Some New Inequalities Between Important Means and Applications to Ky Fan Types Inequalities

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Abstract

In this paper, mainly using the convexity of the function $\frac{a^x-b^x}{c^x-d^x}$ and convexity or concavity of the function $\ln\frac{a^x-b^x}{c^x-d^x}$ on the real line, where $a>b\geq c>d>0$ are fixed real numbers, we obtain some important relations between various important means of these numbers. Also, we apply the obtained results to Ky Fan type inequalities and get some new refinements.

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1 Introduction and Motivation

Suppose that $a>b\geq c>d>0$. It is shown in [6] that the function

$$f(x)=\frac{a^x-b^x}{c^x-d^x} \qquad (-\infty<x<+\infty),$$

is strictly increasing on the real line, moreover $\lim_{x\to+\infty} f(x)=+\infty$ and $\lim_{x\to-\infty} f(x)=0$. By a simple calculation, we have

$$\begin{aligned}
f(x) &= \begin{cases} \frac{a-b}{c-d}\frac{L(c,d)}{L(a,b)} & x=0, \\ \frac{a-b}{c-d} & x=1, \\ \frac{a-b}{c-d}\left(\frac{L_{x-1}(a,b)}{L_{x-1}(c,d)}\right)^{x-1} & x\neq 0,1, \end{cases} \\
\frac{f'(x)}{f(x)} &= \frac{1}{x}\ln\frac{I(a^x,b^x)}{I(c^x,d^x)} \qquad (x\neq 0), \qquad (1)\\
f'(0) &= \frac{a-b}{c-d}\frac{L(c,d)}{L(a,b)}\ln\frac{G(a,b)}{G(c,d)}.
\end{aligned}$$

Note that the notations $L(a,b)$, $L_p(a,b)$ and $G(a,b)$ are well-known means between $a,b>0$. We recall them in the following table:

Name	Notation	Definition
arithmetic mean	$A(a,b)$	$\frac{a+b}{2}$
geometric mean	$G(a,b)$	$\sqrt{ab}$
harmonic mean	$H(a,b)$	$\frac{2}{\frac{1}{a}+\frac{1}{b}}$
logarithmic mean	$L(a,b)$	$\begin{cases} a & a=b \\ \frac{a-b}{\ln a-\ln b} & a\neq b \end{cases}$
identric mean	$I(a,b)$	$\begin{cases} a & a=b \\ \frac{1}{e}\left(\frac{a^a}{b^b}\right)^{\frac{1}{a-b}} & a\neq b \end{cases}$
$p-$logarithmic mean	$L_p(a,b)$	$\begin{cases} a & a=b \\ \left(\frac{a^{p+1}-b^{p+1}}{(p+1)(a-b)}\right)^{\frac{1}{p}} & a\neq b \end{cases}$ $p\neq 0,-1$

Remark 1 (i) *With above notations, we have the following limit cases:*

$$\lim_{p\to 0} L_p(a,b)=I(a,b), \qquad \lim_{p\to -1} L_p(a,b)=L(a,b). \tag{2}$$

(ii) *The following inequalities are well-known in the literature:*

$$H(a,b)\leq G(a,b)\leq L(a,b)\leq I(a,b)\leq A(a,b), \tag{3}$$

and the equality holds in each inequality if and only if $a=b$.

In [3] it is declared that f is strictly convex on the real line, which by considering $f(x)\to 0$ ($x\to -\infty$), it is a stronger result than being strictly increasing, and besides, the function

$$g(x)=\ln\frac{a^x-b^x}{c^x-d^x},$$

is strictly convex if $ad-bc>0$, and is strictly concave if $ad-bc<0$. Since, we can write f and g in terms of means, we can use the convexity or concavity of them in order to get some relations between different means mentioned in the above table.
In this paper, first we study these functions more closely and get some interesting inequalities which are contained in the heart of these functions, and then as applications, using the achieved results, we sharpen some Ky Fan type inequalities.

2 Study of the Functions $\frac{a^x-b^x}{c^x-d^x}$ and $\ln\frac{a^x-b^x}{c^x-d^x}$

In this section, we will prove our claims about the functions f and g. First g:

Theorem 2.1 *Suppose $a>b\geq c>d>0$, and let*

$$g(x)=\ln\frac{a^x-b^x}{c^x-d^x}.$$

Then g is strictly convex if $ad-bc>0$, and is strictly concave if $ad-bc<0$. If $ad-bc=0$, then g turns out to be a linear mapping.

Proof. Suppose that $ad - bc > 0$. Since

$$g(x) = \ln \frac{(\frac{a}{b})^x - 1}{(\frac{c}{d})^x - 1} + x \ln \frac{b}{d},$$

it is sufficient to show that if $a > b > 1$, then $\ln \frac{a^x - 1}{b^x - 1}$ is strictly convex, and since $\ln \frac{a^x-1}{b^x-1} = \ln \frac{e^{x \ln a} - 1}{e^{x \ln b} - 1}$, it is sufficient to show that if $a > b > 0$, then

$$u(x) = \ln \frac{e^{ax} - 1}{e^{bx} - 1}$$

is strictly convex. A simple calculation yields that

$$u''(x) = \frac{b^2 e^{bx}}{(e^{bx} - 1)^2} - \frac{a^2 e^{ax}}{(e^{ax} - 1)^2} \qquad (x \neq 0).$$

So, for $x \neq 0$, $u''(x) > 0$ is equivalent to $\frac{|\sinh \frac{ax}{2}|}{a} > \frac{|\sinh \frac{bx}{2}|}{b}$, or $\frac{\sinh \frac{ax}{2}}{\frac{ax}{2}} > \frac{\sinh \frac{bx}{2}}{\frac{bx}{2}}$. But it is clear that the function $\frac{\sinh x}{x}$ is strictly decreasing on $(-\infty, 0]$ and strictly increasing on $[0, +\infty)$. This yields our claim in this case.
In the case $ad - bc < 0$, rewrite g as follows

$$g(x) = -\ln \frac{(\frac{c}{d})^x - 1}{(\frac{a}{b})^x - 1} + x \ln \frac{b}{d}.$$

According to the above argument, the function $-\ln \frac{(\frac{c}{d})^x - 1}{(\frac{a}{b})^x - 1}$, and so g, is strictly concave.
If $ad - bc = 0$, then $g(x) = x \ln \frac{b}{d}$; a straight line through the origin. This completes the proof. $\square$

Now, consider the function f. If $ad - bc = 0$, then $f(x) = (\frac{b}{d})^x$ which is clearly strictly convex. If $ad - bc > 0$, g is strictly convex, and since the function "exp" is strictly increasing and convex, $f = \exp(\ln(f)) = \exp(g)$ is strictly convex. But, in the case of $ad - bc < 0$, we cannot use the above method. Therefore, we go to prove the convexity of f independently.

Theorem 2.2 *Suppose $a > b \geq c > d > 0$ and*

$$f(x) = \frac{a^x - b^x}{c^x - d^x}.$$

Then f is strictly convex on the real line.

Proof. Since $\frac{a}{d} > \frac{b}{d} \geq \frac{c}{d} > 1$ and $f(x) = \frac{(\frac{a}{d})^x - (\frac{b}{d})^x}{(\frac{c}{d})^x - 1}$, it is sufficient to consider

$$f(x) = \frac{a^x - b^x}{c^x - 1} \qquad (a > b \geq c > 1),$$

and since $f(x) = \frac{e^{x \ln a} - e^{x \ln b}}{e^{x \ln c} - 1}$ and $\ln a > \ln b \geq \ln c > 0$, it is sufficient to consider

$$f(x) = \frac{e^{ax} - e^{bx}}{e^{cx} - 1} \qquad (a > b \geq c > 0).$$

But, $f(x) = \frac{e^{ax}}{e^{cx} - 1} - \frac{e^{bx}}{e^{cx} - 1}$, and so

$$f''(x) = \frac{e^{ax}\{[(a-c)e^{cx} - a]^2 + c^2 e^{cx}\} - e^{bx}\{[(b-c)e^{cx} - b]^2 + c^2 e^{cx}\}}{(e^{cx} - 1)^3}.$$

Therefore, it is sufficient to prove that for any fixed $x > 0$ ($x < 0$), the function

$$h(t) = e^{tx}\{[(t-c)e^{cx} - t]^2 + c^2 e^{cx}\},$$

is strictly increasing (decreasing) on $t \geq c$. But,

$$h'(t) = xe^{tx}\{[(t-c)e^{cx} - t]^2 + c^2 e^{cx}\} + 2(e^{cx} - 1)[(t-c)e^{cx} - t]e^{tx}.$$

Since $e^{tx} > 0$, the sign of $h'(t)$ agrees with the sign of

$$k(t) = \frac{h'(t)}{e^{tx}} = x\{[(t-c)e^{cx} - t]^2 + c^2 e^{cx}\} + 2(e^{cx} - 1)[(t-c)e^{cx} - t].$$

But,

$$k'(t) = 2x(e^{cx} - 1)[(t-c)e^{cx} - t] + 2(e^{cx} - 1)^2,$$

and

$$k''(t) = 2x(e^{cx} - 1)^2.$$

So, if $x > 0$, then $k'(t)$ is strictly increasing, and so for $t > c$, $k'(t) > k'(c)$. Also, if $x < 0$, then $k'(t)$ is strictly decreasing and so for $t > c$, $k'(t) < k'(c)$. But, $k'(c) = 2(e^{cx} - 1)(e^{cx} - 1 - cx)$, and $e^{cx} - 1 - cx > 0$ ($x \neq 0$). So, if $x > 0$, then $k'(c) > 0$ and if $x < 0$, then $k'(c) < 0$. Thus, when $x > 0$, we have $k'(t) > k'(c) > 0$ ($t > c$) and when $x < 0$ we have $k'(t) < k'(c) < 0$ ($t > c$). Therefore, if $x > 0$, then $k(t)$ is strictly increasing and if $x < 0$, then $k(t)$ is strictly decreasing on $[c, \infty)$. So, for $t > c$, if $x > 0$, then $k(t) > k(c)$ and if $x < 0$, then $k(t) < k(c)$. Now, let

$$u(x) = k(c) = xc^2 + xc^2 e^{cx} - 2c(e^{cx} - 1).$$

We have

$$u'(x) = xc^3 e^{cx} - c^2 e^{cx} + c^2,$$

and

$$u''(x) = xc^4 e^{cx}.$$

Thus if $x > 0$, then $u''(x) > 0$, $u'(x) > u'(0) = 0$, and therefore, $u(x) > u(0) = 0$. Also, if $x < 0$, then $u''(x) < 0$ and $u'(x) < u'(0) = 0$, and therefore $u(x) < u(0) = 0$. So for $t > c$, if $x > 0$, then $k(t) > k(c) = u(x) > 0$ and if $x < 0$, then $k(t) < k(c) = u(x) < 0$. So, for $t > c$, the sign of $h'(t)$ is same as the sign of x. Thus if $x > 0$, then the function $h(t)$ is strictly increasing on $t \geq c$ and if $x < 0$, then the function $h(t)$ is strictly decreasing on $t \geq c$. So, if $x > 0$, $h(a) > h(b)$ and therefore $f''(x) > 0$, and besides if $x < 0$, $h(a) < h(b)$ and again $f''(x) > 0$. This completes the proof. □

3 Applications to Special Means

As we said before, the convexity of f is a strong equipment for establishing interesting inequalities. For example, if $a > b \geq c > d > 0$, we have the following nontrivial inequality

$$\frac{\frac{a^b - b^b}{c^b - d^b} - \frac{a^d - b^d}{c^d - d^d}}{b - d} < \frac{\frac{a^a - b^a}{c^a - d^a} - \frac{a^c - b^c}{c^c - d^c}}{a - c},$$

since $m_{d,b} < m_{c,a}$, where $m_{\alpha,\beta}$ denote the slope of the line segment joining points $(\alpha, f(\alpha))$ and $(\beta, f(\beta))$. Some other results are given in the next theorems.

Theorem 3.1 *Suppose $a > b \geq c > d > 0$ and $p, q \neq 0, -1$. Then we have the following inequality*

$$\frac{L_p^p(a,b)}{L_p^p(c,d)} \geq \frac{L_q^q(a,b)}{L_q^q(c,d)} \left(1 + \left(\frac{p-q}{q+1}\right) \ln \frac{I(a^{q+1}, b^{q+1})}{I(c^{q+1}, d^{q+1})}\right), \tag{4}$$

with equality holding if and only if $p = q$.
In particular

$$\exp\left(1 - \frac{L(c,d)}{L(a,b)}\right) < \frac{I(a,b)}{I(c,d)} < \exp\left(\frac{L(a,b)}{L(c,d)} - 1\right), \tag{5}$$

and for $a > b > 0$,

$$\exp\left(1 - \frac{b}{L(a,b)}\right) < \frac{I(a,b)}{b} < \exp\left(\frac{L(a,b)}{b} - 1\right). \tag{6}$$

Proof. Since f is strictly convex, we have

$$f(p+1) \geq f(q+1) + (p-q)f'(q+1),$$

with equality holding if and only if $p = q$. Considering this fact and (1) yields (4).
Now, if we put $p = -1$ and $q = 0$ in the last relation, considering (1), we have

$$\frac{L(c,d)}{L(a,b)} > 1 - \ln \frac{I(a,b)}{I(c,d)},$$

which yields the left hand side of (5). Similarly, if we put $p = -1$ and $q = -2$, then

$$\frac{L(c,d)}{L(a,b)} > \frac{L_{-2}^{-2}(a,b)}{L_{-2}^{-2}(c,d)} \left(1 - \ln \frac{I(a^{-1}, b^{-1})}{I(c^{-1}, d^{-1})}\right).$$

But, $L_{-2}^{-2}(a,b) = \frac{1}{ab}$ and $\frac{1}{d} > \frac{1}{c} \geq \frac{1}{b} > \frac{1}{a} > 0$. So, changing $\frac{1}{d}$, $\frac{1}{c}$, $\frac{1}{b}$ and $\frac{1}{a}$ by a, b, c and d respectively and considering $L(a^{-1}, b^{-1}) = \frac{1}{ab} L(a,b)$, we get the right hand side of (5).
Since

$$\frac{L(a,b)}{b} = \frac{\frac{a}{b} - 1}{\ln \frac{a}{b}} \quad \text{and} \quad \ln \frac{I(a,b)}{b} = -1 + \frac{\frac{a}{b}}{\frac{a}{b} - 1} \ln \frac{a}{b}, \tag{7}$$

putting $x = \frac{a}{b}$, the inequalities in (6) follow from

$$x \ln x + \ln x - 2x + 2 > 0 \qquad (x > 1),$$
$$(x-1)^2 - x \ln^2 x > 0 \qquad (x > 1),$$

respectively. □

Theorem 3.2 *If $a > b \geq c > d > 0$, then*

$$\frac{L(a,b)}{L(c,d)} > 1 + \ln \frac{G(a,b)}{G(c,d)} > \frac{2ab}{ab + cd}, \tag{8}$$

and

$$\frac{L(a,b)}{L(c,d)} > \frac{\ln \frac{G(a,b)}{G(c,d)}}{\ln \frac{I(a,b)}{I(c,d)}}. \tag{9}$$

In particular, if $a > b > 0$, then

$$\frac{L(a,b)}{b} > 1 + \frac{1}{2} \ln \frac{a}{b} > \frac{2a}{a+b} > \frac{\ln \frac{a}{b}}{2 \ln \frac{I(a,b)}{b}}. \tag{10}$$

Proof. Since $f(x) = \frac{a^x - b^x}{c^x - d^x}$ is strictly convex, we have

$$f'(0)x + f(0) < f(x) \qquad (x \neq 0),$$

which by setting $x = 1$ and using (1), we get the first inequality in (8). The second inequality in (8) is equivalent to

$$\ln \frac{G(a,b)}{G(c,d)} > \frac{ab - cd}{ab + cd}, \tag{11}$$

which by putting $x = ab$ and $y = cd$, (11) follows from

$$\frac{1}{2} \ln \frac{x}{y} > \frac{\frac{x}{y} - 1}{\frac{x}{y} + 1} \qquad (x > y > 0). \tag{12}$$

But, (12) is obtained by the facts that the function $h(x) = \frac{1}{2} \ln x - \frac{x-1}{x+1}$ is strictly increasing on $[1, \infty)$ and $\frac{x}{y}$ is greater than 1.
The inequality (9) follows from $f'(0) < f'(1)$ and considering (1).
Considering (7) and putting $x = \frac{a}{b}$, the inequalities in (10) follow from left to right from

$$\ln^2 x + 2 \ln x - 2x + 2 < 0 \qquad (x > 1),$$

$$x \ln x + \ln x - 2x + 2 > 0 \qquad (x > 1),$$

$$(x^2 - 1) \ln x - 4x(1 - x + x \ln x) < 0 \qquad (x > 1),$$

respectively. This completes the proof. □

Now, consider the function $g(x) = \ln \frac{a^x - b^x}{c^x - d^x}$. It is evident from (1), $g'(0) = \ln \frac{G(a,b)}{G(c,d)}$ and for $x \neq 0$, $g'(x) = \frac{1}{x} \ln \frac{I(a^x, b^x)}{I(c^x, d^x)}$.

Theorem 3.3 *Suppose $a > b \geq c > d > 0$ and $p, q \neq 0, -1$. If $ad - bc > 0$, then*

$$\left(\frac{L_p(a,b)}{L_p(c,d)} \right)^p \geq \left(\frac{L_q(a,b)}{L_q(c,d)} \right)^q \left(\frac{I(a^{q+1}, b^{q+1})}{I(c^{q+1}, d^{q+1})} \right)^{\frac{p-q}{q+1}}. \tag{13}$$

If $ad - bc < 0$, the inequality reverses. The equality holds if and only if $ad - bc = 0$ or $p = q$.

Proof. If $ad - bc > 0$, then by Theorem 2.1, g is strictly convex and so considering the tangent line at $x = q + 1$, we have

$$g(p + 1) \geq g(q + 1) + (p - q)g'(q + 1),$$

with equality holding if and only if $p = q$. Now, considering (1), we get (13) with equality if and only if $p = q$.
If $ad - bc < 0$, then g is strictly concave and the argument is similar.
If $ad - bc = 0$, then $g(x) = x \ln \frac{b}{d}$ is linear, and so equality always holds in (13). □

In the cases $p, q = 0, -1$, we conclude the following nice result:

Theorem 3.4 *Suppose $a \geq b \geq c \geq d > 0$. If $ad - bc > 0$, then*

$$\frac{H(a,b)}{H(c,d)} < \frac{G(a,b)}{G(c,d)} < \frac{L(a,b)}{L(c,d)} < \frac{I(a,b)}{I(c,d)} < \frac{A(a,b)}{A(c,d)}. \tag{14}$$

If $ad - bc < 0$, all inequalities reverse, and if $ad - bc = 0$, all inequalities turn out to be equalities.

Proof. **Case I.** $ad - bc > 0$. It is divided into two branches; $a > b \geq c > d$ and $a > b \geq c = d$.
If $a > b \geq c > d$, writing the first inequality in (14) in terms of $\frac{a}{b}$ and $\frac{c}{d}$, it follows from the fact that the function $x + \frac{1}{x}$ is strictly increasing on $[1, \infty)$.
The second one follows from the fact that the slope of the line segment between $(-1, g(-1))$ and $(0, g(0))$ is strictly less than the slope of the line segment between $(0, g(0))$ and $(1, g(1))$.
The third one follows from the fact that the point $(0, g(0))$ is strictly above the tangent line to the graph of g at $x = 1$.
Writing the last inequality in (14) in terms of $\frac{a}{b}$ and $\frac{c}{d}$, and considering (7), it follows from the fact that the function $\frac{x \ln x}{x-1} - \ln(x + 1)$ is strictly decreasing on $[1, \infty)$.
If $a > b \geq c = d$, all denominators in (14) are all equal to c, and so (14) follows from Remark 1, (ii).
Case II. $ad - bc < 0$. We have $a > b \geq c > d$ or $a = b \geq c > d$, and the result follows similarly by using the strict concavity of g.
Case III. $ad - bc = 0$. It turns to two branches; $a > b \geq c > d$ and $a = b = c = d$. To prove the first case, proceed as the case I and use the linearity of g. In the second case, all the fractions in (14) are equal to 1. □

Now, we give a nice example concerning some numerical sequences.

Example. For every $n \in \mathbb{N}$, we have

$$\frac{n+2}{n+1} < 1 + \ln\sqrt{\frac{n+2}{n}} < \frac{\ln(1+\frac{1}{n})}{\ln(1+\frac{1}{n+1})}, \tag{15}$$

and

$$\frac{\ln\sqrt{\frac{n+2}{n}}}{\ln\frac{(n+2)(1+\frac{1}{n+1})^{n+1}}{(n+1)(1+\frac{1}{n})^{n}}} < \frac{\ln(1+\frac{1}{n})}{\ln(1+\frac{1}{n+1})}. \tag{16}$$

Also, we have

$$\frac{2n+3}{2n+1} < \frac{n+2}{n+1}\frac{(1+\frac{1}{n+1})^{n+1}}{(1+\frac{1}{n})^{n}} < \frac{\ln(1+\frac{1}{n})}{\ln(1+\frac{1}{n+1})} < \sqrt{\frac{n+2}{n}} < \frac{(n+2)(2n+1)}{n(2n+3)}. \tag{17}$$

These are obtained from (8), (9) and Theorem 3.4, by putting $a = n + 2$, $b = c = n + 1$ and $d = n$ with considering $ad - bc < 0$.

4 Applications to Ky Fan Type Inequalities

Throughout this section, given n arbitrary nonnegative real numbers $x_1, \cdots, x_n$ belonging to $(0, \frac{1}{2}]$, we denote by A_n and G_n, the unweighted arithmetic and geometric means of $x_1, \cdots, x_n$ respectively, i.e.

$$A_n = \frac{1}{n}\sum_{i=1}^{n} x_i, \qquad G_n = \prod_{i=1}^{n} x_i^{1/n},$$

and by A'_n and G'_n, the unweighted arithmetic and geometric means of $1-x_1, \cdots, 1-x_n$ respectively, i.e.

$$A'_n = \frac{1}{n}\sum_{i=1}^{n}(1 - x_i), \qquad G'_n = \prod_{i=1}^{n}(1 - x_i)^{1/n}.$$

With the above notations, the Ky Fan's inequality [2] asserts that:

$$\frac{A'_n}{G'_n} \leq \frac{A_n}{G_n}, \tag{18}$$

with equality holding if and only if $x_1 = \cdots = x_n$.
In 1988, H. Alzer [1] obtained an additive analogue of Ky Fan's inequality as follows:

$$A'_n - G'_n \leq A_n - G_n, \tag{19}$$

with equality holding if and only if $x_1 = \cdots = x_n$.
Also, in 1995, J.E. Pečarić and H. Alzer [5], using the Dinghas Identity [4], proved that:

$${A_n}^n - {G_n}^n \leq {A'_n}^n - {G'_n}^n, \tag{20}$$

in which if $n = 1, 2$, *equality always holds in* (20), *and if* $n \geq 3$, *the equality is valid if and only if* $x_1 = \cdots x_n$.
Now, we use the obtained results in Theorems 3.2 and 3.4 and find some refinements and inverses of Ky Fan's inequality (18) and its additive analogues (19) and (20). Perhaps, the most interesting results are:

$$\left(\frac{A'_n}{G'_n}\right)^{A'_n+G'_n} \leq \left(\frac{A_n}{G_n}\right)^{A_n+G_n}, \tag{21}$$

$$\left(\frac{A'_n}{G'_n}\right)^{A_n-G_n} \leq \left(\frac{A_n}{G_n}\right)^{A'_n-G'_n}, \tag{22}$$

with equality holding if and only if $x_1 = \cdots = x_n$.
These are easily obtained from the first inequality in (29) which need a great labor to handle them directly.

Theorem 4.1 *Suppose* $x_1, \cdots, x_n \in (0, \frac{1}{2}]$ *not all equal. Then*

$$\frac{A'_n}{G'_n} < \left(\frac{A_n}{G_n}\right)^{\frac{A'_n-G'_n}{A_n-G_n} - \frac{\ln\frac{A'_n}{G'_n}}{\ln\frac{A_n}{G_n}} \ln\sqrt{\frac{A'_nG'_n}{A_nG_n}}} < \tag{23}$$

$$< \left(\frac{A_n}{G_n}\right)^{\frac{A'_n-G'_n}{A_n-G_n} - \frac{\ln\frac{A'_n}{G'_n}}{\ln\frac{A_n}{G_n}} \ln\frac{A'_n}{A_n}} < \left(\frac{A_n}{G_n}\right)^{1-\frac{\ln\frac{A'_n}{G'_n}}{\ln\frac{A_n}{G_n}} \ln\frac{A'_n}{A_n}} < \frac{A_n}{G_n},$$

and

$$\frac{A'_n}{G'_n} < \max\left\{\left(\frac{A'_n}{G'_n}\right)^{1+\ln\sqrt{\frac{A'_nG'_n}{A_nG_n}}}, \left(\frac{A'_n}{G'_n}\right)^{\frac{A_n-G_n}{A'_n-G'_n}}\right\} < \tag{24}$$

$$< \left(\frac{A'_n}{G'_n}\right)^{\left(1+\ln\sqrt{\frac{A'_nG'_n}{A_nG_n}}\right)\frac{A_n-G_n}{A'_n-G'_n}} < \frac{A_n}{G_n},$$

which are some refinements of Ky Fan's inequality (18).
Also,

$$\frac{{A'_n}^n - {G'_n}^n}{{A_n}^n - {G_n}^n} < \frac{\ln\left(\frac{A'_n}{G'_n}\right)^{{A'_n}^n {G'_n}^n}}{\ln\left(\frac{A_n}{G_n}\right)^{{A_n}^n {G_n}^n}}, \tag{25}$$

which gives an inverse to (20).

Proof. Since $A'_n > G'_n > A_n > G_n > 0$, using the first inequality in (8), we get the first inequality in (23) and the last one in (24). The last inequality in (23), and using (19), the first one in (24) are trivial. The other inequalities in (23) and (24) follow from (18) and (19).
For proving (25), note that $f(-n) < f(0)$, where $f(x) = \frac{{A'_n}^x - {G'_n}^x}{{A_n}^x - {G_n}^x}$. □

Theorem 4.2 *Suppose* $x_1, \cdots, x_n \in (0, \frac{1}{2}]$ *not all equal. Then*

$$\max\left\{\frac{{A'_n}^n - {G'_n}^n}{{A_n}^n - {G_n}^n}\left(\frac{A_nG_n}{A'_nG'_n}\right)^{\frac{n}{2}}, \frac{A'_n - G'_n}{A_n - G_n}\left(\frac{A_nG_n}{A'_nG'_n}\right)^{\frac{1}{2}}\right\} < \tag{26}$$

$$< \frac{\ln\frac{A'_n}{G'_n}}{\ln\frac{A_n}{G_n}} < \frac{A'_n - G'_n}{A_n - G_n}\frac{\ln\frac{I(A'_n,G'_n)}{I(A_n,G_n)}}{\ln\sqrt{\frac{A'_nG'_n}{A_nG_n}}} < \min\left\{\frac{A'_n - G'_n}{A_n - G_n}, \frac{\ln\frac{I(A'_n,G'_n)}{I(A_n,G_n)}}{\ln\sqrt{\frac{A'_nG'_n}{A_nG_n}}}\right\} < 1,$$

and

$$\frac{A'_n}{G'_n} < \left(\frac{A'_n}{G'_n}\right)^{\frac{\ln\sqrt{\frac{A'_nG'_n}{A_nG_n}}}{\ln\frac{I(A'_n,G'_n)}{I(A_n,G_n)}}} < \left(\frac{A_n}{G_n}\right)^{\frac{A'_n-G'_n}{A_n-G_n}} < \frac{A_n}{G_n} < \left(\frac{A'_n}{G'_n}\right)^{\left(\frac{A'_nG'_n}{A_nG_n}\right)^{\frac{n}{2}}}, \tag{27}$$

which give some refinements and inverses of Ky Fan's inequality (18).
Moreover,

$$\frac{{A'_n}^n - {G'_n}^n}{{A_n}^n - {G_n}^n} < \frac{\ln(\frac{A'_n}{G'_n})^{(A'_nG'_n)^{\frac{n}{2}}}}{\ln(\frac{A_n}{G_n})^{(A_nG_n)^{\frac{n}{2}}}} < \left(\frac{A'_nG'_n}{A_nG_n}\right)^{\frac{n}{2}}, \tag{28}$$

which gives some inverses of (20).

Proof. For proving the left hand side of (26), use $\frac{G(a,b)}{G(c,d)} > \frac{L(a,b)}{L(c,d)}$ in Theorem 3.4 with $a = A'_n,\ b = G'_n,\ c = A_n,\ d = G_n$ and also with $a = {A'_n}^n,\ b = {G'_n}^n,\ c = {A_n}^n,\ d = {G_n}^n$. For the second inequality in (26), use (9) with $a = A'_n,\ b = G'_n,\ c = A_n,\ d = G_n$. The third and forth inequalities in (26) follow from $\frac{I(A'_n,G'_n)}{I(A_n,G_n)} < \sqrt{\frac{A'_nG'_n}{A_n,G_n}}$ in Theorem 3.4, and (19).
Putting $a = A'_n,\ b = G'_n,\ c = A_n$ and $d = G_n$, the first inequality in (27) follows from Theorem 3.4, the second one follows from (9), the third one follows from (19), and the last one follows from (26) by considering (20).
Inequalities in (28) follow from (26) and (18). □

Theorem 4.3 *Suppose* $x_1, \cdots, x_n \in (0, \frac{1}{2}]$ *not all equal. Then*

$$\frac{A'_n}{G'_n} < \left(\frac{A_n}{G_n}\right)^{\frac{A_n+G_n}{A'_n+G'_n}\frac{A'_n-G'_n}{A_n-G_n}} < \min\left\{\left(\frac{A_n}{G_n}\right)^{\frac{A_n+G_n}{A'_n+G'_n}}, \left(\frac{A_n}{G_n}\right)^{\frac{A'_n-G'_n}{A_n-G_n}}\right\} < \frac{A_n}{G_n}, \tag{29}$$

$$\frac{A'_n}{G'_n} < \left(\frac{A'_n}{G'_n}\right)^{\frac{I(A'_n,G'_n)}{I(A_n,G_n)}\frac{A_n-G_n}{A'_n-G'_n}} < \frac{A_n}{G_n}, \tag{30}$$

which are some refinements of Ky Fan's inequality. Moreover, we have

$$\frac{A_nG_n}{A'_nG'_n} < \min\left\{\left[\frac{\ln(\frac{A'_n}{G'_n})^{\frac{1}{A'_n-G'_n}}}{\ln(\frac{A_n}{G_n})^{\frac{1}{A_n-G_n}}}\right]^2, \left[\frac{\ln(\frac{{A'_n}^n}{{G'_n}^n})^{\frac{1}{{A'_n}^n-{G'_n}^n}}}{\ln(\frac{{A_n}^n}{{G_n}^n})^{\frac{1}{{A_n}^n-{G_n}^n}}}\right]^{\frac{2}{n}}\right\} < 1. \tag{31}$$

Proof. Put $a = A'_n,\ b = G'_n,\ c = A_n,\ d = G_n$. The inequalities in (29) follow from $\frac{L(a,b)}{L(c,d)} > \frac{A(a,b)}{A(c,d)}$ in Theorem 3.4, (19) and $A_n + G_n < A'_n + G'_n$.
The first inequality in (30) follows from Theorem 3.4 and (19), and the second one follows from $\frac{L(a,b)}{L(c,d)} > \frac{I(a,b)}{I(c,d)}$ in Theorem 3.4.
From (29) we obtain $(\frac{A'_n}{G'_n})^{\frac{1}{A'_n - G'_n}} < (\frac{A_n}{G_n})^{\frac{1}{A_n - G_n}}$, considering this relation, we get (31) by solving the first inequality in (26) with respect to $\frac{A_n G_n}{A'_n G'_n}$. □

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