SOLVING THREE PROBLEMS TOGETHER WITH THE SAME METHOD

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Abstract. In this short note, using $u-v$ Substitution Method, we give one of the solution of three interesting problems in [1].

Mihály Bencze put forward the following three interesting problems in [1].

PP.4468 If $x, y > 0$,then

$$\left(\frac{2}{x}+\frac{1}{y}\right)\left(\frac{2}{y}+\frac{1}{x}\right)\left(\frac{2}{1+x}+\frac{1}{1+y}\right)\left(\frac{2}{1+y}+\frac{1}{1+x}\right) \geqslant \frac{81}{(1+x^2y)(1+xy^2)}. \tag{1}$$

PP.4580 If $x, y > 0$,then

$$(2+x^2+y^2)\left(\frac{1}{1+x+y^2}+\frac{1}{1+y+x^2}\right)+\frac{1+x^2}{1+y+y^2}+\frac{1+y^2}{1+x+x^2} \geqslant 4. \tag{2}$$

PP.4581 If $x, y > 0$,then

$$3xy\left(\frac{2}{x^2y^2+x+y}-\frac{1}{(x+y)xy+1}\right) \leqslant 1. \tag{3}$$

The authors will solve these problems with $u-v$ Substitution Method in this paper.
Now, we firstly prove **PP.4468**:

Proof. The inequality (1) is equivalent to

$$\frac{2x^2+2y^2+5xy}{x^2y^2}\cdot\frac{2x^2+2y^2+5xy+9(x+y)+9}{(xy+x+y+1)^2} \geqslant \frac{81}{1+xy(x+y)+x^3y^3}. \tag{4}$$

Taking $x+y=u, xy=v$, from $x, y > 0$, it immediately follows that $u, v > 0$. Owing to $(x+y)^2 \geqslant 4xy$, we have $u^2 \geqslant 4v$ or $u \geqslant 2\sqrt{v}$. So (4) is equivalent to

$$\frac{2u^2+v}{v^2}\cdot\frac{2u^2+9u+v+9}{(u+v+1)^2} \geqslant \frac{81}{1+uv+v^3}. \tag{5}$$

With simple manipulations on (5), we obtain that (5) is equivalent to the next inequality

$$\begin{aligned}(u^2-4v)[4vu^3+(4v^3+18v+4)u^2+(18v^3+20v^2+18v+18)u \\ +20v^4+18v^3+20v+18]+81v(1+v)(1+u+v)(v-1)^2 \geqslant 0\end{aligned} \tag{6}$$

That is obvious since $u, v > 0$ and $u^2 \geqslant 4v$, the equality holding if and only if $u=2, v=1$. On the other words, the equality of (1) holds if and only if $x=y=1$. The proof is completed. □

Secondly, we give the proof of **PP.4580**:

Proof. The inequality (2) is equivalent to

$$\frac{(2+x^2+y^2)(2+x+y+x^2+y^2)}{(1+x+y^2)(1+y+x^2)}+\frac{(1+x^2)(1+x+x^2)+(1+y^2)(1+y+y^2)}{(1+y+y^2)(1+x+x^2)} \geqslant 4 \tag{7}$$

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Considering that

$$(1+x+y^2)(1+y+x^2)-(1+y+y^2)(1+x+x^2)=(x+y)(x-y)^2\geqslant 0, \tag{8}$$

we can strengthen the inequality (7) to be

$$\frac{(2+x^2+y^2)(2+x+y+x^2+y^2)+(1+x^2)(1+x+x^2)+(1+y^2)(1+y+y^2)}{(1+x+y^2)(1+y+x^2)}\geqslant 4. \tag{9}$$

Taking the same substitution in the proof of **PP.4468**, then (9) is equivalent to

$$\frac{(u^2-2v)(u^2-2v+u+2)+(u^2-2v)^2-2v^2+u(u^2-3v)+2(u^2-2v)+u+2}{v^2+u(u^2-3v)+u^2-v+u+1}\geqslant 4. \tag{10}$$

With simple manipulations on (10), we obtain that (10) is equivalent to the following inequality

$$2v^2-(8u^2-7u+8)v+2u^4-2u^3+2u^2-u+2\geqslant 0. \tag{11}$$

Set a function

$$f(v)=2v^2-(8u^2-7u+8)v+2u^4-2u^3+2u^2-u+2, \tag{12}$$

where $v\in(0,\frac{u^2}{4}]$. It's easy to prove the inequality

$$\frac{u^2}{4}\leqslant\frac{8u^2-7u+8}{4}. \tag{13}$$

So the function f is strictly monotone decreasing function with $v\in(0,\frac{u^2}{4}]$, then we can get

$$f(v)\geqslant f(\frac{u^2}{4})=\frac{1}{8}(u-2)^2[(u+1)^2+3]\geqslant 0.$$

Thus, the inequality (11) or (2) is proved. □

Finally, we will obtain the solution of **PP.4581**:

Proof. Taking the same substitution in the proof of **PP.4468**, then (3) is equivalent to

$$3v\left(\frac{2}{v^2+u}-\frac{1}{uv+1}\right)\leqslant 1. \tag{14}$$

With simple manipulations on (14), we find that (14) is equivalent to the following inequality

$$vu^2+(v^3-6v^2+3v+1)u+3v^3+v^2-6v\geqslant 0. \tag{15}$$

Define a function

$$g(u)=vu^2+(v^3-6v^2+3v+1)u+3v^3+v^2-6v, \tag{16}$$

where $u\in[2\sqrt{v},+\infty)$.
Next, we will prove the inequality

$$-\frac{v^3-6v^2+3v+1}{2v}\leqslant 2\sqrt{v}. \tag{17}$$

Setting $\sqrt{v}=t(t>0)$ and with simple manipulations, then the inequality (17) is equivalent to

$$t^6-6t^4+4t^3+3t^2+1\geqslant 0. \tag{18}$$

Also

$$t^6-6t^4+4t^3+3t^2+1=t^2[\frac{t^2}{4}(2t-3)^2+t(3t^2-\frac{115}{12}t+8)+\frac{1}{3}(2t-3)^2]+1. \tag{19}$$

it's obvious that $3t^2-\frac{115}{12}t+8>0$ holds for all $t\in R$, and

$$t^6-6t^4+4t^3+3t^2+1\geqslant 1>0 \tag{20}$$

holds for all $t>0$. Thus, (18) is proved.

Therefore, the function g is strictly monotone increasing function in interval $[2\sqrt{v}, +\infty)$, then we get

$$g(u) \geqslant g(2\sqrt{v}) = \sqrt{v}(\sqrt{v}-1)^2(2v^2+7v\sqrt{v}-2\sqrt{v}+2).$$

From the elementary inequality $v^2+1 \geqslant 2v$, we obtain

$$2v^2+7v\sqrt{v}-2\sqrt{v}+2 \geqslant v^2+7v\sqrt{v}+2v-2\sqrt{v}+1 = v^2+7v\sqrt{v}+v+(\sqrt{v}-1)^2 \geqslant 0. \tag{21}$$

So we get $g(u) \geqslant g(2\sqrt{v}) \geqslant 0$ immediately, namely the inequality (15) is proved. And on the other words, the origin inequality is proved. □

Through the previous three examples, we can find that $u-v$ Substitution Method is very valid for proving binary symmetrical inequalities.

And at the some time, we point out that the following inequality which in **PP.4582** does not hold.

$$\frac{(3xy-2x-2y)(3xy-2x-2y+x^2+y^2)}{(3xy-2x-2y+2y^2)(3xy-2x-2y+2x^2)} + \frac{1}{2+3xy-2x-2y} \geqslant \frac{1}{2}. \tag{22}$$

where $x, y > 0$. Since let $x = \frac{13}{14}$ and $y = \frac{1}{3}$,then the left of the inequality (22) is $\frac{-697267}{223516}$, it's obvious that $\frac{-697267}{223516} < \frac{1}{2}$.

References

[1] Mihály Bencze: **PP.4468**, **PP.4580-4582**, Octogon Mathematical Magazine, Vol.11.No.2, October 2003, page 796 and 813.

[2] Yu-Dong Wu: $u-v$ Substitution Method with One Kind of Inequalities in Triangle, BOTTEMA, What Do We See? —The New Results, The New Methods And The New Theory of Research in Geometry Inequalities in Triangle, Xizang People Press, 2003, 573–578.

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