ON A TRIGONOMETRIC INEQUALITY AND ITS APPLICATIONS

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Abstract. In this paper, we establish the following trigonometric inequality and another two similar inequalities:
$$8xy\cos A(x\cos B+y\cos C)^2\leqslant(x^2+y^2)^2,$$
where A,B,C are the angles of triangle ABC and $x,y\geqslant 0$, with equality holding if and only if $\sqrt{(x^2+y^2)/2}a=xb=yc$. By its applications, we give some mobile point geometric inequalities.

1. Introduction

For $x,y\geqslant 0$, by using the well-known A-G mean inequality and the power mean inequality, we have

$$8xy(x+y)^2\leqslant(x^2+y^2)^2. \tag{1.1}$$

In every triangle ABC, the following inequality holds:

$$\cos A(\cos B+\cos C)^2\leqslant\frac{1}{2}, \tag{1.2}$$

with equality holding if and only if ΔABC is an equilateral triangle.

In this paper, we establish an useful trigonometric inequality that relating inequalities (1.1) and (1.2), and by its applications, we give some mobile point geometric inequalities. In the final, we obtain another two similar inequalities.

2. Main Results

Theorem 2.1. *Let A,B,C are the angles of triangle ABC. If $x,y\geqslant 0$, then we have*

$$8xy\cos A(x\cos B+y\cos C)^2\leqslant(x^2+y^2)^2, \tag{2.1}$$

with equality holding if and only if $x^2+y^2=4xy\cos A$, and $x\sin B=y\sin C$, or $\sqrt{(x^2+y^2)/2}a=xb=yc$.

Proof. If A is an obtuse angle of triangle ABC, then (2.1) is obvious.
If A is an acute angle of triangle ABC, from A,B,C are the angles of triangle ABC and $x,y\geqslant 0$, and utilizing the facts that $\sin^2\theta+\cos^2\theta=1,\cos A=-\cos(B+C)=\sin B\sin C-\cos B\cos C$, we get

$$\begin{aligned}0&\leqslant(x^2+y^2-4xy\cos A)^2+8xy\cos A(x\sin B-y\sin C)^2\\&=(x^2+y^2)^2-8xy(x^2+y^2)\cos A+16(xy)^2\cos^2 A+8xy\cos A(x^2\sin^2 B-2xy\sin B\sin C+y^2\sin^2 C)\\&=(x^2+y^2)^2-8xy\cos A[(x^2+y^2)-2xy\cos A-x^2\sin^2 B+2xy\sin B\sin C-y^2\sin^2 C]\\&=(x^2+y^2)^2-8xy\cos A(x^2\cos^2 B+2xy\cos B\cos C+y^2\cos^2 C)\\&=(x^2+y^2)^2-8xy\cos A(x\cos B+y\cos C)^2.\end{aligned}$$

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That is the inequality (2.1), with equality holding if and only if $x^2+y^2=4xy\cos A$, and $x\sin B = y\sin C$, or $\sqrt{(x^2+y^2)/2}a = xb = yc$. The proof of Lemma 2.1 is completed. ∎

Corollary 2.1. *Let A, B, C are the angles of triangle ABC. If $x, y \geqslant 0$, then we have*

$$8xy\sin\frac{A}{2}\left(x\sin\frac{B}{2}+y\sin\frac{C}{2}\right)^2 \leqslant (x^2+y^2)^2, \tag{2.2}$$

with equality holding if and only if $x^2+y^2=4xy\sin\frac{A}{2}$, and $x\cos\frac{B}{2}=y\cos\frac{C}{2}$.

Proof. Alter $A\to\frac{\pi-A}{2}, B\to\frac{\pi-B}{2}, C\to\frac{\pi-C}{2}$, the corollary follows from Theorem 2.1 and standard arguments. ∎

The following corollary is obvious.

Corollary 2.2. *Let A, B, C are the angles of triangle ABC. If $x, y, z \geqslant 0$, then we have*

$$\begin{aligned}&8xy\cos A(x\cos B+y\cos C)^2+8yz\cos B(y\cos C+z\cos A)^2\\&+8zx\cos C(z\cos A+x\cos B)^2\leqslant(x^2+y^2)^2+(y^2+z^2)^2+(z^2+x^2)^2,\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}&8xy\sin\frac{A}{2}\left(x\sin\frac{B}{2}+y\sin\frac{C}{2}\right)^2+8yz\sin\frac{B}{2}\left(y\sin\frac{C}{2}+z\sin\frac{A}{2}\right)^2\\&+8zx\sin\frac{C}{2}\left(z\sin\frac{A}{2}+x\sin\frac{B}{2}\right)^2\leqslant(x^2+y^2)^2+(y^2+z^2)^2+(z^2+x^2)^2,\end{aligned} \tag{2.4}$$

with both equalities holding if and only if ΔABC is an equilateral triangle and $x=y=z$.

3. Some Mobile Point Geometric Inequalities

Theorem 3.1. *For any point P inside $\triangle A_1A_2A_3$, denote $PA_1=R_1, PA_2=R_2, PA_3=R_3$, and w_1, w_2, w_3 be the bisectors of $\angle A_2PA_3, \angle A_3PA_1, \angle A_1PA_2$. If $x, y \geqslant 0$, then*

$$8xyw_1(xw_2+yw_3)^2 \leqslant R_1(y^2R_2+x^2R_3)^2, \tag{3.1}$$

with equality holding if and only if $\sqrt{(x^2+y^2)/2}\sin\frac{1}{2}\angle A_2PA_3 = x\sin\frac{1}{2}\angle A_3PA_1 = y\sin\frac{1}{2}\angle A_1PA_2$, and P is the circumcenter of triangle $A_1A_2A_3$.

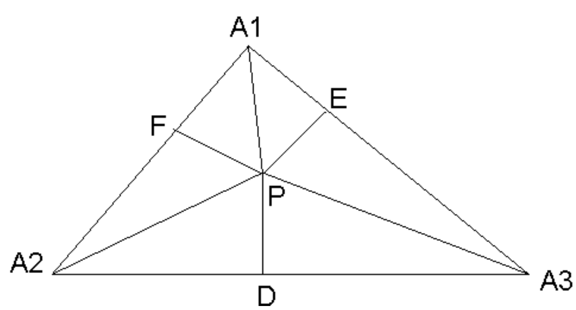

Figure 1.

Proof. As figure 1, let $\angle A_2PA_3=2\alpha$, $\angle A_3PA_1=2\beta$ and $\angle A_1PA_2=2\gamma$, then α, β and γ are the acute angles for $\alpha+\beta+\gamma=\pi$, and

$$w_1\le\sqrt{R_2R_3}\cos\alpha,\ w_2\le\sqrt{R_3R_1}\cos\beta,\ w_3\le\sqrt{R_1R_2}\cos\gamma,$$

with equality holding if and only if P is the circumcenter of triangle $A_1A_2A_3$.

So, now we only need to prove the following inequality:

$$8xy\sqrt{R_2R_3}\cos\alpha\left(x\sqrt{R_3R_1}\cos\beta+y\sqrt{R_1R_2}\cos\gamma\right)^2\leqslant R_1(y^2R_2+x^2R_3)^2, \tag{3.2}$$

or

$$8xy\sqrt{R_2R_3}\cos\alpha\left(x\sqrt{R_3}\cos\beta+y\sqrt{R_2}\cos\gamma\right)^2\leqslant (y^2R_2+x^2R_3)^2. \tag{3.3}$$

Since $R_2>0, R_3>0$, we can take $x\to x\sqrt{R_3}, y\to y\sqrt{R_2}$. Then (3.3) is equivalent to inequality (2.1). From Lemma 2.1, we known inequality (3.1) holds. Thus, the proof of Theorem 3.1 is completed. ∎

Now, we give some corollaries from Theorem 3.1. The proof of Corollary 3.3 will be left to the readers.

Corollary 3.1. *For any point P inside $\triangle A_1A_2A_3$, denote $PA_1=R_1, PA_2=R_2, PA_3=R_3$, and w_1,w_2,w_3 be the bisectors of $\angle A_2PA_3, \angle A_3PA_1, \angle A_1PA_2$, then*

$$8w_1(w_2+w_3)^2\leqslant R_1(R_2+R_3)^2. \tag{3.4}$$

Proof. Let $x=y=1$, the corollary follows from Theorem 3.1 and standard arguments. ∎

Remark 3.1. *Corollary 3.1 is obtained a solution that Jian Liu posed a interesting mobile point geometric inequality conjecture* (3.4) *in* [1].

Corollary 3.2. *For any point P inside $\triangle A_1A_2A_3$, denote $PA_1=R_1, PA_2=R_2, PA_3=R_3$, and r_1,r_2,r_3 the distances from P to BC, CA, AB. If $x,y\geqslant 0$, then we have*

$$8xyr_1(xr_2+yr_3)^2\leqslant R_1(y^2R_2+x^2R_3)^2, \tag{3.5}$$

and

$$8r_1(r_2+r_3)^2\leqslant R_1(R_2+R_3)^2. \tag{3.6}$$

Proof. From $w_i\geqslant r_i,\ i=1,2,3$, Theorem 3.1 and Corollary 3.1, we easily find Corollary 3.2. This is proved. ∎

Corollary 3.3. *For any point P inside $\triangle A_1A_2A_3$, denote $PA_1=R_1, PA_2=R_2, PA_3=R_3$, w_1,w_2,w_3 be the bisectors of $\angle A_2PA_3, \angle A_3PA_1, \angle A_1PA_2$, and r_1,r_2,r_3 the distances from P to BC, CA, AB. If $x,y\geqslant 0$, then we have*

$$\begin{aligned}&8xyr_1(xr_2+yr_3)^2+8yzr_2(yr_3+zr_1)^2+8zxr_3(zr_1+xr_2)^2\\ \leqslant&8xyw_1(xw_2+yw_3)^2+8yzw_2(yw_3+zw_1)^2+8zxw_3(zw_1+xw_2)^2\\ \leqslant&R_1(y^2R_2+x^2R_3)^2+R_2(z^2R_3+x^2R_1)^2+R_3(x^2R_1+z^2R_2)^2,\end{aligned} \tag{3.7}$$

$$\begin{aligned}&8xy\frac{r_1}{R_1}+8yz\frac{r_2}{R_2}+8zx\frac{r_3}{R_3}\\ \leqslant&8xy\frac{w_1}{R_1}+8yz\frac{w_2}{R_2}+8zx\frac{w_3}{R_3}\\ \leqslant&\left(\frac{y^2R_2+x^2R_3}{xr_2+yr_3}\right)^2+\left(\frac{z^2R_3+x^2R_1}{yr_3+zr_1}\right)^2+\left(\frac{x^2R_1+z^2R_2}{zr_1+xr_2}\right)^2,\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}&8r_1(r_2+r_3)^2+8r_2(r_3+r_1)^2+8r_3(r_1+r_2)^2\\ \leqslant&8w_1(w_2+w_3)^2+8w_2(w_3+w_1)^2+8w_3(w_1+w_2)^2\\ \leqslant&R_1(R_2+R_3)^2+R_2(R_3+R_1)^2+R_3(R_1+R_2)^2.\end{aligned} \tag{3.9}$$

4. Some Another Results

Theorem 4.1. *Let A, B, C are the angles of triangle ABC. If $x, y \geqslant 0$, then we have*

$$8xy\sin A(x\cos B - y\sin C)^2 \leqslant (x^2+y^2)^2, \tag{4.1}$$

with equality holding if and only if $x^2+y^2 = 4xy\sin A$ and $x\sin B = y\cos C$; and

$$8xy\cos A(x\sin B - y\sin C)^2 \leqslant (x^2+y^2)^2, \tag{4.2}$$

with equality holding if and only if $x^2+y^2 = 4xy\cos A$, and $x\cos B + y\cos C = 0$.

We only prove inequality (4.1), and the proof of (4.2) will be left to the readers.

Proof. From A, B, C are the angles of triangle ABC and $x, y \geqslant 0$, and utilizing the facts that $\sin^2\theta + \cos^2\theta = 1, \sin A = \sin(B+C) = \sin B\cos C + \cos B\sin C$, we find

$$\begin{aligned}
0 &\leqslant (x^2+y^2-4xy\sin A)^2 + 8xy\sin A(x\sin B - y\cos C)^2 \\
&= (x^2+y^2)^2 - 8xy(x^2+y^2)\sin A + 16(xy)^2\sin^2 A + 8xy\sin A(x^2\sin^2 B - 2xy\sin B\cos C + y^2\cos^2 C) \\
&= (x^2+y^2)^2 - 8xy\sin A[(x^2+y^2) - 2xy\sin A - x^2\sin^2 B + 2xy\sin B\cos C - y^2\cos^2 C] \\
&= (x^2+y^2)^2 - 8xy\sin A(x^2\cos^2 B - 2xy\cos B\sin C + y^2\sin^2 C) \\
&= (x^2+y^2)^2 - 8xy\sin A(x\cos B - y\sin C)^2,
\end{aligned}$$

that is inequality(4.1), with equality holding if and only if $x^2+y^2 = 4xy\sin A$ and $x\sin B = y\cos C$. ∎

By the same way of the section 2, from Theorem 4.1, the following corollary holds.

Corollary 4.1. *Let A, B, C are the angles of an acute triangle. If $x > 0$, then we have*

$$8xy\cos\frac{A}{2}(x\sin\frac{B}{2} - y\cos\frac{C}{2})^2 \leqslant (x^2+y^2)^2, \tag{4.3}$$

with equality holding if and only if $x^2+y^2 = 4xy\cos\frac{A}{2}$ and $x\cos\frac{B}{2} = y\sin\frac{C}{2}$; and

$$8xy\sin\frac{A}{2}(x\cos\frac{B}{2} - y\cos\frac{C}{2})^2 \leqslant (x^2+y^2)^2, \tag{4.4}$$

with equality holding if and only if $x^2+y^2 = 4xy\sin\frac{A}{2}$, and $x\sin\frac{B}{2} + y\sin\frac{C}{2} = 0$.

References

[1] Jian Liu. **CIQ.47**. *Research in Inequalities Communication*. No.7 (2003), page 94. (Chinese)

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