han D. R., He B. S., A new accuracy criterion for approximate proximal point algorithms. J. of Mathematical Analysis and Applications, 2001, 263:343-354.

[6] He B. S., Inexact implicit methods for monotone general variational inequalities. Mathematical Programming, 1999,86: 199-217.

[7] N. J. Huang, Generalized nonlinear variational inclusions with noncompact valued mappings, Appl. Math. Lett., 9 (1996): 25-29.

[8] N. J. Huang, On the generalized implicit quasi-variational inequalities, J. of Mathematical Analysis and Applications, 216 (1997): 197-210.

[9] L. C. Zeng, Iterative algorithm for finding approximate solutions to completely generalized strongly nonlinear quasi-variational inequality, J. of Mathematical Analysis and Applications, 201(1996): 180-191.

[10] K. R. Kazimi, Mann and Ishikawa type perturbed iterative algorithms for generalized quasi-variational inclusions, J. of Mathematical Analysis and Applications, 209 (1997): 572-584.

[11] S. B. Naddler, Multivalued contraction mappings, Pacific J. Math., 30 (1969): 175 -488.

[12] M. A. Noor, Th. M. Rassias, Resolvent equations for nonlinear mixed variational inequalities, Nonlinear Anal. 42(1) (2000): 71-83.

[13] M. A. Noor, Some recent advances in variational inequalities, Part 1, basic concepts, New Zealand J. Math. 26 (1997): 53-80.

[14] M. A. Noor, Some recent advances in variational inequalities, Part 2, basic concepts, New Zealand J. Math. 26 (1997): 229-255.

[15] M. A. Noor, K. I. Noor, Some aspects of variational inequalities, J. Comput. Appl. Math. 47(1993), 285-312.