STRONG STABILITY OF PERIODIC DISCRETE SYSTEMS IN BANACH SPACES

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Abstract. Let $q > 1$ be a fixed integer number. We prove that a discrete q-periodic evolution family

$$\mathcal{U} = \{U(m,n) : (m,n) \in \mathbb{Z}_+ \times \mathbb{Z}_+, \quad m \geq n \geq 0\}$$

on a complex Banach space $\mathcal{X}$ is uniformly asymptotically stable, that is, $U(m,n) \to 0$ in the norm of $\mathcal{L}(\mathcal{X})$ when $(m-n) \to \infty$, if and only if for each $\mu \in \mathbb{R}$ and each $x \in X$ one has

$$\max_{1\leq j\leq q-1} \sup_{m\geq 1} \left\| \sum_{k=1}^{m} e^{-i\mu kq} U(kq,j)x \right\| := M(\mu, x) < \infty.$$

In particular, we obtain the following result of Datko type. The family $\mathcal{U}$ is uniformly asymptotically stable if and only if for each $x \in \mathcal{X}$ one has

$$\max_{1\leq j\leq q-1} \sum_{k=1}^{\infty} ||U(kq,j)x|| < \infty.$$

1. Introduction

Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup on a complex Hilbert space $\mathcal{H}$ and $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ its infinitesimal generator. It is well-known that if the operator resolvent of A exists and is uniformly bounded on the imaginary axis then the semigroup $\mathbf{T}$ is exponentially stable, that is, its uniform growth bound $\omega_0(\mathbf{T}) := \inf_{t>0} \frac{\ln(||T(t)||)}{t}$ is negative. This result is usually referred to as Gearhart's theorem. Another variant of this theorem, independent proofs and related results can be found in [10], [11], [12], [8], [14] and [17]. See also the references therein. Applications of this theorem in the study of the stability of solitary waves for a large class of Hamiltonian partial differential equations of mathematical physics can be found in [5] and the references therein. Because the resolvent operator-valued map $R(\cdot, A)$ is also the Laplace transform of $T(\cdot)$ on the closed right half-plane, the main hypothesis of Gearhart's theorem can be written as

$$\sup_{\mu\in\mathbb{R}} \left\| \int_0^\infty e^{-i\mu t} T(t)x\,dt \right\| = M_x < \infty \text{ for all } x \in \mathcal{H}. \tag{1.1}$$

It is well-known that Gherhart's theorem does not work in general Banach spaces, see [9] for a counterexample.

We now examine the same problem from a different perspective. Let A be a linear and bounded operator on a complex Banach space. It is well-known, see for

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example [1], that the uniformly continuous semigroup $\{e^{tA}\}$ is exponentially stable if and only if

$$\sup_{s\geq 0}\left\|\int_0^s e^{-i\mu t}e^{tA}xdt\right\| < \infty \tag{1.2}$$

for every $\mu \in \mathbb{R}$ and every $x \in \mathcal{X}$. The classical solution of the Cauchy problem

$$\dot{u}(t) = Au(t) + e^{i\mu t}x, \text{ for all } t \geq 0, \quad u(0) = 0, \tag{A,μ,x}$$

where $\mu \in \mathbb{R}$ and $x \in \mathcal{X}$ is given by

$$u(s) = e^{i\mu s}\int_0^s e^{-i\mu t}e^{i\mu A}xdt, \quad s \geq 0.$$

We can say that the linear system

$$\dot{u}(t) = Au(t) \quad t \geq 0 \tag{1.3}$$

is exponentially stable if and only if for each $\mu \in \mathbb{R}$ and each $x \in \mathcal{X}$ the solution of the Cauchy problem (A, μ, x) is bounded. Unfortunately this nice result cannot be extended to the general case of strongly continuous semigroups (cf. [15], [16]).

Incorporating conditions of type (1.1) and (1.2), Jan van Neerven has shown in [13] that if $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is a strongly continuous semigroup on a Banach space $\mathcal{X}$ such that for each $x \in \mathcal{X}$

$$\sup_{\mu\in\mathbb{R}}\sup_{s\geq 0}\left\|\int_0^s e^{-i\mu t}T(t)xdt\right\| = M(x) < \infty, \tag{1.4}$$

then the operator resolvent of the generator A of $\mathbf{T}$ exists and is uniformly bounded on the open right half-plane. Another proof of this result was given by Vu Phong [16] in which the above result was cast in the framework of Cauchy problems. This proof utilised a lemma proved in [15] as well. Combining this with Gearhardt's theorem it follows that if (1.4) holds and $\mathcal{X}$ is a complex Hilbert space then the semigroup, $\mathbf{T}$ is uniformly exponentially stable. In the general framework of Banach spaces this latter fact is not true, see for example [2]. In order to introduce non-autonomous results of this type we recall the notion of an evolution family of bounded linear operators. Let $\mathcal{X}$ be a Banach space. A family $\mathcal{U} = \{U(t,s) : t \geq s \geq 0\}$ of bounded linear operators acting on $\mathcal{X}$ is said to be a *strongly continuous evolution family on* $\mathcal{X}$ if $U(t,t) = I$ for all $t \geq 0$, $U(t,s)U(s,r) = U(t,r)$ for all $t \geq s \geq r \geq 0$ and the function $(t,s) \mapsto U(t,s)$ is strongly continuous on the set $\{(t,s) \in \mathbb{R}\times\mathbb{R} : t \geq s \geq 0\}$. Such a family is *exponentially bounded* if there exist $\omega \in \mathbb{R}$ and $M \geq 1$ such that

$$||U(t,s)|| \leq Me^{\omega(t-s)} \text{ for all } t \geq s \geq 0.$$

Let θ be a positive fixed number. The strongly continuous evolution family $\mathcal{U}$ is called θ-periodic if $U(t+\theta, s+\theta) = U(t,s)$ for all $t \geq s \geq 0$. Here I denotes the identity operator on $\mathcal{X}$. If $U(t+\rho, s+\rho) = U(t,s)$ for every positive ρ and every $t \geq s \geq 0$ then the family $\{U(t,0) : t \geq 0\}$ is a strongly continuous semigroup. Thus it is natural to ask if a θ-periodic strongly continuous evolution family $\mathcal{U}$ on a complex Hilbert space $\mathcal{H}$ satisfying (1.4), with $T(t)$ replaced by $U(s,,s-t)$, is exponentially stable. That is, there exist the constants $N > 0$ and $\nu > 0$ such that

$$||U(t,s)|| \leq Ne^{-\nu(t-s)}$$

for all $t \geq s \geq 0$.

To the best of our knowledge it is still unknown whether or not this is true. A partial positive answer to this question is given in this paper for the case of discrete evolution families on complex Banach spaces. Moreover, we will not require in (1.4) the uniform boundedness condition with respect to μ on $\mathbb{R}$.

2. The Autonomous Case

Let $\mathbb{Z}_+ := \{0, 1, \dots\}$. For each real number μ and each complex number x we consider the following discrete equations:

$$x_{n+1} = ax_n, \quad n \in \mathbb{Z}_+; \tag{2.1}$$

$$y_{n+1} = ay_n + e^{i\mu n}x, \quad n \in \mathbb{Z}_+, \quad y_0 = 0, \tag{A, μ, x}$$

where the complex numbers x_0 and a are given.

After standard calculations we obtain the "solutions" (x_n) and (y_n) of the equations (2.1) and (A, μ, x), given by:

$$x_n = a^n x_0$$

and

$$\begin{aligned} y_n &= e^{i\mu(n-1)} \sum_{k=0}^{n-1} e^{-i\mu k} a^k x \\ &= \begin{cases} (1 - e^{-i\mu}a)^{-1}[e^{i\mu(n-1)} - e^{-i\mu}a^n]x, & \text{if} \quad a \neq e^{i\mu} \\ ne^{i\mu n}b + ne^{-i\mu(n-1)}x, & \text{if} \quad a = e^{i\mu} \end{cases} \end{aligned} \tag{2.2}$$

respectively.

Now we can easily state the following result concerning the asymptotic stability of equation (2.1):

Proposition 1. *Let $x_0 \neq 0$. The following statements are equivalent:*

(1) $\lim_{n \to \infty} x_n = 0$.
(2) *The modulus of a is less than* 1.
(3) *For each real number μ and each complex number x the sequence (y_n) is bounded.*

Before stating abstract results of this type, we firstly examine the two dimensional case.

Let A be a quadratic complex matrix of order 2 and let

$$X = \begin{pmatrix} x \\ x' \end{pmatrix}, \quad X_n = \begin{pmatrix} x_n \\ x'_n \end{pmatrix}, \quad \text{and} \quad Y_n = \begin{pmatrix} y_n \\ y'_n \end{pmatrix}.$$

Let us consider the scalar equation in λ

$$\det(\lambda I_2 - A) = 0. \tag{2.3}$$

Here I_2 denotes the identity matrix of order two. If the equation (2.3) has different roots λ_1 and λ_2, then it is well-known that there exists an invertible quadratic complex matrix T such that

$$T^{-1}AT = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{2.4}$$

In the sequel we consider the following 2-dimensional linear systems

$$X_{n+1} = AX_n, \quad n \in \mathbb{Z}_+; \tag{2.5}$$

$$(A,\mu,X) \qquad Y_{n+1} = AY_n + e^{i\mu n}X, \quad Y_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad n \in \mathbb{Z}_+;$$

If we suppose that for each X_0, the solution (X_n) of the system (2.5) tends to 0 then $|\lambda_1| < 1$ and $|\lambda_2| < 1$. This easily follows because using (2.4) we get:

$$T^{-1}X_n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} T^{-1}X_0. \tag{2.6}$$

Note also that (X_n) tends to 0 if and only if $(T^{-1}X_n)$ tends to 0 when n tends to ∞.

On the other hand

$$Y_n = e^{i\mu(n-1)} \sum_{k=0}^{n-1} (e^{-i\mu}A)^k X.$$

Again using (2.4), we get

$$T^{-1}Y_n = e^{i\mu(n-1)} \sum_{k=0}^{n-1} e^{-i\mu k} \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k. \end{pmatrix} T^{-1}X. \tag{2.7}$$

We remark that $(T^{-1}Y_n)$ is bounded if and only if (Y_n) is bounded as well.

If $\lambda_1 = \lambda_2 := \alpha$ then there exists an invertible quadratic matrix T and a complex number β such that

$$T^{-1}AT = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}.$$

As above we obtain

$$T^{-1}X_n = \begin{pmatrix} \alpha^n & n\beta\alpha^{n-1} \\ 0 & \alpha^n \end{pmatrix} T^{-1}X_0 \tag{2.8}$$

and

$$T^{-1}Y_n = e^{i\mu(n-1)} \sum_{k=0}^{n-1} e^{-i\mu k} \begin{pmatrix} \alpha^k & k\beta\alpha^{k-1} \\ 0 & \alpha^k \end{pmatrix} T^{-1}X. \tag{2.9}$$

Using the representation of the solutions of (2.5) and (A,μ,X) incorporated in (2.6), (2.7), (2.8) and (2.9), and using the second formula for y_n in (2.2) we may state the following result.

Proposition 2. *Suppose that the equation (2.4) has the roots λ_1 and λ_2. Then the following statements are equivalent:*

1. *The system (2.5) is strongly stable, that is $A^nX_0 \to 0$ as $n \to \infty$, for every X_0.*
2. *The spectrum of the matrix A, (i.e. the set $\sigma(A) := \{\lambda_1, \lambda_2\}$) belongs to the open disk $D(0,1) := \{\lambda \in C : |\lambda| < 1\}$.*
3. *For each real number μ the solution (Y_n) of (A,μ,X) is bounded for every X.*

Throughout the paper, $\mathcal{L}(\mathcal{X})$ will denote the set of bounded linear operators acting on the complex Banach space $\mathcal{X}$. If T is a bounded linear operator on $\mathcal{X}$, $\rho(T)$ will denote the resolvent set of T relative to $\mathcal{L}(\mathcal{X})$ (that is, the set of all complex scalars λ for which $\lambda I - T$ is invertible in $\mathcal{L}(\mathcal{X})$) and $\sigma(T) := \mathbf{C} \setminus \rho(T)$ will denote the spectrum of T. The spectrum radius of T will be denoted by $r(T)$. It is well-known that

$$r(T) := \sup\{|\lambda| : \quad \lambda \in \sigma(T)\} = \inf_{n\geq 1} ||T^n||^{\frac{1}{n}}. \tag{2.10}$$

In order to obtain discrete Cauchy problems, we replace in (A, μ, X) the continuous time t by the discrete time $n \in \mathbb{Z}_+$. More precisely we replace the derivative $\dot{u}(t)$ by the difference $u_{n+1} - u_n$ and the operator A by the "discrete Laplacian" $T - I$. The discrete version of (A, μ, X) then reads

$$u_{n+1} = Tu_n + e^{i\mu n}x \text{ for all } n \in \mathbb{Z}_+, \quad u_0 = 0, \tag{T,μ,x}$$

and the continuous semigroup $\mathbf{T}$ will be replaced by the discrete semigroup

$$\mathcal{T} := \{T(n)\}_{n\in\mathbb{Z}_+} \text{ where } T(n) \equiv T^n.$$

With the above stipulations, we may state the following abstract result.

Theorem 1. *The following three statements are equivalent:*

1. $\lim\limits_{n\to\infty} T^n = 0$ *in the norm of* $\mathcal{L}(\mathcal{X})$*, that is,* $\mathcal{T}$ *is uniformly asymptotically stable.*
2. *For each real number* μ *and each* $x \in \mathcal{X}$ *the solution of* (T, μ, x)*, is bounded.*
3. *The spectral radius of* T *is less than* 1.

Proof. **3** ⇒ **1**. From (2.10) it follows that $r(T) \leq ||T||$. If $r(T) = ||T||$ the result can be easily obtained. Then we can suppose that $r(T) < ||T||$. Let $0 < \omega < 1$ such that $r(T) < \omega$. There exists $n_0 \in \mathbb{Z}_+$, $n_0 > 1$ such that $||T^{n_0}|| < \omega^{n_0}$. Let $n = mn_0 + r_0$ with $m \in \mathbb{Z}_+, r_0 \in \mathbb{Z}_+$ and $r_0 < n_0$. It is clear that $n \to \infty$ if and only if $m \to \infty$. Then

$$||T^n|| \leq ||T^{n_0}||^m ||T^{r_0}|| \to 0 \text{ when } n \to \infty.$$

1. ⇒ **3**. Is obvious.
2. ⇒ **3**. After a simple calculation we obtain

$$y_n = e^{i\mu(n-1)} \sum_{k=0}^{n-1} e^{-i\mu k} T^k x.$$

Then the sequence (y_n) is bounded if and only if

$$\sup_{n\geq 1} \left\| \sum_{k=0}^{n-1} e^{-i\mu k} T^k x \right\| = M(\mu, x) < \infty. \tag{2.11}$$

It is known that if (2.11) holds for each real number μ and each $x \in \mathcal{X}$ then $r(T) < 1$. For a proof of the later fact, see [3].
3. ⇒ **2**. If $r(T) < 1$ then for each real number μ, one has $r(e^{-i\mu}T) < 1$. Under these conditions it is well-known that the series $\sum\limits_{k\geq 0} e^{-i\mu k} T^k$ is convergent in $\mathcal{L}(\mathcal{X})$. Now it is easily to see that (2.11) holds.

□

3. The Periodic Time-Varying Case

A family $\mathcal{U} = \{U(m,n) : (m,n) \in \mathbb{Z}_+ \times \mathbb{Z}_+\}$ of bounded linear operators acting on a complex Banach space X is called *discrete periodic evolution family* if

1. $U(m,n)U(n,p) = U(m,p)$ for all $m \geq n \geq p \geq 0$;
2. $U(m,m) = I$ for all $m \in \mathbb{Z}_+$ and
3. there exists an integer number $q > 1$ such that $U(m+q, n+q) = U(m,n)$ for all $m, n \in \mathbb{Z}_+$, $m \geq n$.

Let $q \in \mathbb{Z}_+$ be fixed and by $\mathcal{S}_q(\mathbb{Z}_+, X)$ we will denote the set of all X-valued and q-periodic sequences on $\mathbb{Z}_+$. For each q-periodic $\mathcal{L}(X)$-valued sequence $V = (V_n)$, each $z = (z_n) \in \mathcal{S}_q(\mathbb{Z}_+, X)$ and each real number μ let us consider the following discrete Cauchy problem:

$$(V,\mu,z) \qquad y_{n+1} = V_n y_n + e^{i\mu n} z_n \text{ for all } n \in \mathbb{Z}_+ \quad y_0 = 0.$$

Let

$$U(n,k) := \begin{cases} V_{n-1}V_{n-2}\cdots V_k, & \text{if} \quad k \leq n-1 \\ I, & \text{if} \quad k = n, \end{cases}$$

then, the family $\{U(n,k)\}_{n \geq k \geq 0}$ is a discrete q-periodic evolution family and the solution (y_n) of (V,μ,z) is given by:

$$y_n = \sum_{k=1}^{n} e^{i\mu(k-1)} U(n,k) z_k. \tag{3.1}$$

We begin with the following lemmas which will prove useful later.

Lemma 1. *Let T be a bounded linear operator acting on the Banach space X and $\mu \in \mathbb{R}$. If*

$$\sup_{n \in \{1,2,\cdots\}} \sum_{k=1}^{n} ||e^{-i\mu k} T^k|| < \infty$$

then T is power bounded and $e^{i\mu} \in \sigma(T)$.

Proof. See for example Lemma 2 from [4]. □

Lemma 2. *Let $\mu \in \mathbb{R}$, $q \in \mathbb{Z}$, $q > 1$ and $S_1(\mu) := \sum_{k=1}^{q} k(q-k)e^{i\mu k}$. Then $S_1(\mu) = 0$ if and only if*

$$(q-1)(1+e^{i\mu}) = 2(e^{i\mu} + e^{2i\mu} + \cdots + e^{(q-1)i\mu}).$$

Proof. If $z := e^{i\mu} \pm 1$ then $S_1(\mu) \neq 0$. Let $L := \sum_{k=1}^{q} kz^{k-1}$ and $P := \sum_{k=1}^{q} k^2 z^{k-1}$. Then $S_1(\mu) = 0$ if and only if $q(1-z)L = (1-z)Q$ or equivalently if

$$(q+1)(1+z+\cdots+z^{q-1}) = 2L.$$

If multiply again with $(1-z)$ we get the conclusion of this lemma. □

Let $T := U(q,0)$. The following result may be stated.

Theorem 2. *Let $q > 1$ be a fixed integer number. The following three statements are equivalent.*

1. *The spectral radius of T is less than* 1.
2. *$U(m,n) \to 0$ in the norm of $\mathcal{L}(X)$ when $(m-n) \to \infty$.*

3. *For each real number μ and each sequence (z_n) in $\mathcal{S}_q(\mathbb{Z}_+, \mathcal{X})$ with $z_0 = 0$ the solution (y_n) of the problem (V, μ, z) is bounded.*

Proof. **3**. $\Rightarrow$ **1**. Let $x \in \mathcal{X}$ be fixed and (z_n) in $\mathcal{S}_q(\mathbb{Z}_+, \mathcal{X})$ be given by:

$$z_k = U(k, 0)x, \quad k = 1, \ldots, q-1 \text{ and } z_0 = z_q = 0.$$

For each $m \in \mathbb{Z}_+$ one has:

$$\begin{aligned} y_{mq} &= \sum_{j=0}^{m-1} \sum_{k=jq}^{(j+1)q} e^{i\mu(k-1)} U(mq, k) z_k \\ &= \sum_{j=0}^{m-1} \sum_{\rho=1}^{q-1} e^{i\mu(jq+\rho-1)} U((m-j)q, \rho) z_\rho \\ &= e^{-i\mu} \sum_{\rho=1}^{q-1} e^{i\mu\rho} \sum_{j=0}^{m-1} e^{i\mu jq} T^{m-j} x \\ &= e^{i\mu(qm-1)} S(\mu) \sum_{j=0}^{m-1} e^{-i\mu qj} T^j x. \end{aligned}$$

If are taking $z_k = k(q-k)U(k, 0)x$, $k = 1, 2 \cdots q$ obtain same result with $S_1(\mu)$ instead of $S(\mu)$. Because $S(\mu) := \sum_{\rho=1}^{q-1} e^{i\mu\rho}$ and $S_1(\mu)$ cannot be null simultaneously, the sequence $(y_{mq})_m$ is bounded if and only if for each real number μ and each $x \in \mathcal{X}$ one has

$$\sup_{m \in \mathbb{Z}_+} \left\| \sum_{j=0}^{m} e^{-i\mu qj} T^j x \right\| := K(\mu, q, x) < \infty,$$

that is, $r(T) < 1$.

1. $\Rightarrow$ **2**. We prove that there exist $N > 0$ and $\nu > 0$ such that

$$||U(m, n)|| \leq N e^{-\nu(m-n)} \text{ for every } m \geq n \geq 0. \tag{3.2}$$

Let $\omega > 0$ such that $r(T) < e^{-\omega}$. Then $r(e^{\omega t}T) < 1$ and there exists $K > 1$ such that

$$\sup_{n \in \mathbb{Z}_+} ||e^{\omega n} T^n|| \leq K.$$

If $m = pq + r$ with $p \in \mathbb{Z}_+, r \in \mathbb{Z}_+$ and $r < q$ then

$$||U(m, 0)|| \leq ||U(r, 0)|| \cdot ||U(pq, 0)|| \leq R_q e^{-\omega m},$$

where

$$R_q := e^{\omega q} \sup_{0 \leq r \leq q} ||U(r, 0)||.$$

Let

$$L_q := \sup_{0 \leq r \leq p \leq 2q} ||U(p, k)||.$$

If $n \leq m \leq n + q$ then

$$||U(m, n)|| \leq L_q e^q e^{-(m-n)}.$$

If $m > n + q$ and $n = p_1 q + r_1$ with $p_1 \in \mathbb{Z}_+, r_1 \in \mathbb{Z}_+$ and $0 \leq r_1 < q$ then

$$||U(m, n)|| = ||U(m, (p_1 + 1)q) U(q, r_1)|| \leq L_q R_q e^{-\omega(m-n)}.$$

We can choose $N = \max\{R_q, e^q L_q, R_q L_q\}$ and $\nu = \max\{\omega, 1\}$.

2. $\Rightarrow$ **3**. Using (3.1) and (3.2) we obtain

$$||y_n|| \leq N||z||_\infty \sum_{k=1}^{n} e^{-\nu(n-k)} = \frac{Ne^\nu}{e^\nu - 1}||z||_\infty$$

for every $n \in \mathbb{Z}_+$, and the theorem is proved.

□

Corollary 1. *Let $q > 1$ be a fixed integer number. A discrete q-periodic evolution family $\mathcal{U} = \{U(m,n)\}_{m \geq n \geq 0}$ on a complex Banach space $\mathcal{X}$ is uniformly asymptotically stable if and only if for each real number μ and each $x \in \mathcal{X}$ the following inequality holds:*

$$\max_{1 \leq j \leq q-1} \sup_{m \geq 1} \left\|\sum_{k=1}^{m} e^{-i\mu kq} U(kq, j)x\right\| := M(\mu, x) < \infty. \tag{3.3}$$

Proof. The sequence (y_n) given in (3.1) is bounded if and only if the sequence $(y_{mq})_m$ is bounded. As above one has:

$$\begin{aligned} y_{mq} &= \sum_{j=0}^{m-1} \sum_{\rho=0}^{q} e^{i\mu(jq+\rho-1)} U((m-j)q, \rho) z_\rho \\ &= \sum_{\rho=0}^{q} \sum_{j=0}^{m-1} e^{i\mu jq} e^{i\mu(\rho-1)} U((m-j)q, \rho) z_\rho \\ &= \sum_{\rho=0}^{q} e^{i\mu mq} e^{i\mu(\rho-1)} \sum_{k=1}^{m} e^{-i\mu kq} U(kq, \rho) z_\rho. \end{aligned} \tag{3.4}$$

Using (3.3) and (3.4) it follows that:

$$||y_{mq}|| \leq \sum_{\rho=1}^{q-1} M(\mu, z_\rho) < \infty.$$

Now we apply the above Theorem 2 to complete the proof. □

The Datko theorem says that a strongly continuous and exponentially bounded evolution family $\mathcal{U} = \{U(t,s)\}_{t \geq s \geq 0}$ is uniformly exponentially stable if and only if

$$\sup_{s \geq 0} \int_s^\infty ||U(t,s)x||dt = K(x) < \infty$$

for every $x \in \mathcal{X}$. See [6] for details.

Arguing as in the proof of Corollary 1 we can obtain the following discrete variant of the Datko theorem.

Corollary 2. *Let $q > 1$ be a fixed integer number. A discrete q-periodic evolution family $\mathcal{U} = \{U(m,n)\}_{m \geq n \geq 0}$ on a complex Banach space $\mathcal{X}$ is uniformly asymptotically stable if and only if for each $x \in \mathcal{X}$ one has*

$$\max_{1 \leq j \leq q-1} \sum_{k=1}^{\infty} ||U(kq, j)x|| < \infty.$$

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