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MONOTONICITY RESULTS AND INEQUALITIES INVOLVING THE GAMMA FUNCTION

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Abstract. In this paper,we will prove that the function

$$f(x) = \frac{[\Gamma(x+1)]^{\frac{1}{x}}}{(x+b)^{\alpha}}$$

is strictly decreasing in $(0,\infty)$,where$\alpha \geq 1$ and $0 \leq b \leq \frac{6}{5}$ are two constants and the function

$$g(x) = \frac{[\Gamma(x+1)]^{\frac{1}{x}}}{(x+c)^{\beta}}$$

is strictly increasing in $(0,\infty)$, where $\beta \leq 1$ and $0 \leq c \leq \frac{4}{3}$ are two constants.According to the above results, if n is a positive integer and

$$\Phi(n) = (n!)^{\frac{1}{n}}$$

, then

$$1 < (1+\frac{1}{n})^{\beta} < \frac{\phi(n+1)}{\phi(n)} < 1 + \frac{1}{n+\frac{6}{5}} < 1 + \frac{1}{n}$$

where $0 < \beta < 1$. The lower bound $(1+\frac{1}{n})^{\beta}$ and upper bound $1+\frac{1}{n+\frac{6}{5}}$ are the best possible. From this, the well-known H.Minc and L.Sathre's inequality is improved.

1. Introduction

In [14], H.Minc and L.Sathre's proved that, if n is a positive integer and

$$\Phi(n) = (n!)^{\frac{1}{n}}$$

, then

$$1 < \frac{\phi(n+1)}{\phi(n)} < 1 + \frac{1}{n}, \tag{1}$$

which can be rearranged as

$$[\Gamma(1+n)]^{\frac{1}{n}} < [\Gamma(2+n)]^{\frac{1}{n+1}} \tag{2}$$

and

$$\frac{[\Gamma(1+n)]^{\frac{1}{n}}}{n} > \frac{[\Gamma(2+n)]^{\frac{1}{n+1}}}{n+1} \tag{3}$$

Where $\Gamma(x)$ denotes the well-known gamma function usually defined by

$$\Gamma(z) = \int_0^{\infty} e^{-t}t^{z-1}dt \tag{4}$$

for $\Re(z) > 0$.

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In [2,13], H. Alzer and J.S. Martins refined the right inequality in (1) and showed that, if n is a positive integer, then , for all positive real numbers r, we have

$$\frac{n}{n+1} < (\frac{\frac{1}{n}\sum_{i=1}^{n} i^r}{\frac{1}{n+1}\sum_{i=1}^{n+1} i^r}) < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} \tag{5}$$

Both bounds in (5) are the best possible. There have been many extensions and generalizations of the inequalities in (50, please refer to [4,6,12,15,17,19,21,22] and the references therein. The inequalities in (1) were refined and generalized in [9,20,24,25,26] and the following inequalities were obtained:

$$\frac{n+k+1}{n+m+k+1} < \frac{(\prod_{i=k+1}^{n+k} i)^{\frac{1}{n}}}{(\prod_{i=k+1}^{n+m+k} i)^{\frac{1}{n+m}}} \leq \sqrt{\frac{n+k}{n+m+k}} \tag{6}$$

Where k is a nonnegative integer, n and m are natural numbers. For $n = m = 1$, the equality in (6) is valid.

In [11], the following monotonicity results for the gamma function were established: The function $[\Gamma(1+\frac{1}{x})]^x$ decreases with $x > 0$ and $x[\Gamma(1+\frac{1}{x})]^x$ increases with $x > 0$, recovering the inequalities in (1) which refer to integer values of n. These are equivalent to the function $[\Gamma(1+x)]^{\frac{1}{x}}$ being increasing and $\frac{[\Gamma(1+x)]^{\frac{1}{x}}}{x}$ being decreasing on $(0,\infty)$, respectively. In addition, it was proved that the function $x^{1-\gamma}[\Gamma(1+\frac{1}{x})]^x$ decreases for $0 < x < 1$, where $\gamma = 0.57721566490153286...$ denotes the Euler's constants, which is equivalent to $\frac{[\Gamma(1+x)]^{\frac{1}{x}}}{x^{1-\gamma}}$ being increasing on $(1,\infty)$.

In [9], the following monotonicity result was obtained: The function

$$\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{\frac{1}{x}}}{x+y+1} \tag{7}$$

is decreasing in $x \geq 1$ for fixed $y \geq 0$. Then, for positive real numbers x and y, we have

$$\frac{x+y+1}{x+y+2} \leq \frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{\frac{1}{x}}}{[\Gamma(x+y+2)/\Gamma(y+1)]^{\frac{1}{x+1}}} \tag{8}$$

Inequality (8) extends and generalizes inequality (6), since $\Gamma(n+1) = n!$.

In [9,10,20], the authors proposed the following

Open problem 1. For positive real numbers x and y, we have

$$\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{\frac{1}{x}}}{[\Gamma(x+y+2)/\Gamma(y+1)]^{\frac{1}{x+1}}} \leq \sqrt{\frac{x}{x+y}} \tag{9}$$

Where Γ denotes the gamma function. If $x = 1$ and $y = 0$, the equality in (9) holds.

Open problem 2. For any positive real number z, define $z! = z(z-1)...z$, where $z = z-[z-1]$, and $[z]$ denotes the Gauss function whose value is the largest integer not more than z. Let $x > 0$ and $y \geq 0$ be real numbers, then

$$\frac{x+y}{x+y+1} \leq \frac{\sqrt[x]{x!}}{\sqrt[x+y]{(x+y)!}} \leq \sqrt{\frac{x}{x+y}} \tag{10}$$

Equality holds in the right hand side of (10) when $x = y = 1$.

Hence the inequalities in (9) and (10) are equivalent to the following monotonicity results in some sense for $x \geq 1$, which are obtained in [5] by Ch.-P. Chen and F. Qi: The function $\frac{[\Gamma(x+1)]^{\frac{1}{x}}}{x+1}$

is strictly decreasing on $[1,\infty)$, the function $\frac{[\Gamma(x+1)]^{\frac{1}{x}}}{\sqrt{x}}$ is strictly increasing on $[2,\infty)$, and the function $\frac{[\Gamma(x+1)]^{\frac{1}{x}}}{\sqrt{x+1}}$ is strictly increasing on $[1,\infty)$, respectively.

In this paper, we will obtain the following monotonicity results:

Theorem 1. The function $f(x)=\frac{[\Gamma(x+1)]^{\frac{1}{x}}}{(x+b)^{\alpha}}$ is strictly decreasing in $(0,\infty)$, where $\alpha\geq 1$ and $0\leq b\leq\frac{6}{5}$ are two constants.

Theorem 2. The function $g(x)=\frac{[\Gamma(x+1)]^{\frac{1}{x}}}{(x+c)^{\beta}}$ is strictly increasing in $(0,\infty)$, where $\beta<1$ and $0\leq c\leq\frac{4}{3}$ are two constants.

Corollary 1.([18]) The function $f(x)=\frac{[\Gamma(x+1)]^{\frac{1}{x}}}{(x+1)}$ is strictly decreasing on $[1,\infty)$.

Corollary 2.([18]) The function $g(x)=\frac{[\Gamma(x+1)]^{\frac{1}{x}}}{\sqrt{x+1}}$ is strictly increasing on $(0,\infty)$.

Corollary 3. Let $0<x<y$, then we have

$$(\frac{y+c}{x+c})^{\beta}<\frac{[\Gamma(y+1)]^{\frac{1}{y}}}{[\Gamma(x+1)]^{\frac{1}{x}}}<(\frac{y+b}{x+b})^{\alpha} \tag{11}$$

where $\alpha\geq 1$, $0\leq b\leq\frac{6}{5}$ and $\beta<1$, $0\leq c\leq\frac{4}{3}$.

Theorem 3. If n is a positive integer and $\Phi(n)=(n!)^{\frac{1}{n}}$, then

$$1<(1+\frac{1}{n})^{\beta}<\frac{\Phi(n+1)}{\Phi(n)}<1+\frac{1}{n+\frac{6}{5}}<1+\frac{1}{n} \tag{12}$$

where $0<\beta<1$. The lower bound $(1+\frac{1}{n})^{\beta}$ and upper $1+\frac{1}{n+\frac{6}{5}}$ are the best possible. From this, the well-known H. Minc and L. Sathre's inequality is improved.

2. Preliminaries

In this section, we present some useful formulas related to the derivatives of the logarithm of the gamma function.

In [23, pp. 103-105], the following formula was given:

$$\frac{\Gamma'(z)}{\Gamma(z)}+\gamma=\int_0^{\infty}\frac{e^{-t}-e^{zt}}{1-e^{-t}}dt=\int_0^1\frac{1-t^{z-1}}{1-t}dt \tag{13}$$

where $\gamma=0.57721566490153286...$ denotes the Euler's constants. See [23, p. 94]. Formula (13) can be used to calculate $\Gamma'(k)$ for $k\in N$. We call $\Psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$ the digamma or psi function. See [3, p. 71].

It is well known that the Bernoulli numbers B_n are generally defined [23, p. 1] by

$$\frac{1}{e^t-1}+\frac{1}{2}-\frac{1}{t}=\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{t^{2n}}{(2n)!}\cdot B_n \tag{14}$$

In particular, we have the following

$B_1=\frac{1}{6}$, $B_2=\frac{1}{30}$, $B_3=\frac{1}{42}$, ...

In [23, p. 45], the following summation formula is given

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{2k+1}}=\frac{\pi^{2k+1}E_k}{2^{2k+2}(2k)!} \tag{15}$$

for nonnegative integer k, where E_k denotes Euler's number, which implies

$$B_n = \frac{2(2n)!}{(2\pi)^{2n}} \cdot \sum_{m=1}^{\infty} \frac{1}{m^{2n}} \qquad n \in N \tag{16}$$

The formula (16) can also be found in [7, p. 1237].

Lemma 1. For a real number $x > 0$ and natural number m, we have

$$ln\Gamma(x) = \frac{1}{2} \cdot \ln(2\pi) + (x - \frac{1}{2}) \cdot \ln x - x + \sum_{n=1}^{m} (-1)^{n-1} \frac{B_n}{2(2n-1)n} \cdot \frac{1}{x^{2n-1}}$$

$$+(-1)^m \theta_1 \cdot \frac{B_{m+1}}{(2m+1)(2m+2)} \cdot \frac{1}{x^{2m+1}}, \qquad 0 < \theta_1 < 1 \tag{17}$$

$$\frac{d}{dx} ln\Gamma(x) = \ln x - \frac{1}{2x} + \sum_{n=1}^{m} (-1)^n \frac{B_n}{2n} \cdot \frac{1}{x^{2n}}$$

$$+(-1)^{m+1} \theta_2 \cdot \frac{B_{m+1}}{2m+2} \cdot \frac{1}{x^{2m+2}}, \qquad 0 < \theta_2 < 1 \tag{18}$$

$$\frac{d^2}{dx^2} ln\Gamma(x) = \frac{1}{x} + \frac{1}{2x^2} + \sum_{n=1}^{m} (-1)^{n-1} \frac{B_n}{x^{2n+1}}$$

$$+(-1)^m \theta_3 \cdot \frac{B_{m+1}}{x^{2m+3}}, \qquad 0 < \theta_3 < 1 \tag{19}$$

Remark 1. The formulas and their proofs in Lemma 1 are well-known and can be found in many textbooks on Analysis; See, for instance, [8, Section 54 and Section 541].

3. Proofs of theorems

Proof of Theorem 1. Taking the logarithm and straightforward calculation gives

$$lnf(x) = \frac{1}{x} \cdot \ln\Gamma(x+1) - \alpha \ln(x+b) \tag{20}$$

Differentiating with respect to x on both sides of (20) and rearranging leads to

$$x^2 \frac{f'(x)}{f(x)} = -ln\Gamma(x+1) + x\frac{d}{dx} Ln\Gamma(x+1) - \frac{\alpha x^2}{x+b} \tag{21}$$

and, using (19)

$$(x^2 \frac{f'(x)}{f(x)})' = x\frac{d^2}{dx^2} ln\Gamma(x+1) - \frac{\alpha x^2 + 2b\alpha x}{(x+b)^2}$$

$$< x[\frac{1}{x+1} + \frac{1}{2(x+1)^2} + \frac{1}{6(x+1)^3}] - \frac{\alpha x^2 + 2b\alpha x}{(x+b)^2}$$

$$= \frac{G(x)}{6(x+1)^3(x+b)^2} \tag{22}$$

$$G(x) = (6-6a)x^5 + (12b - 12b\alpha - 18\alpha + 15)x^4 + (6b^2 + 30b - 36b\alpha - 18\alpha + 10)x^3$$

$$+(15b^2 + 20b - 36b\alpha - 6\alpha)x^2 + (10b^2 - 12b\alpha)x \tag{23}$$

$$G'(0) = 10b^2 - 12b\alpha$$

$$G''(0) = 2(15b^2 + 20b - 36b\alpha - 6\alpha)$$

$$G^{(3)}(0) = 6(6b^2 + 30b - 36b\alpha - 18\alpha + 10)$$

$$G^{(4)}(0) = 24(12b - 12b\alpha - 18\alpha + 15)$$

$$G^{(5)}(0) = 720(1 - \alpha)$$

For $x > 0$, $\alpha \geq 1$ and $0 \leq b \leq \frac{6}{5}$, we have $G^{(5)}(x) \leq 0$, $G^{(4)}(x)$ decreasing, $G^{(4)}(x) \leq G^{(4)}(0) \leq 0$; $G^{(3)}(x)$ decreasing, $G^{(3)}(x) \leq G^{(3)}(0) \leq 0$, and thus $G''(x)$ decreasing, $G''(x) \leq G''(0) \leq 0$, and then $G'(x)$ decreasing, $G(x) \leq G(0) = 0$. Therefore the function $\tau(x) = x^2\frac{f'(x)}{f(x)}$ is strictly decreasing in $(0, \infty)$, $\tau(x) < \tau(0) = 0$, and then $f'(x) < 0$, hence $f(x)$ is strictly decreasing in $(0, \infty)$.The proof of Theorem 1 is complete.

Proof of Theorem 2. Taking the logarithm and a simple calculation yields

$$lng(x) = \frac{1}{x} \cdot \ln \Gamma(x+1) - \beta \ln(x+c) \tag{24}$$

Differentiating with respect to x on both sides of (24) and rearranging leads to

$$x^2\frac{g'(x)}{g(x)} = -\ln \Gamma(x+1) + x\frac{d}{dx}ln\Gamma(x+1) - \frac{\beta x^2}{x+c} \tag{25}$$

and, using (19)

$$(x^2\frac{g'(x)}{g(x)})' = x\frac{d^2}{dx^2}ln\Gamma(x+1) - \frac{\beta x^2 + 2c\beta x}{(x+c)^2}$$

$$> x[\frac{1}{x+1} + \frac{1}{2(x+1)^2}] - \frac{\beta x^2 + 2c\beta x}{(x+c)^2} = \frac{\mu(x)}{2(x+1)^2(x+c)^2} \tag{26}$$

$$\mu(x) \triangleq (2-2\beta)x^4 + (4c+3-4\beta-4c\beta)x^3$$

$$+(2c^2+6c-2\beta-8c\beta)x^2 + (3c^2-4c\beta)x \tag{27}$$

$$\mu'(0) = 3c^2 - 4c\beta$$

$$\mu''(0) = 2(2c^2+6c-2\beta-8c\beta)$$

$$\mu^{(3)}(0) = 6(4c+3-4\beta-4c\beta)$$

$$\mu^{(4)}(x) = 24(2-2\beta)$$

For $x > 0$, $\beta < 1$ and $0 \leq c \leq \frac{4}{3}$, we have $\mu^{(4)}(x) < 0$, $\mu^{(3)}(x)$ increasing, $\mu^{(3)}(x) \geq \mu^{(3)}(0) \geq 0$, and then $\mu''(x)$ increasing, $\mu''(x) \geq \mu''(0) \geq 0$, and thus $\mu'(x)$ increasing, $\mu'(x) \geq \mu'(0) \geq 0$, hence $\mu(x)$ increasing. $\mu(x) \geq \mu(0) \geq 0$. Therefore the function $\xi(x) = x^2\frac{g'(x)}{g(x)}$ is strictly increasing in $(0, \infty)$, $\xi(x) > \xi(0) = 0$, and then $g'(x) > 0$; hence $g(x)$ is strictly increasing in $(0, \infty)$.

The proof of theorem 2 is complete.

Proof of Theorem 3. If let $x = n$, $y = n+1$, $b = \frac{6}{5}$, $c = 0$, and $\alpha = 1$, $0 < \beta < 1$, $n \in N$ in inequality (11), then we obtain the inequality (12). The proof is complete.

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