

# A REFINEMENT OF GRÜSS TYPE INEQUALITY FOR THE BOCHNER INTEGRAL OF VECTOR-VALUED FUNCTIONS IN HILBERT SPACES AND APPLICATIONS

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ABSTRACT. A refinement of Grüss type inequality for the Bochner integral of vector-valued functions in real or complex Hilbert spaces is given. Related results are obtained. Application for finite Fourier transforms of vector-valued functions and some particular inequalities are provided.

## 1. INTRODUCTION

In 1934, G. Grüss [5] showed that

$$(1.1) \quad |T(f, g)| \leq \frac{1}{4} (M - m)(N - n),$$

provided  $m, M, n, N$  are real numbers with the property

$$-\infty < m \leq f \leq M < \infty, \quad -\infty < n \leq g \leq N < \infty \quad \text{a.e. on } [a, b]$$

and  $T(f, g)$  is the Čebyšev functional

$$T(f, g) := \frac{1}{b-a} \int_a^b f(t) g(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b g(t) dt.$$

The constant  $\frac{1}{4}$  is best possible in (1.1) in the sense that it cannot be replaced by a smaller one.

An extension of this classical result to real or complex inner product spaces has been obtained by S.S. Dragomir in [2]:

**Theorem 1.** *Let  $(H; \langle \cdot, \cdot \rangle)$  be an inner product space over the real or complex number field  $\mathbb{K}$  and  $e \in H$ ,  $\|e\| = 1$ . If  $\varphi, \phi, \gamma, \Gamma \in \mathbb{R}$  and  $x, y \in H$  are such that*

$$(1.2) \quad \operatorname{Re} \langle \phi e - x, x - \varphi e \rangle \geq 0 \quad \text{and} \quad \operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle \geq 0$$

or, equivalently (see [4])

$$(1.3) \quad \left\| x - \frac{\varphi + \phi}{2} e \right\| \leq \frac{1}{2} |\phi - \varphi| \quad \text{and} \quad \left\| y - \frac{\gamma + \Gamma}{2} e \right\| \leq \frac{1}{2} |\Gamma - \gamma|,$$

then

$$(1.4) \quad |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{4} |\phi - \varphi| |\Gamma - \gamma|.$$

The constant  $\frac{1}{4}$  is best possible in (1.4).

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A further extension for Bochner integrals of vector-valued functions in real or complex Hilbert spaces was obtained by S.S. Dragomir in 2001, [3].

**Theorem 2.** *Let  $(H; \langle \cdot, \cdot \rangle)$  be a real or complex Hilbert space,  $\Omega \subset \mathbb{R}^n$  be a Lebesgue measurable set and  $\rho : \Omega \rightarrow [0, \infty)$  a Lebesgue measurable function with  $\int_{\Omega} \rho(s) ds = 1$ . We denote by  $L_{2,\rho}(\Omega, H)$  the set of all Bochner measurable functions  $f$  on  $\Omega$  such that  $\|f\|_{2,\rho}^2 := \int_{\Omega} \rho(s) \|f(s)\|^2 ds < \infty$ . If  $f, g$  belong to  $L_{2,\rho}(\Omega, H)$  and there exist the vectors  $x, X, y, Y \in H$  such that*

$$(1.5) \quad \int_{\Omega} \rho(t) \operatorname{Re} \langle X - f(t), f(t) - x \rangle dt \geq 0, \\ \int_{\Omega} \rho(t) \operatorname{Re} \langle Y - g(t), g(t) - y \rangle dt \geq 0,$$

then we have the inequality

$$(1.6) \quad \left| \int_{\Omega} \rho(t) \langle f(t), g(t) \rangle dt - \left\langle \int_{\Omega} \rho(t) f(t) dt, \int_{\Omega} \rho(t) g(t) dt \right\rangle \right| \\ \leq \frac{1}{4} \|X - x\| \|Y - y\|.$$

The constant  $\frac{1}{4}$  is sharp in the sense mentioned above.

**Remark 1.** *We must state that the functions under the integrals (1.5) and (1.6) are Bochner integrable on  $\Omega$  since they are Bochner measurable and we can state the following obvious results*

$$\begin{aligned} & \rho(t) |\operatorname{Re} \langle X - f(t), f(t) - x \rangle| \\ & \leq \rho(t) |\langle X - f(t), f(t) - x \rangle| \\ & \leq \rho(t) \|f(t)\|^2 + (\|X\| + \|x\|) \rho(t) \|f(t)\| + |\langle X, x \rangle| \rho(t) \end{aligned}$$

for a.e.  $t \in \Omega$ ;

$$\int_{\Omega} \rho(t) \|f(t)\| dt \leq \|f\|_{2,\rho}$$

and

$$\int_{\Omega} \rho(t) |\langle f(t), g(t) \rangle| dt \leq \|f\|_{2,\rho} \|g\|_{2,\rho}.$$

**Remark 2.** *A practical sufficient condition for (1.5) to hold is*

$$\operatorname{Re} \langle X - f(t), f(t) - x \rangle \geq 0, \quad \operatorname{Re} \langle Y - g(t), g(t) - y \rangle \geq 0$$

or, equivalently

$$\left\| f(t) - \frac{X+x}{2} \right\| \leq \frac{1}{2} \|X-x\| \quad \text{and} \quad \left\| g(t) - \frac{Y+y}{2} \right\| \leq \frac{1}{2} \|Y-y\|,$$

for a.e.  $t \in \Omega$ .

An interesting particular inequality that has not been mentioned in [3] can be obtained by considering  $H = \mathbb{C}$ ,  $\langle x, y \rangle := x \cdot \bar{y}$  and  $g = \bar{f}$ , to give

$$(1.7) \quad \left| \int_{\Omega} \rho(s) f^2(s) ds - \left( \int_{\Omega} \rho(s) f(s) ds \right)^2 \right| \leq \frac{1}{4} |A - a|^2,$$

provided

$$(1.8) \quad \int_{\Omega} \rho(s) \operatorname{Re} \left[ (A - f(s)) \left( \overline{f(s)} - \bar{a} \right) \right] ds \geq 0$$

or, sufficiently,

$$(1.9) \quad \operatorname{Re} \left[ (A - f(s)) \left( \overline{f(s)} - \bar{a} \right) \right] \geq 0$$

for a.e.  $s \in \Omega$ .

Note that the alternative result

$$(1.10) \quad 0 \leq \int_{\Omega} \rho(s) |f(s)|^2 ds - \left| \int_{\Omega} \rho(s) f(s) ds \right|^2 \leq \frac{1}{4} |A - a|^2,$$

provided (1.8) or (1.9) hold, has been stated in [3].

The main aim of this paper is to obtain an improvement of the Grüss inequality (1.6) and establish some Grüss type results in providing upper bounds for the quantities

$$\left| \int_{\Omega} \rho(t) \langle f(t), g(t) \rangle dt - \left\langle \int_{\Omega} \rho(t) f(t) dt, \int_{\Omega} \rho(t) g(t) dt \right\rangle \right|$$

and

$$\left\| \int_{\Omega} \rho(t) \alpha(t) f(t) dt - \int_{\Omega} \rho(t) \alpha(t) dt \cdot \int_{\Omega} \rho(t) f(t) dt \right\|,$$

under various assumptions for  $\rho$ ,  $\alpha \in L_{2,\rho}(\Omega, \mathbb{K})$  and  $f \in L_{2,\rho}(\Omega, H)$ .

Applications in approximating the finite Fourier transform of vector-valued functions in Hilbert spaces are provided. Inequalities for some particular vector-valued functions are given as well.

## 2. SOME INEQUALITIES OF GRÜSS TYPE

The following lemma holds.

**Lemma 1.** *Assume that  $f \in L_{2,\rho}(\Omega, H)$  and there exist the vectors  $x, X \in H$  such that*

$$(2.1) \quad \int_{\Omega} \rho(t) \operatorname{Re} \langle X - f(t), f(t) - x \rangle dt \geq 0$$

or, equivalently,

$$(2.2) \quad \int_{\Omega} \rho(t) \left\| f(t) - \frac{X+x}{2} \right\|^2 dt \leq \frac{1}{4} \|X-x\|^2.$$

Then we have the inequality

$$(2.3) \quad \begin{aligned} 0 &\leq \int_{\Omega} \rho(t) \|f(t)\|^2 dt - \left\| \int_{\Omega} \rho(t) f(t) dt \right\|^2 \\ &\leq \frac{1}{4} \|X-x\|^2 - \int_{\Omega} \rho(t) \operatorname{Re} \langle X - f(t), f(t) - x \rangle dt \\ &\leq \frac{1}{4} \|X-x\|^2. \end{aligned}$$

The constant  $\frac{1}{4}$  in the second and third inequalities cannot be replaced by a smaller quantity.

*Proof.* Since, for any  $y, x, X \in H$

$$\left\| y - \frac{X+x}{2} \right\|^2 - \frac{1}{4} \|X-x\|^2 = \operatorname{Re} \langle y - X, y - x \rangle,$$

hence

$$\begin{aligned} & \int_{\Omega} \rho(t) \operatorname{Re} \langle X - f(t), f(t) - x \rangle dt \\ &= \int_{\Omega} \rho(t) \left[ \frac{1}{4} \|X-x\|^2 - \left\| f(t) - \frac{X+x}{2} \right\|^2 \right] dt \\ &= \frac{1}{4} \|X-x\|^2 - \int_{\Omega} \rho(t) \left\| f(t) - \frac{X+x}{2} \right\|^2 dt \end{aligned}$$

showing that, indeed, (2.1) and (2.2) are equivalent.

Define (see also [3])

$$I_1 := \left\langle X - \int_{\Omega} \rho(t) f(t) dt, \int_{\Omega} \rho(t) f(t) dt - x \right\rangle$$

and

$$I_2 := \int_{\Omega} \rho(t) \langle X - f(t), f(t) - x \rangle dt.$$

Then, obviously

$$I_1 = \int_{\Omega} \rho(t) [\langle X, f(t) \rangle + \langle f(t), x \rangle] dt - \langle X, x \rangle - \left\| \int_{\Omega} \rho(t) f(t) dt \right\|^2$$

and

$$I_2 = \int_{\Omega} \rho(t) [\langle X, f(t) \rangle + \langle f(t), x \rangle] dt - \langle X, x \rangle - \int_{\Omega} \rho(t) \|f(t)\|^2 dt.$$

Consequently,

$$(2.4) \quad I_1 - I_2 = \int_{\Omega} \rho(t) \|f(t)\|^2 dt - \left\| \int_{\Omega} \rho(t) f(t) dt \right\|^2.$$

Taking the real value in (2.4), we can state the following identity as well [3]

$$\begin{aligned} (2.5) \quad & \int_{\Omega} \rho(t) \|f(t)\|^2 dt - \left\| \int_{\Omega} \rho(t) f(t) dt \right\|^2 \\ &= \operatorname{Re} \left\langle X - \int_{\Omega} \rho(t) f(t) dt, \int_{\Omega} \rho(t) f(t) dt - x \right\rangle \\ & \quad - \int_{\Omega} \rho(t) \operatorname{Re} \langle X - f(t), f(t) - x \rangle dt \end{aligned}$$

that is of interest in itself.

Using the well known inequality in inner product spaces

$$(2.6) \quad \operatorname{Re} \langle z, y \rangle \leq \left\| \frac{z+y}{2} \right\|^2$$

with equality iff  $z = y$ , we may state that

$$\operatorname{Re} \left\langle X - \int_{\Omega} \rho(t) f(t) dt, \int_{\Omega} \rho(t) f(t) dt - x \right\rangle \leq \frac{1}{4} \|X-x\|^2$$

and by the identity (2.5), we deduce the second inequality in (2.3).

The third inequality follows by the assumption (2.1).

Now, assume that (2.3) holds with the constants  $C, D > 0$ . That is,

$$(2.7) \quad \begin{aligned} 0 &\leq \int_{\Omega} \rho(t) \|f(t)\|^2 dt - \left\| \int_{\Omega} \rho(t) f(t) dt \right\|^2 \\ &\leq C \|X - x\|^2 - \int_{\Omega} \rho(t) \operatorname{Re} \langle X - f(t), f(t) - x \rangle dt \\ &\leq D \|X - x\|^2. \end{aligned}$$

If we choose  $\Omega = [a, b] \subset \mathbb{R}$ ,  $H = \mathbb{R}$ ,  $f : [a, b] \rightarrow \mathbb{R}$ ,

$$f(t) = \begin{cases} -1 & \text{if } t \in [a, \frac{a+b}{2}] \\ 1 & \text{if } t \in (\frac{a+b}{2}, b] \end{cases}$$

then for  $X = 1$ ,  $x = -1$  and  $\rho : [a, b] \rightarrow \mathbb{R}$ ,  $\rho(t) = 1$ ,  $t \in [a, b]$  the condition (2.1) holds and by (2.7) we deduce

$$1 \leq 4C \leq 4D$$

giving  $C \geq \frac{1}{4}$  and  $D \geq \frac{1}{4}$ , and the lemma is proved. ■

The following refinement of the Grüss inequality holds.

**Theorem 3.** *Assume that  $f, g \in L_{2,\rho}(\Omega, H)$  and there exists the vectors  $x, X, y, Y \in H$  such that*

$$(2.8) \quad \begin{aligned} \int_{\Omega} \rho(t) \operatorname{Re} \langle X - f(t), f(t) - x \rangle dt &\geq 0, \\ \int_{\Omega} \rho(t) \operatorname{Re} \langle Y - g(t), g(t) - y \rangle dt &\geq 0 \end{aligned}$$

or, equivalently,

$$(2.9) \quad \begin{aligned} \int_{\Omega} \rho(t) \left\| f(t) - \frac{X+x}{2} \right\|^2 dt &\leq \frac{1}{4} \|X-x\|^2, \\ \int_{\Omega} \rho(t) \left\| g(t) - \frac{Y+y}{2} \right\|^2 dt &\leq \frac{1}{4} \|Y-y\|^2. \end{aligned}$$

Then we have the inequality

$$(2.10) \quad \begin{aligned} &\left| \int_{\Omega} \rho(t) \langle f(t), g(t) \rangle dt - \left\langle \int_{\Omega} \rho(t) f(t) dt, \int_{\Omega} \rho(t) g(t) dt \right\rangle \right| \\ &\leq \frac{1}{4} \|X-x\| \|Y-y\| - \left[ \int_{\Omega} \rho(t) \operatorname{Re} \langle X-f(t), f(t)-x \rangle dt \right. \\ &\quad \left. \times \int_{\Omega} \rho(t) \operatorname{Re} \langle Y-g(t), g(t)-y \rangle dt \right]^{1/2} \\ &\leq \frac{1}{4} \|X-x\| \|Y-y\|. \end{aligned}$$

The constant  $\frac{1}{4}$  in both inequalities is sharp in the sense that it cannot be replaced by a smaller quantity.

*Proof.* We start with the following Korkine type identity (see also [3])

$$(2.11) \quad \int_{\Omega} \rho(t) \langle f(t), g(t) \rangle dt - \left\langle \int_{\Omega} \rho(t) f(t) dt, \int_{\Omega} \rho(t) g(t) dt \right\rangle \\ = \frac{1}{2} \int_{\Omega} \int_{\Omega} f(t) \rho(s) \langle f(t) - f(s), g(t) - g(s) \rangle dt ds.$$

Taking the modulus and using the Schwarz inequality in inner product spaces, we have

$$(2.12) \quad \left| \int_{\Omega} \rho(t) \langle f(t), g(t) \rangle dt - \left\langle \int_{\Omega} \rho(t) f(t) dt, \int_{\Omega} \rho(t) g(t) dt \right\rangle \right| \\ \leq \frac{1}{2} \int_{\Omega} \int_{\Omega} \rho(t) \rho(s) \|f(t) - f(s)\| \|g(t) - g(s)\| dt ds.$$

Using the Cauchy-Bunyakovski-Schwarz inequality for double integrals, we have

$$(2.13) \quad \frac{1}{2} \int_{\Omega} \int_{\Omega} \rho(t) \rho(s) \|f(t) - f(s)\| \|g(t) - g(s)\| dt ds \\ \leq \left( \frac{1}{2} \int_{\Omega} \int_{\Omega} \rho(t) \rho(s) \|f(t) - f(s)\|^2 dt ds \right)^{\frac{1}{2}} \\ \times \left( \frac{1}{2} \int_{\Omega} \int_{\Omega} \rho(t) \rho(s) \|g(t) - g(s)\|^2 dt ds \right)^{\frac{1}{2}}.$$

Since a simple computation shows that

$$(2.14) \quad \frac{1}{2} \int_{\Omega} \int_{\Omega} \rho(t) \rho(s) \|f(t) - f(s)\|^2 dt ds \\ = \int_{\Omega} \rho(t) \|f(t)\|^2 dt - \left\| \int_{\Omega} \rho(t) f(t) dt \right\|^2$$

and

$$(2.15) \quad \frac{1}{2} \int_{\Omega} \int_{\Omega} \rho(t) \rho(s) \|g(t) - g(s)\|^2 dt ds \\ = \int_{\Omega} \rho(t) \|g(t)\|^2 dt - \left\| \int_{\Omega} \rho(t) g(t) dt \right\|^2,$$

then by (2.11) – (2.15), we deduce

$$(2.16) \quad \left| \int_{\Omega} \rho(t) \langle f(t), g(t) \rangle dt - \left\langle \int_{\Omega} \rho(t) f(t) dt, \int_{\Omega} \rho(t) g(t) dt \right\rangle \right|^2 \\ \leq \left( \int_{\Omega} \rho(t) \|f(t)\|^2 dt - \left\| \int_{\Omega} \rho(t) f(t) dt \right\|^2 \right) \\ \times \left( \int_{\Omega} \rho(t) \|g(t)\|^2 dt - \left\| \int_{\Omega} \rho(t) g(t) dt \right\|^2 \right) =: M.$$

Using Lemma 1, we may deduce

$$(2.17) \quad M \leq \left( \frac{1}{4} \|X - x\|^2 - \int_{\Omega} \rho(t) \operatorname{Re} \langle X - f(t), f(t) - x \rangle dt \right) \\ \times \left( \frac{1}{4} \|Y - y\|^2 - \int_{\Omega} \rho(t) \operatorname{Re} \langle Y - g(t), g(t) - y \rangle dt \right) =: N.$$

By the elementary inequality

$$(m^2 - n^2)(p^2 - q^2) \leq (mp - nq)^2, \quad m, n, p, q \in \mathbb{R},$$

we may state that

$$(2.18) \quad N \leq \left( \frac{1}{4} \|X - x\| \|Y - y\| - \left[ \int_{\Omega} \rho(t) \operatorname{Re} \langle X - f(t), f(t) - x \rangle dt \right. \right. \\ \left. \left. \times \int_{\Omega} \rho(t) \operatorname{Re} \langle Y - g(t), g(t) - y \rangle dt \right]^{\frac{1}{2}} \right)^2$$

and thus by (2.16) – (2.18) we can conclude that

$$(2.19) \quad \left| \int_{\Omega} \rho(t) \langle f(t), g(t) \rangle dt - \left\langle \int_{\Omega} \rho(t) f(t) dt, \int_{\Omega} \rho(t) g(t) dt \right\rangle \right| \\ \leq \left| \frac{1}{4} \|X - x\| \|Y - y\| - \left[ \int_{\Omega} \rho(t) \operatorname{Re} \langle X - f(t), f(t) - x \rangle dt \right. \right. \\ \left. \left. \times \int_{\Omega} \rho(t) \operatorname{Re} \langle Y - g(t), g(t) - y \rangle dt \right]^{\frac{1}{2}} \right|$$

and since, by Lemma 1,

$$\left[ \int_{\Omega} \rho(t) \operatorname{Re} \langle X - f(t), f(t) - x \rangle dt \cdot \int_{\Omega} \rho(t) \operatorname{Re} \langle Y - g(t), g(t) - y \rangle dt \right]^{\frac{1}{2}} \\ \leq \frac{1}{4} \|X - x\| \|Y - y\|,$$

hence from (2.19) we deduce (2.10).

The sharpness of the constant  $\frac{1}{4}$  follows by Lemma 1, and we omit the details. ■

**Remark 3.** *The inequality (2.10) is obviously a refinement of (1.6), which has been obtained in [3].*

The following result of Grüss type also holds.

**Theorem 4.** *Assume that  $\alpha \in L_{2,\rho}(\Omega, H)$ ,  $f \in L_{2,\rho}(\Omega, H)$  and there exists the scalars  $a, A \in \mathbb{K}$  ( $\mathbb{K} = \mathbb{C}, \mathbb{R}$ ) and the vectors  $x, X \in H$  such that*

$$(2.20) \quad \int_{\Omega} \rho(t) \operatorname{Re} \left[ (A - \alpha(t)) (\overline{\alpha(t)} - \bar{a}) \right] dt \geq 0 \quad \text{and} \\ \int_{\Omega} \rho(t) \operatorname{Re} \langle X - f(t), f(t) - x \rangle dt \geq 0$$

or, equivalently,

$$(2.21) \quad \begin{aligned} \int_{\Omega} \rho(t) \left| \alpha(t) - \frac{A+a}{2} \right|^2 dt &\leq \frac{1}{4} |A-a|^2, \\ \int_{\Omega} \rho(t) \left\| f(t) - \frac{X+x}{2} \right\|^2 dt &\leq \frac{1}{4} \|X-x\|^2, \end{aligned}$$

then we have the inequality

$$(2.22) \quad \begin{aligned} &\left\| \int_{\Omega} \rho(t) \alpha(t) f(t) dt - \int_{\Omega} \rho(t) \alpha(t) dt \cdot \int_{\Omega} \rho(t) f(t) dt \right\| \\ &\leq \frac{1}{4} |A-a| \|X-x\| - \left( \int_{\Omega} \rho(t) \operatorname{Re} \left[ (A-\alpha(t)) (\overline{\alpha(t)} - \bar{a}) \right] dt \right. \\ &\quad \left. \times \int_{\Omega} \rho(t) \operatorname{Re} \langle X-f(t), f(t)-x \rangle dt \right)^{1/2} \\ &\leq \frac{1}{4} |A-a| \|X-x\|. \end{aligned}$$

The constant  $\frac{1}{4}$  in both inequalities is sharp in the sense that it cannot be replaced by a smaller quantity.

*Proof.* We observe that the following Korkine type identity holds

$$\begin{aligned} &\int_{\Omega} \rho(t) \alpha(t) f(t) dt - \int_{\Omega} \rho(t) \alpha(t) dt \cdot \int_{\Omega} \rho(t) f(t) dt \\ &= \frac{1}{2} \int_{\Omega} \int_{\Omega} \rho(t) \rho(s) (\alpha(t) - \alpha(s)) (f(t) - f(s)) dt ds. \end{aligned}$$

Using a similar approach to the one in Theorem 3, we have successively

$$(2.23) \quad \begin{aligned} &\left\| \int_{\Omega} \rho(t) \alpha(t) f(t) dt - \int_{\Omega} \rho(t) \alpha(t) dt \cdot \int_{\Omega} \rho(t) f(t) dt \right\| \\ &\leq \frac{1}{2} \int_{\Omega} \int_{\Omega} \rho(t) \rho(s) |\alpha(t) - \alpha(s)| \|f(t) - f(s)\| dt ds \\ &\leq \left[ \frac{1}{2} \int_{\Omega} \int_{\Omega} \rho(t) \rho(s) |\alpha(t) - \alpha(s)|^2 dt ds \right. \\ &\quad \left. \times \frac{1}{2} \int_{\Omega} \int_{\Omega} \rho(t) \rho(s) \|f(t) - f(s)\|^2 dt ds \right]^{\frac{1}{2}} \\ &= \left[ \int_{\Omega} \rho(t) |\alpha(t)|^2 dt - \left| \int_{\Omega} \rho(t) \alpha(t) dt \right|^2 \right]^{\frac{1}{2}} \\ &\quad \times \left[ \int_{\Omega} \rho(t) \|f(t)\|^2 dt - \left\| \int_{\Omega} \rho(t) f(t) dt \right\|^2 \right]^{\frac{1}{2}} \\ &\leq \left( \frac{1}{4} |A-a|^2 - \int_{\Omega} \rho(t) \operatorname{Re} \left[ (A-\alpha(t)) (\overline{\alpha(t)} - \bar{a}) \right] dt \right)^{\frac{1}{2}} \\ &\quad \times \left( \frac{1}{4} \|X-x\|^2 - \int_{\Omega} \rho(t) \operatorname{Re} \langle X-f(t), f(t)-x \rangle dt \right)^{\frac{1}{2}} \end{aligned}$$



$$\begin{aligned}
&\leq \left| \frac{1}{4} |A - a| \|X - x\| - \left( \int_{\Omega} \rho(t) \operatorname{Re} \left[ (A - \alpha(t)) (\overline{\alpha(t)} - \bar{a}) \right] dt \right)^{\frac{1}{2}} \right. \\
&\quad \left. \times \left( \int_{\Omega} \rho(t) \operatorname{Re} \langle X - f(t), f(t) - x \rangle dt \right)^{\frac{1}{2}} \right| \\
&= \frac{1}{4} |A - a| \|X - x\| - \left( \int_{\Omega} \rho(t) \operatorname{Re} \left[ (A - \alpha(t)) (\overline{\alpha(t)} - \bar{a}) \right] dt \right. \\
&\quad \left. \times \int_{\Omega} \rho(t) \operatorname{Re} \langle X - f(t), f(t) - x \rangle dt \right)^{\frac{1}{2}}
\end{aligned}$$

and the first inequality in (2.22) is proved.

The second inequality and the sharpness of the constant  $\frac{1}{4}$  are obvious and we omit the details. ■

### 3. PRE-GRÜSS TYPE INEQUALITIES

The following result provides an inequality of *pre-Grüss type* that may be useful in applications when one of the factors is known and some bounds for the second factor are provided.

**Theorem 5.** *Assume that  $f, g \in L_{2,\rho}(\Omega, H)$  and there exists the vectors  $x, X \in H$  such that either (2.1) or (2.2) hold true. Then we have the inequality:*

$$\begin{aligned}
(3.1) \quad &\left| \int_{\Omega} \rho(t) \langle f(t), g(t) \rangle dt - \left\langle \int_{\Omega} \rho(t) f(t) dt, \int_{\Omega} \rho(t) g(t) dt \right\rangle \right| \\
&\leq \left( \frac{1}{4} \|X - x\|^2 - \int_{\Omega} \rho(t) \operatorname{Re} \langle X - f(t), f(t) - x \rangle dt \right)^{\frac{1}{2}} \\
&\quad \times \left( \int_{\Omega} \rho(t) \|g(t)\|^2 dt - \left\| \int_{\Omega} \rho(t) g(t) dt \right\|^2 \right)^{\frac{1}{2}} \\
&\leq \frac{1}{2} \|X - x\| \left( \int_{\Omega} \rho(t) \|g(t)\|^2 dt - \left\| \int_{\Omega} \rho(t) g(t) dt \right\|^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

The proof follows by Lemma 1 and the inequality (2.16) and we omit the details.

Similarly, we can state the following pre-Grüss inequality related to the case when our function is scalar.

**Theorem 6.** *Assume that  $\alpha \in L_{2,\rho}(\Omega, H)$ ,  $f \in L_{2,\rho}(\Omega, H)$  and there exists the vectors  $x, X \in H$  such that (2.1) or, equivalently (2.2) hold true. Then we have the*

inequality

$$\begin{aligned}
(3.2) \quad & \left\| \int_{\Omega} \rho(t) \alpha(t) f(t) dt - \int_{\Omega} \rho(t) \alpha(t) dt \cdot \int_{\Omega} \rho(t) f(t) dt \right\| \\
& \leq \left( \frac{1}{4} \|X - x\|^2 - \int_{\Omega} \rho(t) \operatorname{Re} \langle X - f(t), f(t) - x \rangle dt \right)^{\frac{1}{2}} \\
& \quad \times \left( \int_{\Omega} \rho(t) |\alpha(t)|^2 dt - \left| \int_{\Omega} \rho(t) \alpha(t) dt \right|^2 \right)^{\frac{1}{2}} \\
& \leq \frac{1}{2} \|X - x\| \left( \int_{\Omega} \rho(t) |\alpha(t)|^2 dt - \left| \int_{\Omega} \rho(t) \alpha(t) dt \right|^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

The proof follows by Lemma 1 and the inequality (2.23) and we omit the details.

**Remark 4.** Assume that  $\Omega = [a, b] \subset \mathbb{R}$  and  $\rho(t) = \frac{1}{b-a}$ . Then, from (3.2) we get

$$\begin{aligned}
(3.3) \quad & \left\| \frac{1}{b-a} \int_{\Omega} \alpha(t) f(t) dt - \frac{1}{b-a} \int_{\Omega} \alpha(t) dt \cdot \frac{1}{b-a} \int_{\Omega} f(t) dt \right\| \\
& \leq \left[ \frac{1}{4} \|X - x\|^2 - \frac{1}{b-a} \int_{\Omega} \operatorname{Re} \langle X - f(t), f(t) - x \rangle dt \right]^{\frac{1}{2}} \\
& \quad \times \left[ \frac{1}{b-a} \int_{\Omega} |\alpha(t)|^2 dt - \left| \frac{1}{b-a} \int_{\Omega} \alpha(t) dt \right|^2 \right]^{\frac{1}{2}},
\end{aligned}$$

provided  $\alpha \in L_2([a, b], \mathbb{K})$ ,  $f \in L_2([a, b], H)$  and

$$(3.4) \quad \int_{\Omega} \operatorname{Re} \langle X - f(t), f(t) - x \rangle dt \geq 0$$

or, equivalently,

$$(3.5) \quad \int_{\Omega} \left\| f(t) - \frac{X+x}{2} \right\|^2 dt \leq \frac{1}{4} \|X - x\|^2.$$

We observe that, in practical applications the condition (3.4) and (3.5) may be replaced with the more convenient sufficient conditions

$$(3.6) \quad \operatorname{Re} \langle X - f(t), f(t) - x \rangle \geq 0 \quad \text{for a.e. } t \in [a, b],$$

or, equivalently,

$$(3.7) \quad \left\| f(t) - \frac{X+x}{2} \right\| \leq \frac{1}{2} \|X - x\| \quad \text{for a.e. } t \in [a, b].$$

#### 4. INEQUALITIES FOR THE FINITE FOURIER TRANSFORM

Let  $(H; \langle \cdot, \cdot \rangle)$  be a real or complex Hilbert space and  $g : [a, b] \rightarrow H$  be a Bochner integrable function on  $[a, b]$ . Define its *finite Fourier transform* by

$$(4.1) \quad \mathcal{F}(g)(t) := \int_a^b e^{-2\pi its} g(s) ds.$$

We also consider the *exponential mean* of two complex numbers (see also [6])

$$E(z, w) := \begin{cases} \frac{e^z - e^w}{z - w} & \text{if } z \neq w \\ \exp(w) & \text{if } z = w \end{cases}, \quad z, w \in \mathbb{C}.$$

The following result may be stated.

**Theorem 7.** *Assume that  $f \in L_2([a, b], H)$  satisfies either (3.4) or, equivalently, (3.5). Then we have the inequality*

$$(4.2) \quad \begin{aligned} & \left\| \mathcal{F}(f)(t) - E(-2\pi ita, -2\pi itb) \int_a^b f(s) ds \right\| \\ & \leq \frac{1}{2} \|X - x\| \left[ 1 - \frac{\sin^2[\pi t(b-a)]}{(b-a)^2 \pi^2 t^2} \right]^{\frac{1}{2}} (b-a) \\ & \leq \frac{b-a}{2} \|X - x\| \end{aligned}$$

for each  $t \in [a, b]$  ( $t \neq 0$ ).

*Proof.* We apply the pre-Grüss inequality (3.3) to get

$$(4.3) \quad \begin{aligned} & \left\| \frac{1}{b-a} \int_a^b e^{-2\pi its} f(s) ds - \frac{1}{b-a} \int_a^b e^{-2\pi its} ds \cdot \frac{1}{b-a} \int_a^b f(s) ds \right\| \\ & \leq \frac{1}{2} \|X - x\| \left[ \frac{1}{b-a} \int_a^b |e^{-2\pi its}|^2 ds - \left| \frac{1}{b-a} \int_a^b e^{-2\pi its} ds \right|^2 \right]^{\frac{1}{2}}. \end{aligned}$$

However,

$$\begin{aligned} \int_a^b e^{-2\pi its} ds &= (b-a) E(-2\pi ita, -2\pi itb), \\ |e^{-2\pi its}|^2 &= 1, \\ \int_a^b e^{2\pi its} ds &= \frac{e^{2\pi itb} - e^{2\pi ita}}{2\pi it}, \end{aligned}$$

and

$$\begin{aligned} \left| \int_a^b e^{-2\pi its} ds \right|^2 &= \left| \int_a^b e^{2\pi its} ds \right|^2 \\ &= \frac{1}{4\pi^2 t^2} \left[ |e^{2\pi itb}|^2 - 2 \operatorname{Re} [e^{2\pi itb} \cdot e^{-2\pi ita}] + |e^{2\pi ita}|^2 \right] \\ &= \frac{1}{2\pi^2 t^2} [1 - \cos[2\pi t(b-a)]] \\ &= \frac{\sin^2[\pi t(b-a)]}{\pi^2 t^2}. \end{aligned}$$

Utilising (4.3), we deduce the desired inequality (4.2). ■

**Remark 5.** *The above inequality (4.2) extends for vector-valued functions the corresponding result from [6].*

From Theorem 5 for  $\Omega = [a, b]$  and  $\rho(t) = \frac{1}{b-a}$ , we may deduce the following inequality that will be utilised in Theorem 8 to point out another type of inequality for Fourier transforms:

$$\begin{aligned}
(4.4) \quad & \left| \frac{1}{b-a} \int_a^b \langle f(t)g(t) \rangle dt - \left\langle \frac{1}{b-a} \int_a^b f(t) dt, \frac{1}{b-a} \int_a^b g(t) dt \right\rangle \right| \\
& \leq \left( \frac{1}{4} \|X - x\|^2 - \frac{1}{b-a} \int_a^b \operatorname{Re} \langle X - f(t), f(t) - x \rangle dt \right)^{\frac{1}{2}} \\
& \quad \times \left( \frac{1}{b-a} \int_a^b \|g(t)\|^2 dt - \left\| \frac{1}{b-a} \int_a^b g(t) dt \right\|^2 \right)^{\frac{1}{2}} \\
& \leq \frac{1}{2} \|X - x\| \left[ \frac{1}{b-a} \int_a^b \|g(t)\|^2 dt - \left\| \frac{1}{b-a} \int_a^b g(t) dt \right\|^2 \right]^{\frac{1}{2}},
\end{aligned}$$

provided (3.4) or (3.5) hold true.

In the following we use the notation  $\langle f, g \rangle$  for the function  $\ell : [a, b] \rightarrow \mathbb{K}$ ,  $\ell(t) := \langle f(t), g(t) \rangle$ ,  $t \in [a, b]$ , where  $f, g \in L_2([a, b], H)$ .

The following result may be stated as well.

**Theorem 8.** *Let  $f, h \in L_2([a, b], H)$ . If  $f$  satisfies either (3.4) or, equivalently, (3.5), then we have the inequality:*

$$\begin{aligned}
(4.5) \quad & \left\| \mathcal{F}(\langle f, h \rangle)(t) - \left\langle \frac{1}{b-a} \int_a^b f(s) ds, \tilde{\mathcal{F}}(h)(t) \right\rangle \right\| \\
& \leq \frac{1}{2} \|X - x\| \left[ \frac{1}{b-a} \int_{\Omega} \|h(s)\|^2 ds - \left\| \frac{1}{b-a} \tilde{\mathcal{F}}(h)(t) \right\|^2 \right]^{\frac{1}{2}} (b-a),
\end{aligned}$$

for any  $t \in [a, b]$ , where

$$\tilde{\mathcal{F}}(h)(t) := \int_a^b e^{2\pi its} h(s) ds,$$

is the inverse Fourier transform.

*Proof.* If we apply the inequality (4.4) to  $g(s) = e^{2\pi its} h(s)$ ,  $t \in [a, b]$ , then we get

$$\begin{aligned}
& \left| \frac{1}{b-a} \int_a^b e^{-2\pi its} \langle f(s)h(s) \rangle ds - \left\langle \frac{1}{b-a} \int_a^b f(s) ds, \frac{1}{b-a} \int_a^b e^{2\pi its} h(s) ds \right\rangle \right| \\
& \leq \frac{1}{2} \|X - x\| \left[ \frac{1}{b-a} \int_{\Omega} \|h(s)\|^2 ds - \left\| \frac{1}{b-a} \int_a^b e^{2\pi its} h(s) ds \right\|^2 \right]^{\frac{1}{2}},
\end{aligned}$$

which is obviously equivalent to (4.5). ■

## 5. INEQUALITIES FOR PARTICULAR VECTOR-VALUED FUNCTIONS

Let  $H$  be a real or complex Hilbert space and  $\mathcal{L}(H)$  be the linear space of all linear and bounded operators acting on  $H$ . The norms of vectors in  $H$  and of operators in  $\mathcal{L}(H)$  will be denoted by  $\|\cdot\|$ .

1. Choose  $\Omega = [0, 1]$ ,  $\rho(t) = 1$  and  $f(t) = e^{tA}z$  for  $t \in \Omega$ ,  $A$  being an invertible bounded linear operator acting on  $H$  and  $z$  be fixed in  $H$ . Since, for each  $t \in [0, 1]$  one has

$$\|e^{tA}z\| \leq e^{t\|A\|}\|z\| \leq e^{\|A\|}\|z\|,$$

then it follows that

$$\|e^{tA}z - \frac{1}{2}e^{\|A\|}z\| \leq e^{t\|A\|}\|z\| + \frac{1}{2}e^{\|A\|}\|z\| \leq \frac{3}{2}e^{\|A\|}\|z\|.$$

Let  $X := 2e^{\|A\|}z$  and  $x := -e^{\|A\|}z$ . An application of inequalities (2.3) gives:

$$(5.1) \quad 0 \leq \int_0^1 \|e^{tA}z\|^2 dt - \left\| \int_0^1 e^{tA}z dt \right\|^2 \leq \frac{9}{4}e^{2\|A\|}\|z\|^2.$$

On the other hand

$$\int_0^1 e^{tA}z dt = A^{-1}(e^A - I)z,$$

and so in view of (5.1) we get

$$\begin{aligned} \|A^{-1}(e^A - I)z\|^2 &\leq \int_0^1 \|e^{tA}z\|^2 dt \\ &\leq \|A^{-1}(e^A - I)\|^2\|z\|^2 + \frac{9}{4}e^{2\|A\|}\|z\|^2, \end{aligned}$$

and, moreover:

$$\sup_{\|z\| \leq 1} \int_0^1 \|e^{tA}x\|^2 dt \leq \|A^{-1}(e^A - I)\|^2 + \frac{9}{4}e^{2\|A\|}.$$

2. Let  $\Omega$ ,  $\rho$  and  $z$  be as above. Consider  $f(t) = e^{(1-t)B}(B - A)e^{tB}z$  for each  $t \in \Omega$ , where  $A$  and  $B$  belong to  $\mathcal{L}(H)$ . After a simple calculation, [1], we obtain:

$$\int_0^1 f(t) dt = \frac{1}{2} [e^B - e^A] z.$$

On the other hand it is clear that  $\|f(t)\| \leq g(t)$  where

$$g(t) := e^{(1-t)\|B\|}\|B - A\|e^{t\|A\|}\|z\|.$$

Consider here only the case when  $\|A\| \geq \|B\|$ . In this case the map  $g$  is non-decreasing and so

$$\|f(t)\| \leq g(1) = \|B - A\|e^{\|A\|}\|z\|$$

for all  $t \in \Omega$ . The inequality (2.1) holds for

$$X := 2\|B - A\|e^{\|A\|}z \text{ and } x := -\|B - A\|e^{\|A\|}z.$$

From the inequality (2.3) it then follows that

$$\begin{aligned} \frac{1}{4}\|(e^B - e^A)z\|^2 &\leq \int_0^1 \|e^{(1-t)B}(B - A)e^{tA}z\|^2 dt \\ &\leq \frac{1}{4}[\|e^B - e^A\|^2 + 9\|B - A\|^2 e^{2\|A\|}]\|z\|^2. \end{aligned}$$

In particular, for  $B = I + A$  and  $\|A\| \geq \|I + A\|$  we get:

$$(e^2 + 2e - 3)\|e^A z\|^2 \leq 9e^{2\|A\|}\|z\|^2,$$

or equivalently

$$\sqrt{e^2 + 2e - 3} \|e^A\| \leq 3e^{\|A\|}.$$

3. Let  $\mathbf{T} = \{T(t)\}_{t \in \mathbb{R}}$  be a strongly continuous group of linear and bounded operators acting on a Hilbert space  $H$  and let  $A : D(A) \subset H \rightarrow H$  be its infinitesimal generator. We suppose that  $\mathbf{T}$  is exponentially stable, that is, there exist the positive constant  $N$  and  $\nu$  such that  $\|T(t)z\| \leq Ne^{-\nu|t|}\|z\|$  for all  $t \in \mathbb{R}$ . Then it is well-known that  $A$  has an inverse in  $\mathcal{L}(H)$ . Consider  $\Omega = \mathbb{R}$ ,  $\rho(t) := \nu e^{-\nu|t|}$  and  $f(t) := e^{\nu|t|}T(t)z$  for a fixed  $z \in D(A)$  and  $t \in \mathbb{R}$ . An application of the inequality (2.3) for  $X := 2Nz$  and  $x := -Nz$  gives the inequality:

$$0 \leq \int_{-\infty}^{\infty} e^{\nu|t|} \|T(t)z\|^2 dt \leq \frac{9N^2}{4\nu} \|z\|^2,$$

where the fact that

$$\int_{-\infty}^{\infty} T(t)z dt = A^{-1}T(t)z|_{-\infty}^{\infty} = 0$$

has been used. In particular, if  $A$  is a real or complex quadratic  $n$ -dimensional matrix and

$$\nu_0 := \sup\{\Re(\lambda) : \det(\lambda I_n - A) = 0\} < \nu < 0,$$

then there exists a positive constant  $N$  such that

$$\int_{-\infty}^{\infty} e^{\nu|t|} \|e^{tA}\|^2 dt \leq \frac{9N^2}{4\nu}.$$

Here  $n$  is a positive integer,  $I_n$  is the  $n$ -dimensional identity matrix and we can take

$$N := \sup\{e^{-\nu|t|} \|e^{tA}\| : t \in \mathbb{R}\}.$$

4. Let  $\mathbf{S} = \{S(t)\}_{t \geq 0}$  be a strongly continuous semigroup of linear and bounded operators acting on a Hilbert space  $H$  and let  $G : D(G) \subset H \rightarrow H$  be its infinitesimal generator. We suppose that  $\mathbf{S}$  is exponentially stable, that is, there exist the positive constants  $K$  and  $\alpha$  such that  $\|S(t)\| \leq Ke^{-\alpha t}$  for all  $t \geq 0$ . Then it is well-known that  $G$  has an inverse in  $\mathcal{L}(H)$ . Consider  $\Omega = [0, \infty)$ ,  $\rho(t) := \alpha e^{-\alpha t}$  and  $f(t) := e^{\alpha t}S(t)z$  for fixed  $z \in D(G)$  and  $t \geq 0$ . An application of the inequality (2.3) for  $X := 2Kz$  and  $x := -Kz$  yields:

$$\|G^{-1}z\|^2 \leq \frac{1}{\alpha} \int_0^{\infty} e^{\alpha t} \|S(t)z\|^2 dt \leq \|G^{-1}z\|^2 + \frac{9N^2}{4\alpha^2}.$$

5. A densely defined linear operator  $A$  on a Hilbert space  $H$  is said to be sectorial if  $(0, \infty)$  resides in the resolvent set of  $A$  and there exists  $M > 0$  such that

$$(t+1)\|R(t, A)\| \leq M \text{ for all } t > 0,$$

where  $R(t, A) := (tI - A)^{-1}$  is the resolvent operator of  $A$ .

Consider  $\Omega := [0, \infty)$ ,  $\rho(t) := (t+1)^{-2}$  and  $f(t) = (t+1)^2 R(t, A)^2 z$  for a fixed  $z \in H$  and every  $t \geq 0$ . In order to find suitable  $X$  and  $x$  we remark that:

$$\|f(t)\| \leq (t+1)^2 \|R(t, A)\| \|R(t, A)z\| \leq M^2 \|z\|.$$

An application of the inequality (2.3) for  $X := 2M^2z$  and  $x := -M^2z$  yields:

$$\|A^{-1}z\|^2 \leq \int_0^\infty (t+1)^2 \|R(t, A)^2 z\|^2 dt \leq \|A^{-1}z\|^2 + \frac{9}{4} M^4 \|z\|^2,$$

where the identity

$$\int_0^\infty \rho(t) f(t) dt = -R(t, A)z \Big|_0^\infty = A^{-1}z$$

was used.

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