AGM, Ky Fan and Alzer's Inequalities via Binomial Series

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Abstract

In this article, using binomial series, we get some interesting recursive identities concerning AGM, Ky Fan and Alzer's inequalities, from which, all of these are handled by induction at once.

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Throughout this article, let $\lambda_1, \lambda_2, \cdots \lambda_n > 0$ with $\sum_{i=1}^n \lambda_i = 1$, and A_n and G_n be the arithmetic and geometric means of $x_1, \cdots, x_n > 0$ respectively, i.e.

$$A_n = \sum_{i=1}^n \lambda_i x_i, \qquad G_n = \prod_{i=1}^n x_i^{\lambda_i}. \tag{1}$$

Also, if $x_i \in (0, 1/2]$, we denote by A'_n and G'_n, the arithmetic and geometric means of $1-x_1, \cdots, 1-x_n$ respectively, i.e.

$$A'_n = \sum_{i=1}^n \lambda_i (1-x_i), \qquad G'_n = \prod_{i=1}^n (1-x_i)^{\lambda_i}. \tag{2}$$

There are three important inequalities concerning these means:
AGM inequality [4]:

$$G_n \le A_n, \tag{3}$$

Ky Fan's inequality [3,1]:

$$\frac{A'_n}{G'_n} \le \frac{A_n}{G_n}, \tag{4}$$

Alzer's inequality [2,1]:

$$A'_n - G'_n \leq A_n - G_n. \tag{5}$$

Equality holds in each of them if and only if $x_1 = \cdots = x_n$.
In this article, first using binomial series, we get some recursive identities relating these means, and then using these identities, we prove (3-5) by induction on n. This shows the power of binomial expansion from one hand and the close relations between these inequalities from the other. All we need is the following trivial lemma:

Lemma 1. *If $a, b > 0$ and $|\frac{a}{b} - 1| < 1$, then for each real λ*

$$(1-\lambda)a + \lambda b = a^{1-\lambda}b^{\lambda} + \sum_{k=2}^{\infty}(-1)^{k-1}\binom{1-\lambda}{k}(b-a)^k b^{1-k}. \tag{6}$$

Proof. Using binomial series [5], we have

$$\left(\frac{a}{b}\right)^{1-\lambda} = 1 + \sum_{k=1}^{\infty}\binom{1-\lambda}{k}\left(\frac{a}{b}-1\right)^k,$$

which by multiplying each side by b we get (6).

Corollary 2.

(*i*) If $x_n \geq A_{n-1}$, then

$$A_n - G_n = \left(A_{n-1}^{1-\lambda_n} - G_{n-1}^{1-\lambda_n}\right)x_n^{\lambda_n} + \sum_{k=2}^{\infty}(-1)^{k-1}\binom{1-\lambda_n}{k}(x_n - A_{n-1})^k x_n^{1-k}, \tag{7}$$

and

$$\frac{A_n}{G_n} = \left(\frac{A_{n-1}}{G_{n-1}}\right)^{1-\lambda_n} + \frac{1}{G_n}\sum_{k=2}^{\infty}(-1)^{k-1}\binom{1-\lambda_n}{k}(x_n - A_{n-1})^k x_n^{1-k}, \tag{8}$$

where $A_{n-1} = \sum_{i=1}^{n-1}\frac{\lambda_i}{1-\lambda_n}x_i$ and $G_{n-1} = \prod_{i=1}^{n-1}x_i^{\frac{\lambda_i}{1-\lambda_n}}$.

(*ii*) If $x_i \in (0, 1/2]$, then

$$A'_n - G'_n = ({A'}_{n-1}^{1-\lambda_n} - {G'}_{n-1}^{1-\lambda_n})(1-x_n)^{\lambda_n} + \sum_{k=2}^{\infty}(-1)^{k-1}\binom{1-\lambda_n}{k}(A_{n-1} - x_n)^k(1-x_n)^{1-k}, \tag{9}$$

and

$$\frac{A'_n}{G'_n} = \left(\frac{A'_{n-1}}{G'_{n-1}}\right)^{1-\lambda_n} + \frac{1}{G'_n}\sum_{k=2}^{\infty}(-1)^{k-1}\binom{1-\lambda_n}{k}(A_{n-1} - x_n)^k(1-x_n)^{1-k}, \tag{10}$$

where $A'_{n-1} = \sum_{i=1}^{n-1}\frac{\lambda_i}{1-\lambda_n}(1-x_i)$ and $G'_{n-1} = \prod_{i=1}^{n-1}(1-x_i)^{\frac{\lambda_i}{1-\lambda_n}}$.

Proof.

(*i*) Since $\left|\frac{A_{n-1}}{x_n} - 1\right| < 1$, (7) follows from (6) by taking $a = A_{n-1}, b = x_n$, and $\lambda = \lambda_n$ in the preceding lemma and considering $G_n = G_{n-1}^{1-\lambda_n}x_n^{\lambda_n}$. Now, (8) follows from (7) by simply dividing each side of it by G_n.

(*ii*) Since $\left|\frac{A'_{n-1}}{1-x_n}-1\right|<1$, (9) and (10) follow similarly by taking $a=A'_{n-1}, b=1-x_n$, and $\lambda=\lambda_n$ in the preceding lemma.

Now we prove (3-5) by induction on n. If $n=1$, there is nothing to prove. Suppose $n>1$ and the assertions hold for $n-1$. If $x_1=\cdots=x_n$, obviously equality holds in (3-5). Let not all x_i's are equal. By an rearrangement if necessary we can suppose that $x_n=\max\limits_{1\le i\le n} x_i$. Now since $x_n-A_{n-1}>0$ and $(-1)^{k-1}\binom{1-\lambda_n}{k}>0$ $(k\ge 2)$, (3) and (4) follow from (7), (8) and (10) with strict inequalities.
Finally, using the mean value theorem, we have

$$\begin{aligned}\left(A_{n-1}^{1-\lambda_n}-G_{n-1}^{1-\lambda_n}\right)x_n^{\lambda_n} &= (1-\lambda_n)\left(\frac{x_n}{\theta_{n-1}}\right)^{\lambda_n}(A_{n-1}-G_{n-1})\\ &\ge (1-\lambda_n)\left(\frac{1-x_n}{\theta'_{n-1}}\right)^{\lambda_n}({A'}_{n-1}-{G'}_{n-1})\\ &= \left({A'}_{n-1}^{1-\lambda_n}-{G'}_{n-1}^{1-\lambda_n}\right)(1-x_n)^{\lambda_n},\end{aligned}$$

where $\theta_{n-1}\in[G_{n-1},A_{n-1}]$ and $\theta'_{n-1}\in[G'_{n-1},A'_{n-1}]$; since $x_n\ge A_{n-1}\ge\theta_{n-1}$ and $1-x_n\le G'_{n-1}\le\theta'_{n-1}$. Now comparing (7) and (9) we get strict inequality in (5), and the proof is complete.

REFERENCES

1. H. Alzer, The inequality of Ky Fan and related results, *Acta Appl. Math.*, **38** (1995), 305-354.
2. H. Alzer, Ungleichungen für geometrische und arithmetische Mittelwerte, *Proc. Kon. Nederl. Akad. Wetensch.*, **91** (1988), 365-374.
3. E.F. Beckenbach and R. Bellman, *Inequalities*, Springer-Verlag, Berlin, 1961.
4. P.S. Bullen, D. S. Mitrinović, and P. M. Vasić, *Means and Their Inequalities*, Reidel, Dordrecht, 1988.
5. E. Hewitt and K. Stromberg, *Real and abstract analysis*, Springer-Verlag, New York, 1975.