A MONOTONICITY RESULT OF A FUNCTION INVOLVING THE EXPONENTIAL FUNCTION AND AN APPLICATION

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Abstract. Let $x>0$, then $\frac{1}{x^2}-\frac{e^{-x}}{(1-e^{-x})^2}$ is strictly decreasing. This result can be applied to solve the 69th problem in [2, p. 295] and [3, p. 217].

In [2, pp. 702–708], the author collected 152 unsolved problems on inequalities. The 69th problem [2, p. 295 and p. 704] states: What is the best possible constant c such that the inequality

$$\frac{1}{x^2}-c<\frac{e^{-x}}{(1-e^{-x})^2}<\frac{1}{x^2} \tag{1}$$

is valid for all real $x\in(0,1)$?

This problem originated from [3, p. 217] maybe.

In [4], it is proved that the best constant c in (1) is $\frac{1}{12}$.

In [1], it is proved that inequality (1) holds in the interval $(0,\infty)$ if and only if $c\geq\frac{1}{12}$.

In the following, we shall present a general result.

Theorem 1. *The function*

$$f(x)=\frac{1}{x^2}-\frac{e^{-x}}{(1-e^{-x})^2} \tag{2}$$

is strictly decreasing in $(0,\infty)$.

Proof. Straightforward computing yields

$$\begin{aligned} f'(x)&=\frac{2-2e^{3x}+(x^3-6)e^x+(x^3+6)e^{2x}}{x^3(e^x-1)}\\ &\triangleq\frac{g(x)}{x^3(e^x-1)}, \end{aligned} \tag{3}$$

$$\begin{aligned} g'(x)&=[(2x^3+3x^2+12)e^x-6e^{2x}+x^3+3x^2-6]e^x\\ &\triangleq e^x h(x), \end{aligned} \tag{4}$$

$$h'(x)=(12+6x+9x^2+2x^3)e^x-12e^{2x}+3x(2+x), \tag{5}$$

$$h''(x)=(18+24x+15x^2+2x^3)e^x-24e^{2x}+6(1+x), \tag{6}$$

$$h'''(x)=(42+54x+21x^2+2x^3)e^x-48e^{2x}+6, \tag{7}$$

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$$
\begin{aligned}
h^{(4)}(x) &= (96 - 96e^x + 96x + 27x^2 + 2x^3)e^x \\
&\triangleq e^x \phi(x),
\end{aligned}
\tag{8}
$$

$$
\phi'(x) = 6(16 - 16e^x + 9x + x^2), \tag{9}
$$

$$
\phi''(x) = 54 - 96e^x + 12x, \tag{10}
$$

$$
\phi'''(x) = 12 - 96e^x, \tag{11}
$$

and

$$
\begin{aligned}
\phi''(0) &= -42, & \phi'(0) &= 0, & \phi(0) &= 0, \\
h^{(4)}(0) &= 0, & h'''(0) &= 0, & h''(0) &= 0, \\
h'(0) &= 0, & h(0) &= 0, & g'(0) &= 0.
\end{aligned}
\tag{12}
$$

It is clear that $\phi'''(x) < 0$ in $(0, \infty)$, then $\phi''(x)$ is decreasing, $\phi''(x) < 0$, $\phi'(x)$ is decreasing, $\phi'(x) < 0$, $\phi(x)$ is decreasing, $\phi(x) < 0$, $h^{(4)}(x) < 0$, $h'''(x)$ is decreasing, $h'''(x) < 0$, $h''(x)$ is decreasing, $h''(x) < 0$, $h'(x)$ is decreasing, $h'(x) < 0$, $h(x)$ is decreasing, $h(x) < 0$, $g'(x) < 0$, $g(x)$ is decreasing. Since $g(0) = 0$, $g(x) < 0$ which is equivalent to $f'(x) < 0$ in $(0, \infty)$. Hence the function $f(x)$ is strictly decreasing in $(0, \infty)$. The proof is complete. □

Remark 1. Using the power series expansion of e^x at $x = 0$, we can expand the function $g(x)$ defined in (3) at $x = 0$ into a power series as $g(x) = \sum_{i=7}^{\infty} a_i x^i$ with $a_i < 0$ for $i \geq 7$. This means $g(x) < 0$, and then $f'(x) < 0$ in $(0, \infty)$. Hence $f(x)$ is strictly decreasing in $(0, \infty)$.

As an application of Theorem 1, we have

Corollary 1. *In the interval* $(0, 1)$*, we have*

$$
\frac{1}{x^2} - \frac{1}{12} < \frac{e^{-x}}{(1 - e^{-x})^2} < \frac{1}{x^2} - \frac{e^2 - 3e + 1}{(e - 1)^2}. \tag{13}
$$

The constants $\frac{1}{12}$ *and* $\frac{e^2 - 3e + 1}{(e-1)^2}$ *in* (13) *are the best possible.*

On the whole real line,

$$
\frac{1}{x^2} - \frac{1}{12} < \frac{e^{-x}}{(1 - e^{-x})^2} < \frac{1}{x^2}. \tag{14}
$$

The constant $\frac{1}{12}$ *is also the best possible.*

Proof. Using the power series expansion of e^x at $x = 0$ and direct computing gives

$$
\lim_{x \to 0^+} \left[\frac{1}{x^2} - \frac{e^{-x}}{(1 - e^{-x})^2} \right] = \lim_{x \to 0^+} \frac{\frac{x^4}{12} + o(x^4)}{x^4 + o(x^4)} = \frac{1}{12}. \tag{15}
$$

Inequality (13) follows readily from Theorem 1 and $f(1) = \frac{e^2 - 3e + 1}{(e-1)^2}$.

Inequality (14) follows from Theorem 1 and $\lim_{x \to \infty} f(x) = 0$ easily. □

Remark 2. In the final, it is natural to pose the following open problem: Find the range of α such that the function

$$
\frac{1}{x^\alpha} - \frac{e^{-x}}{(1 - e^{-x})^2} \tag{16}
$$

is monotonic in $(0, \infty)$.

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