SOME IMPROVED RESULTS ON THE EULER'S INEQUALITY

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Abstract. In this paper, we establish an extension of Euler's inequality relating two triangles, which imply an interesting sharpness of Euler's inequality, some results given by Z.-H. Zhang et al in [4] are improved. In addition, we present the sharpness and reverse of Euler's inequality in different forms.

1. Introduction

In what follows, for a given triangle ABC denote by A, B, C the measures of its angles, a, b, c the lengths of its sides, and let R, r and s denote respectively the circumradius, the inradius and the semi-perimeter of the triangle ABC. Similarly define the triangle $A'B'C'$.

The so-called Euler's inequality is one of the oldest geometric inequalities, it was presented by L. Euler in 1765 [1], as follows

$$R \geq 2r, \tag{1.1}$$

with the equality holds if and only if triangle is equilateral.

Euler's inequality has stimulated the interest of many researchers, so far there are a number of papers have been written on its generalizations and applications (see [2],[3]). Recently, Z.-H. Zhang et al [4] establish following inequality relating two triangles as an extension of Euler's inequality

$$\frac{R}{r'} \geq \frac{2}{3}\left(\frac{a}{a'} + \frac{b}{b'} + \frac{c}{c'}\right). \tag{1.2}$$

As a special case, an interesting sharpness of Euler's inequalities was derived from (1.2), i.e.

$$\frac{R}{r} \geq \frac{2}{3}\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right). \tag{1.3}$$

In this note we shall improve the Euler's inequality in different forms, the further sharpness of inequalities (1.2) and (1.3) will be discussed firstly in the next section.

2. Extension and Sharpness of the Euler's Inequality

Theorem 2.1. *For any triangle ABC and triangle $A'B'C'$ we have the following inequality*

$$R\left(\frac{1}{r'} + \frac{1}{R'}\right) \geq \frac{a}{a'} + \frac{b}{b'} + \frac{c}{c'}\ , \tag{2.1}$$

with the equality holds if and only if triangle ABC and triangle $A'B'C'$ are equilateral.

In order to prove Theorem 2.1 we need following Lemmas

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Lemma 2.1. (Klamkin's inequality [5]) *Let x, y, z be positive real numbers, and A, B, C denote the angles of triangle. Then*

$$x\sin A + y\sin B + z\sin C \le \frac{1}{2}(xy+yz+zx)\sqrt{\frac{x+y+z}{xyz}}, \tag{2.2}$$

with the equality holds if and only if $x=y=z$ and the triangle is equilateral.

Lemma 2.2. (Gerretsen's inequality [6]) *In every triangle ABC we have the following inequality*

$$16Rr-5r^2 \le s^2 \le 4R^2+4Rr+3r^2, \tag{2.3}$$

with the equality holds if and only if triangle ABC is equilateral.

Now we prove the Theorem 2.1

Proof. Substituting $x=\frac{1}{\sin A'}$, $y=\frac{1}{\sin B'}$, $z=\frac{1}{\sin C'}$ into (2.2) and using the law of sines, we get that

$$\frac{a}{a'}+\frac{b}{b'}+\frac{c}{c'} \le R\left(\frac{1}{a'b'}+\frac{1}{b'c'}+\frac{1}{c'a'}\right)\sqrt{a'b'+b'c'+c'a'}. \tag{2.4}$$

On the other hand, by using the identity of triangle

$$\frac{1}{a'b'}+\frac{1}{b'c'}+\frac{1}{c'a'}=\frac{1}{2R'r'}, \quad a'b'+b'c'+c'a'=s'^2+4R'r'+r'^2, \tag{2.5}$$

and Gerretsen's inequality $s'^2 \le 4R'^2+4R'r'+3r'^2$, we obtain that

$$\left(\frac{1}{a'b'}+\frac{1}{b'c'}+\frac{1}{c'a'}\right)\sqrt{a'b'+b'c'+c'a'} \le \frac{1}{r'}+\frac{1}{R'}. \tag{2.6}$$

Combining (2.4) and (2.6) we deduce inequality (2.1) immediately.

The conditions of equality for (2.2), (2.3) show that the equality in (2.1) holds if and only if triangle ABC and triangle $A'B'C'$ are equilateral. The proof of Theorem 2.1 is complete. □

Remark 2.1. *Euler's inequality $R' \ge 2r'$ reveal that*

$$R\left(\frac{1}{r'}+\frac{1}{R'}\right) \le \frac{3}{2}\left(\frac{R}{r'}\right). \tag{2.7}$$

This means that inequality (2.1) has sharpened the inequality (1.2).

In particular let $a'=b$, $b'=c$, $c'=a$ in Theorem 2.1, we obtain a valuable sharpness of inequality (1.3)

Theorem 2.2. *In every triangle ABC we have the following inequality*

$$\frac{R}{r} \ge \frac{a}{b}+\frac{b}{c}+\frac{c}{a}-1, \tag{2.8}$$

with the equality holds if and only if triangle ABC is equilateral.

Remark 2.2. *From the obvious inequality*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-1 \ge \frac{2}{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right), \tag{2.9}$$

it is easy to see that inequality (2.8) has sharpened the inequality (1.3). Actually we can prove that the parameter $\mu=1$ is best possible for a class of inequalities as follows

$$\frac{R}{r} \ge \mu\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+2-3\mu. \tag{2.10}$$

Since (2.10) is equivalent to the following inequality

$$\frac{2abc}{(a+b-c)(b+c-a)(c+a-b)} \geq \mu\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+2-3\mu. \tag{2.11}$$

Substituting $a=b=1$, $c=\varepsilon$ *into (2.11), and then let* $\varepsilon \to 0$, *we deduce* $\mu \leq 1$, *therefore constant* $\mu = 1$ *is best possible in the sense that it cannot be replaced by a larger constant.*

The next improvement of Euler's inequality will involve exponential parameter.

Theorem 2.3. *Let* λ *be real numbers and* $\lambda \geq 2$. *Then for any triangle* ABC *we have the following inequality*

$$R^{\lambda} \geq (2r)^{\lambda} + \left|\sqrt{2}(a-b)(b-c)(c-a)\right|^{\frac{\lambda}{3}}, \tag{2.12}$$

with the equality holds if and only if triangle ABC *is equilateral.*

Proof. From the identity of triangle

$$(a-b)^2(b-c)^2(c-a)^2 = -4r^2(s^2-2R^2-10Rr+r^2)^2+16r^2R(R-2r)^3, \tag{2.13}$$

we get that

$$|\,(a-b)(b-c)(c-a)\,| \leq 4r(R-2r)\sqrt{R(R-2r)},$$

that is

$$R-2r \geq (4r\sqrt{R})^{-\frac{2}{3}}\,|\,(a-b)(b-c)(c-a)\,|^{\frac{2}{3}},$$

multiplying both sides of above inequality by $R+2r$, it becomes

$$R^2-4r^2 \geq (R+2r)(4r\sqrt{R})^{-\frac{2}{3}}\,|\,(a-b)(b-c)(c-a)\,|^{\frac{2}{3}},$$

utilizing the arithmetic-geometric mean inequality and Euler's inequality $R \geq 2r$, we deduce that

$$R^2-4r^2 \geq 2\sqrt{2Rr}(4r\sqrt{R})^{-\frac{2}{3}}\,|\,(a-b)(b-c)(c-a)\,|^{\frac{2}{3}}$$

$$= \left(\frac{2R}{r}\right)^{\frac{1}{6}}|\,(a-b)(b-c)(c-a)\,|^{\frac{2}{3}} \geq \left|\sqrt{2}(a-b)(b-c)(c-a)\right|^{\frac{2}{3}},$$

therefore

$$(R^2-4r^2)^{\frac{\lambda}{2}} \geq \left|\sqrt{2}\,(a-b)(b-c)(c-a)\right|^{\frac{\lambda}{3}}. \tag{2.14}$$

On the other hand, by Euler's inequality $2r/R \leq 1$ and $\lambda \geq 2$, it follows that

$$R^{\lambda}-(2r)^{\lambda} = R^{\lambda}\left[1-\left(\frac{2r}{R}\right)^{\lambda}\right] = R^{\lambda}\left[1-\left(\frac{2r}{R}\right)^{2}\left(\frac{2r}{R}\right)^{\lambda-2}\right] \tag{2.15}$$

$$\geq R^{\lambda}\left[1-\left(\frac{2r}{R}\right)^{2}\right] \geq R^{\lambda}\left[1-\left(\frac{2r}{R}\right)^{2}\right]^{\frac{\lambda}{2}} = (R^2-4r^2)^{\frac{\lambda}{2}}.$$

Combining (2.15) and (2.14), we obtain inequality (2.12). The conditions of equality for (2.12) follows from the conditions of equality of Euler's inequality (1.1). The proof of Theorem 2.3 is complete. ∎

3. Sharpness and Reverse of the Euler's Inequality

In this section we shall use the notation of cyclic sum, such as

$$\sum f(a) = f(a) + f(b) + f(c), \quad \sum f(a,b) = f(a,b) + f(b,c) + f(c,a).$$

Base on the well-known inequality [7]

$$\sum \frac{b+c}{a} \geq 4 \sum \frac{a}{b+c} \geq 6, \tag{3.1}$$

we establish the following sharpness and reverse of the Euler's inequality.

Theorem 3.1. *In every triangle ABC, we have the following inequality*

$$\frac{2}{9}\left(\sum \frac{s-a}{a}\right)\left(\sum \frac{a}{s-a}\right) \geq \frac{R}{r} \geq \frac{2}{9}\left(\sum \frac{b+c}{a}\right)\left(\sum \frac{a}{b+c}\right), \tag{3.2}$$

with the equality holds if and only if triangle ABC is equilateral.

Proof. Since

$$\left(\sum \frac{s-a}{a}\right)\left(\sum \frac{a}{s-a}\right) = \left(-3 + s\sum \frac{1}{a}\right)\left(-3 + s\sum \frac{1}{s-a}\right) \tag{3.3}$$

$$= \left(-3 + \frac{s}{abc}\sum bc\right)\left[-3 + \frac{s}{(s-a)(s-b)(s-c)}\left(-s^2 + \sum bc\right)\right],$$

using the identity of triangle

$$abc = 4sRr, \quad (s-a)(s-b)(s-c) = sr^2, \quad \sum bc = s^2 + 4Rr + r^2, \tag{3.4}$$

together with (3.3), we have

$$\left(\sum \frac{s-a}{a}\right)\left(\sum \frac{a}{s-a}\right) = \frac{(s^2 - 8Rr + r^2)(2R - r)}{2Rr^2} \tag{3.5}$$

$$= \frac{9R}{2r} + \frac{(s^2 - 16Rr + 5r^2)(2R - r) + r(R - 2r)(7R - 2r)}{2Rr^2}.$$

By Gerretsen's inequality $s^2 \geq 16Rr - 5r^2$ and Euler's inequality $R \geq 2r$, we obtain the left hand side of inequality in (3.2).

Since

$$\left(\sum \frac{b+c}{a}\right)\left(\sum \frac{a}{b+c}\right) = \left(-3 + 2s\sum \frac{1}{a}\right)\left(-3 + 2s\sum \frac{1}{b+c}\right) \tag{3.6}$$

$$= \left(-3 + \frac{2s}{abc}\sum bc\right)\left[-3 + \frac{2s}{(a+b)(b+c)(c+a)}\left(4s^2 + \sum bc\right)\right].$$

Combining (3.6) and the identity of triangle

$$abc = 4sRr, \quad (a+b)(b+c)(c+a) = 2s(s^2 + 2Rr + r^2), \quad \sum bc = s^2 + 4Rr + r^2, \tag{3.7}$$

we obtain

$$\left(\sum \frac{b+c}{a}\right)\left(\sum \frac{a}{b+c}\right) = \frac{(s^2 - 2Rr + r^2)(s^2 - Rr - r^2)}{Rr(s^2 + 2Rr + r^2)} \tag{3.8}$$

$$= \frac{9R}{2r} + \frac{2s^4 - (9R^2 + 6Rr)s^2 - 18R^3r - 5R^2r^2 + 2Rr^3 - 2r^4}{2Rr(s^2 + 2Rr + r^2)}.$$

Defining

$$f(s^2) = 2s^4 - (9R^2 + 6Rr)s^2 - 18R^3r - 5R^2r^2 + 2Rr^3 - 2r^4.$$

By Euler's inequality $R \geq 2r$, we have

$$\begin{aligned} f(16Rr - 5r^2) &= -162R^3r + 456R^2r^2 - 288Rr^3 + 48r^4 \\ &= -2r(R - 2r)[81R(R - 2r) + 96Rr + 12r^2] \leq 0, \end{aligned}$$

and

$$\begin{aligned} f(4R^2 + 4Rr + 3r^2) &= -4R^4 - 14R^3r + 24R^2r^2 + 32Rr^3 + 16r^4 \\ &= -2(R - 2r)(2R^3 + 11R^2r + 10Rr^2 + 4r^3) \leq 0, \end{aligned}$$

applying Gerretsen's inequality $16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2$ together with the monotonicity of the function f, we infer that $f(s^2) \leq 0$.

Combining (3.8) and $f(s^2) \leq 0$, we deduce the right hand side of inequality in (3.2).

The conditions of equality for the Gerretsen's inequality and Euler's inequality shows that the equality in (3.2) holds if and only if the triangle ABC is equilateral. This complete the proof of Theorem 3.1. □

Remark 3.1. *According to inequality (3.1), it is easy to see that the right hand side of inequality in (3.2) has sharpened the well-known Milisavljević's inequality* [8]

$$\frac{R}{r} \geq \frac{1}{3} \sum \frac{b + c}{a}. \tag{3.9}$$

In the following we shall establish another interesting double inequality, which also is the sharpness of the Euler's inequality.

Theorem 3.2. *In every triangle ABC, we have the following inequality*

$$\frac{1}{16R} \left(\sum |a - b| \right)^2 + 2r \leq R \leq 2r + \frac{1}{16r} \left(\sum |a - b| \right)^2. \tag{3.10}$$

Proof. Since (3.10) is symmetrical on variable a, b, c, without loss of generality we may assume that $a \geq b \geq c$. Then we have

$$\begin{aligned} R - 2r - \frac{1}{16R} \left(\sum |a - b| \right)^2 &= R - 2r - \frac{1}{4R}(a - c)^2 \\ = R\left[1 - \frac{2r}{R} - (\sin A - \sin C)^2 \right] &= R\left[1 - 8 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2} - (\sin A - \sin C)^2 \right] \\ = R\left[1 - 4 \sin\frac{B}{2}\left(\cos\frac{A - C}{2} - \sin\frac{B}{2} \right) - 4 \sin^2\frac{B}{2}\left(1 - \cos^2\frac{A - C}{2}\right)\right] \\ = R\left(1 - 2 \sin\frac{B}{2} \cos\frac{A - C}{2}\right)^2 &= R\,(1 - \cos A - \cos C)^2 \geq 0. \end{aligned}$$

Therefore the left hand side of inequality in (3.10) is proved.

Since

$$\begin{aligned} \left(\sum |a - b| \right)^2 &= \sum (a - b)^2 + \sum |a - b| \left(|b - c| + |c - a| \right) \\ &\geq \sum (a - b)^2 + \sum (a - b)^2 = 4 \sum a^2 - 4 \sum ab, \end{aligned} \tag{3.11}$$

substituting the following identity into (3.11)

$$a^2 + b^2 + c^2 = 2(s^2 - 4Rr - r^2), \quad ab + bc + ca = s^2 + 4Rr + r^2, \tag{3.12}$$

we obtain

$$\left(\sum |a-b|\right)^2 \geq 4\,(s^2 - 12Rr - 3r^2), \tag{3.13}$$

by Gerretsen's inequality $s^2 \geq 16Rr - 5r^2$, it follows that

$$\left(\sum |a-b|\right)^2 \geq 16r(R-2r). \tag{3.14}$$

Clearly inequality (3.14) is equivalent to the right hand side of inequality in (3.10). The proof of Theorem 3.2 is complete. □

Remark 3.2. *From process of above proof , it is easy to see that the equality in the right hand side of inequality (3.10) holds if and only if triangle ABC is equilateral, the equality in the left hand side of inequality (3.10) holds if and only if* $\cos(\max(A,B,C)) + \cos(\min(A,B,C)) = 1$.

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