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A Note On an Open Problem for Integral Inequality

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Abstract. In ([4]:F. Qi, Several integral inequalities, J. Inequal. Pure and Appl. Math. 1(2), (2000), Art. 19.) Feng Qi posed an open problem. In this article, we will give the solution and further extensions of this problem. Reverse inequalities to the posed one are considered.

1. Introduction

In the paper [4], F, Qi proved the following proposition.

Proposition 1. Suppose that $f \in \mathcal{C}^n[a,b]$ satisfies $f^{(i)}(a) \geq 0$ and $f^{(n)} \geq n!$ for $x \in [a,b]$, where $0 \leq i \leq n-1$ and $n \in N$, Then

$$\int_a^b [f(x)]^{n+2} dx \geq (\int_a^b f(x)dx)^{n+1} \tag{1}$$

Next, he proposed the following open problem:

Open problem. Under what conditions does the inequality

$$\int_a^b [f(x)]^t dx \geq (\int_a^b f(x)dx)^{t-1} \tag{2}$$

hold for $t > 1$?

In the joint paper [5], K. W. Yu and F. Qi obtained one answer to the above problem: inequality (2) is valid for all $f \in \mathcal{C}[a,b]$ such that $\int_a^b f(x)dx \geq (b-a)^{t-1}$ for given $t > 1$. Many authors considered different generalizations of open problem cf. [1,2,3,6].

In this note, we are going to give further extension answer to the above problem that is :

Theorem 1. Let $f(x)$ be differentiable on (a,b) and $f(a) = 0$, we have

(i)If $f'(x) \geq 0$ and $t \leq 1$, then

$$\int_a^b [f(x)]^t dx \leq (\int_a^b f(x)dx)^{t-1} \tag{3}$$

(ii) If $f'(x) \geq 0$ and $1 \leq t \leq 2$, then

$$\int_a^b [f(x)]^t dx \geq (\int_a^b f(x)dx)^{t-1} \tag{4}$$

(iii) If $0 \leq f'(x) \leq (t-2)(x-a)^{t-3}$ and $2 \leq t \leq 3$, then

$$\int_a^b [f(x)]^t dx \leq (\int_a^b f(x)dx)^{t-1} \tag{5}$$

(iv) If $f'(x) \geq (t-2)(x-a)^{t-3}$ and $t \geq 3$, then

$$\int_a^b [f(x)]^t dx \geq (\int_a^b f(x)dx)^{t-1} \tag{6}$$

2. Proof of theorem1

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Proof of Theorem 1. For $x \in [a, b]$, Set

$$F(x) = \int_a^x [f(\xi)]^t d\xi - (\int_a^x f(\xi)d\xi)^{t-1} \tag{7}$$

Then $F(a) = 0$ and $F'(x) = f(x)G(x)$, where

$$G(x) = [f(x)]^{t-1} - (t-1)(\int_a^x f(\xi)d\xi)^{t-2} \tag{8}$$

Clearly, $G(a) = 0$ and direct calculation produces

$$G'(x) = (t-1)f(x)H(x) \tag{9}$$

$$H(x) = [f(x)]^{t-3}f'(x) - (t-2)(\int_a^x f(\xi)d\xi)^{t-3} \tag{10}$$

Since $f'(t) \geq 0$ and $f(a) = 0$, thus $f(t)$ is increasing and $f(x) \geq 0$. Also we have

$$f(x)(x-a) \geq \int_a^x f(\xi)d\xi = f(\eta)(x-a) \geq 0 \qquad (a \leq \eta \leq x) \tag{11}$$

(i) When $f'(x) \geq 0$ and $t \leq 1$, we have $H(x) \geq 0$ and $G'(x) \leq 0$, $G(x)$ decreases and $G(x) \leq 0$ because of $G(a) = 0$, hence $F'(x) = f(x)G(x) \leq 0$, $F(x)$ is decreasing. Since $F(a) = 0$, we have $F(x) \leq 0$, and $F(b) \leq 0$. Therefore, the inequality (3) holds.

(ii) When $f'(x) \geq 0$ and $1 \leq t \leq 2$, we have $H(x) \geq 0$ and $G'(x) \geq 0$, $G(x)$ increases, $G(x) \geq 0$, $F'(x) \geq 0$, and $F(x)$ is increasing, then $F(x) \geq 0$ and $F(b) \geq 0$, the inequality (4) follows.

(iii) When $0 \leq f'(x) \leq (t-2)(x-a)^{t-3}$ and $2 \leq t \leq 3$, from inequality (11) we have

$$(f(x)(x-a))^{t-3} - (\int_a^x f(\xi)d\xi)^{t-3} \leq 0 \tag{12}$$

Therefore, we get

$$H(x) \leq (t-2)((f(x)(x-a))^{t-3} - (\int_a^x f(\xi)d\xi)^{t-3}) \leq 0 \tag{13}$$

Thus $G'(x) \leq 0$, $G(x)$ decreases, $G(x) \leq 0$, $F'(x) \leq 0$, and $F(x)$ is decreasing, then $F(x) \leq 0$ and $F(b) \leq 0$, the inequality (5) holds.

(iv) When $f'(x) \geq (t-2)(x-a)^{t-3}$ and $t \geq 3$, from inequality (11) we have

$$(f(x)(x-a))^{t-3} - (\int_a^x f(\xi)d\xi)^{t-3} \geq 0 \tag{14}$$

Therefore, we get

$$H(x) \geq (t-2)((f(x)(x-a))^{t-3} - (\int_a^x f(\xi)d\xi)^{t-3}) \geq 0 \tag{15}$$

Thus $G'(x) \geq 0$, $G(x)$ increases, $G(x) \geq 0$, $F'(x) \geq 0$, and $F(x)$ is increasing, then $F(x) \geq 0$ and $F(b) \geq 0$, the inequality (6) follows.

Proof of Theorem 1 is complete.

In inequality (5), take $t = 3$,$a = 0$ and $b = 1$, then we obtain

Corollary 1.([7], P.624) Let $f(x)$ be a continuous function on the closed interval $[0, 1]$ and $f(0) = 0$, its derivative of the first order is bounded by $0 \leq f'(x) \leq 1$, for $x \in [0, 1]$, then

$$\int_0^1 [f(x)]^3 dx \leq (\int_0^1 f(x)dx)^2 \tag{16}$$

Equality in (16) holds if and only if $f(x) \equiv 0$ or $f(x) = x$.

In inequality (5) and (6), take $t = 3$, then we have

Corollary 2.([4]) Let $f(x)$ be differentiable on (a, b) and $f(a) = 0$.If $0 \leq f'(x) \leq 1$ then

$$\int_a^b [f(x)]^3 dx \leq (\int_a^b f(x)dx)^2 \tag{17}$$

If $f'(x) \geq 1$, then inequality (17) reverses. The equality in (17) holds if and only if $f(x) \equiv 0$ or $f(x) = x - a$.

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