ALGEBRAIC APPROACH TO THE FRACTIONAL DERIVATIVES

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Abstract

In this paper we introduce an alternative definition of the fractional derivatives and also a characteristic class of so called ideal functions, which admit arbitrary fractional derivatives (also integrals). Further are found the expansions of the functions $\frac{xe^x}{e^x-1}$, $\frac{1}{\cos x}$, $x\tanh x$, and some other functions of the form $\sum\limits_{k=-\infty}^{\infty} a_k \frac{x^k}{k!}$, which enables us to calculate any fractional derivative of these functions at $x=0$. These calculations lead to representations of the Bernoulli and Euler numbers B_k and E_k for any complex number k, via fractional derivatives of some functions at $x=0$.

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1 Some new theoretical results for the fractional derivatives.

Several authors have considered and introduced different methods for calculating of fractional derivatives of a given function (see [1]). An old idea for more than 170 years is to use power series and to apply the fractional derivatives to each summand. Later this method was considered by J. Liouville and B. Riemann. Recently it was developed [3, 4], such that

$$\Big(\sum_{i=-\infty}^{\infty} a_i \frac{x^i}{i!}\Big)^{(\alpha)} = \sum_{i=-\infty}^{\infty} a_i \frac{x^{i-\alpha}}{(i-\alpha)!}, \tag{1.1}$$

where $x! = \Gamma(x+1)$. Note that $(-1)! = (-2)! = \cdots = \pm\infty$. The summands for $i \in Z^-$ play important role, although these summands are equal to zero. Indeed, only a special choice of the "left part" yields to satisfactory results, called "natural" representation. For example, the natural representations for the functions e^x, $\sin x$ and $\cos x$ are the following:

$$e^x = \sum_{i=-\infty}^{\infty} \frac{x^i}{i!}, \tag{1.2}$$

$$\sin(x) = \sum_{i=-\infty}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!}, \tag{1.3}$$

$$\cos(x) = \sum_{i=-\infty}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!}, \tag{1.4}$$

such that we obtained [3, 4] from them the usual classical fractional derivatives. In this paper we consider the "natural" representations for the functions $x \cot x$, $\frac{xe^x}{e^x-1}$, $\frac{x}{\sin x}$ and some other functions. In this section we represent an improved version of this idea, by distinguishing a class of analytical functions which have "natural" representations for $i \in Z$.

Now let us assume that an analytical function f can be written in the form

$$f(x) = \sum_{i=-\infty}^{\infty} a_i \frac{x^{\alpha+i}}{(\alpha+i)!}, \qquad (x \neq 0 \text{ if } \alpha \notin Z), \tag{1.5}$$

which means that the sum of the right side (including the summation of divergent series) converges to $f(x)$, for $x \neq 0$. The formal calculation of the $(\alpha+i)$-th derivative at $x = 0$ yields that $f^{(\alpha+i)}(0) = a_i$. Hence

$$f(x) = \sum_{i=-\infty}^{\infty} f^{(\alpha+i)}(0) \frac{x^{\alpha+i}}{(\alpha+i)!}, \tag{1.6}$$

where α is an arbitrary real or complex number. More generally

$$f(x) = \sum_{i=-\infty}^{\infty} f^{(\alpha+i)}(x_0) \frac{(x-x_0)^{\alpha+i}}{(\alpha+i)!}, \qquad (x \neq x_0 \text{ if } \alpha \notin Z), \tag{1.7}$$

which generalizes the ordinary Taylor's series.

On the other hand, if f admits fractional derivatives (integrations are also included) of arbitrary order, let $g = f^{(\alpha)}$. If we write g as a Laurent's series

$$g(x) = \sum_{i=-\infty}^{\infty} a_i \frac{x^i}{i!},$$

then

$$f(x) = g^{(-\alpha)}(x) = \sum_{i=-\infty}^{\infty} a_i \frac{x^{\alpha+i}}{(\alpha+i)!}$$

and f can be written in the required form. Namely, we proved the following proposition.

Proposition 1.1. *If f admits fractional derivatives of arbitrary order, then f satisfies the equalities (1.6) and (1.7).*

The previous discussion naturally yields to the following definition of fractional derivatives.

Definition 1.1. *Assume that an analytical mapping f can be written in the following form*

$$f(x) = \sum_{i=-\infty}^{\infty} C_i \frac{(x-x_0)^{\alpha+i}}{(\alpha+i)!}, \qquad (x \neq x_0 \;\; if \;\; \alpha \notin Z),$$

for each $\alpha \in R$ (or $\alpha \in C$). Then f together with the above representations is called ideal function and we define $f^{(\alpha+i)}(x_0) := C_i$, $(i \in Z)$.

The functions e^x, $\sin(x)$ and $\cos(x)$ are ideal and the corresponding representations for arbitrary α are the following (see [3, 4])

$$e^x = \sum_{i=-\infty}^{\infty} \frac{x^{\alpha+i}}{(\alpha+i)!},$$

$$\sin(x) = \sum_{i=-\infty}^{\infty} \sin\frac{(\alpha+i)\pi}{2} \cdot \frac{x^{\alpha+i}}{(\alpha+i)!},$$

$$\cos(x) = \sum_{i=-\infty}^{\infty} \cos\frac{(\alpha+i)\pi}{2} \cdot \frac{x^{\alpha+i}}{(\alpha+i)!}.$$

Using the new definition we come to a *natural representation* of any ideal function. Namely, let f be such an analytical function and let in Taylor series it is written as $\sum_{i=0}^{\infty} a_i \frac{x^i}{i!}$. The problem is how to find the coefficients $a_{-1}, a_{-2}, \ldots$ form the "zero part" of f. Let $g = f^{(-\alpha)}$ and let b_i, $(i \in Z)$ and $\alpha \notin Z$ are such that

$$g(x) = \sum_{i=-\infty}^{\infty} b_i \frac{x^{\alpha+i}}{(\alpha+i)!}.$$

Then

$$f(x) = g^{(\alpha)}(x) = \sum_{i=-\infty}^{\infty} b_i \frac{x^i}{i!}.$$

Obviously $b_i = a_i$ for $i \in N_0 = N \cup \{0\}$ and we define $a_i = b_i$ for $i \in Z^-$. Hence f is written in the required "natural representation" $\sum_{i=-\infty}^{\infty} a_i \frac{x^i}{i!}$. Moreover, the coefficients $a_{-1}, a_{-2}, \ldots$ do not depend on the choice of α. Note that if we know the natural representation of an ideal function f, then all fractional derivatives of f are known. So the main problem in examining of an ideal function is to find its natural representation. Note that natural representation may exists also if the function is not ideal. In that case we can use it for calculating the fractional derivatives according to the old definition of [3, 4].

Looking at this theory axiomatically, we have a class $\mathcal{I}$ (ideal functions) of analytical functions f, such that

(i) for each $f \in \mathcal{I}$ and each $\alpha \in (C \setminus Z)$ there exists the decomposition

$$f(x) = \sum_{i=-\infty}^{\infty} C_i \frac{(x-x_0)^{\alpha+i}}{(\alpha+i)!}, \tag{1.8}$$

which has the same meaning discussed for (1.5), such that t should be replaced also by $t - x_0$, and where we define $C_i = f^{(\alpha+i)}(x_0)$; if $\alpha \in Z$, we choose the natural representation

$$f(x) = \sum_{i=-\infty}^{\infty} C_i \frac{(x-x_0)^i}{i!}.$$

(ii) If $f \in \mathcal{I}$, then

$$f^{(\beta)} := \sum_{i=-\infty}^{\infty} C_i \frac{(x-x_0)^{\alpha+i-\beta}}{(\alpha+i-\beta)!} \in \mathcal{I}, \quad (\beta \in C),$$

where f is given by (1.8).

To the end of this section we give some properties of the ideal functions.

1. The set of ideal functions $\mathcal{I}$ can be separated in a quotient set $\mathcal{I}/\sim$, where the equivalence relation $\sim$ is defined by $f \sim g$ iff there exists $\alpha \in C$, such that $f^{(\alpha)} = g$. Each such class determines unique sequence a_i, $(i \in Z)$, such that $\sum_{i=-\infty}^{\infty} a_i \frac{x^i}{i!} \in \mathcal{I}$. Namely, then

$$\sum_{i=-\infty}^{\infty} a_i \frac{x^{\alpha+i}}{(\alpha+i)!} \sim \sum_{i=-\infty}^{\infty} a_i \frac{x^i}{i!},$$

for arbitrary $\alpha \in C$. Moreover, from the definition of ideal functions it follows that $x^n f(x)$ is an ideal function if $f(x)$ is an ideal function and n is a positive integer.

2. $\mathcal{I}$ is a nonempty set because $e^x, \sin x, \cos x \in \mathcal{I}$. The zero function is also an ideal function. The set of ideal functions is a vector space, such that if $f, g \in \mathcal{I}$, then $\lambda f, f+g \in \mathcal{I}$. Note that also $f(\lambda x) \in \mathcal{I}$ if $\lambda \neq 0$ and $f(x+\lambda) \in \mathcal{I}$.

3. If P is a polynomial, $P(x) = \sum_{k=-\infty}^{n} a_k \frac{x^k}{k!}$, with known coefficients a_0, $a_1, \cdots, a_n$, then a_{-1}, a_{-2}, $a_{-3}, \cdots$ are not uniquely determined such that P is an ideal function. In the remark of section 4 is constructed a wide class of polynomials $\mathcal{P}$ which are ideal functions.

2 Expanding of the Bernoulli numbers.

Now we shall expand the Bernoulli numbers B_k for integer k and also for complex number k. Although the Bernoulli numbers are defined by the development

$$\frac{x}{e^x-1} = B_0 + B_1\frac{x^1}{1!} + B_2\frac{x^2}{2!} + B_3\frac{x^3}{3!} + \cdots$$

it appears more convenient to consider modified Bernoulli numbers B_i^* via the development of the function $\frac{x}{e^x-1} + x = \frac{xe^x}{e^x-1}$ as

$$\frac{xe^x}{e^x-1} = B_0^* + B_1^*\frac{x^1}{1!} + B_2^*\frac{x^2}{2!} + B_3^*\frac{x^3}{3!} + \cdots$$

Now we get: $B_0^* = 1$, $B_1^* = \frac{1}{2}$, $B_2^* = \frac{1}{6}$, $B_4^* = -\frac{1}{30}$, $B_6^* = \frac{1}{42}$, $B_8^* = -\frac{1}{30}$, $B_{10}^* = \frac{5}{66}, \ldots$, while $B_3^* = B_5^* = B_7^* = B_9^* = \cdots = 0$. Note that $B_i = B_i^*$ for $i \neq 1$ and $B_1 = -B_1^*$. According to the formula (2.4.3) in [2, p.19] we have

$$B_{2m}^* = -2m\zeta(1-2m), \qquad m = 1,2,3\cdots \tag{2.1}$$

where ζ is the Riemann Zeta function, i.e. its analytical continuation. Since, $\zeta(-2m) = 0$ for $m = 1,2,3,\cdots$, from (2.1) we can write

$$B_p^* = -p\cdot\zeta(1-p), \qquad p = 1,2,3,\cdots. \tag{2.2}$$

Moreover, we accept by definition that for each complex number α,

$$B_\alpha^* = -\alpha\cdot\zeta(1-\alpha), \qquad \text{for} \qquad |\zeta(1-\alpha)| \neq \infty \tag{2.3}$$

Further we shall consider the recurrent relations for the modified Bernoulli numbers. Starting from the ordinary equality

$$xe^x = (e^x-1)\cdot(B_0^* + B_1^*\frac{x^1}{1!} + B_2^*\frac{x^2}{2!} + B_3^*\frac{x^3}{3!} + \ldots)$$

and equalizing the coefficients of both sides in front of x^{n+1} we obtain:

$$\frac{1}{n!} = \frac{B_n^*}{n!1!} + \frac{B_{n-1}^*}{(n-1)!2!} + \frac{B_{n-2}^*}{(n-2)!3!} + \ldots + \frac{B_0^*}{0!(n+1)!}$$

and hence by multiplication with $(n+1)!$ we get

$$n+1 = B_n^*\binom{n+1}{n} + B_{n-1}^*\binom{n+1}{n-1} + B_{n-2}^*\binom{n+1}{n-2} + \ldots + B_0^*\binom{n+1}{0}, \quad n \in N_0.$$

This formula together with $B_0^* = 1$ is the recurrent relation in finding the sequence of Bernoulli numbers whose indices are nonnegative integers. Rewriting this equality in the form

$$n+1 = \sum_{k=1}^{\infty}\binom{n+1}{n+1-k}B_{n+1-k}^* \tag{2.4}$$

we prove the following proposition.

Proposition 2.1. *The equality (2.4) is true for each integer number* n.

Proof. If n is a nonnegative integer, we saw that it is true in the previous discussion. It is also true for $n=-1$ because it reduces to $0=0$. So, we should prove it for $n \in \{-2,-3,-4,\cdots\}$.

Using the equality (2.2) we obtain

$$\sum_{k=1}^{\infty}\binom{n+1}{k}B^*_{n+1-k}=\sum_{k=1}^{\infty}\binom{n+1}{k}\sum_{i=1}^{\infty}(-1)^{i+1}\frac{n+1-k}{2^{n+1-k}-1}i^{n-k}=$$

$$=\sum_{k=1}^{\infty}\binom{n+1}{k}\frac{n+1-k}{2^{n+1-k}-1}\sum_{i=1}^{\infty}(-1)^{i+1}i^{n-k}=$$

$$=\sum_{k=1}^{\infty}\binom{n+1}{k}\frac{n+1-k}{2^{n+1-k}-1}\cdot\zeta(k-n)(1-2^{1-k+n}).$$

Now, let us consider the function $f(x)=\sum\limits_{k=1}^{\infty}\binom{n+1}{k}(k-n-1)x^{k-n}$ and later find $f(\frac{1}{j})$. We notice that

$$\frac{f(x)}{x^2}=\sum_{k=1}^{\infty}\binom{n+1}{k}(k-n-1)x^{k-n-2}$$

and hence

$$\int\frac{f(x)}{x^2}dx=\sum_{k=1}^{\infty}\binom{n+1}{k}x^{k-n-1}=\Big(\frac{1}{x}\Big)^{n+1}=$$

$$=\sum_{k=1}^{\infty}\binom{n+1}{k}x^k=\frac{1}{x^{n+1}}[(1+x)^{n+1}-1]=\Big(1+\frac{1}{x}\Big)^{n+1}-\frac{1}{x^{n+1}}.$$

Hence it is easy to obtain

$$f(x)=x^2\Big[\Big(1+\frac{1}{x}\Big)^{n+1}-\frac{1}{x^{n+1}}\Big]'=(n+1)\Big[\frac{1}{x^n}-\Big(1+\frac{1}{x}\Big)^n\Big],$$

and since $n\le -2$, we obtain finally

$$\sum_{k=1}^{\infty}\binom{n+1}{k}B^*_{n+1-k}=\sum_{j=1}^{\infty}f\Big(\frac{1}{j}\Big)=(n+1)\sum_{j=1}^{\infty}[j^n-(1+j)^n]=(n+1).$$

3 Natural representation of $1/\cos x$ and Euler numbers.

In this section we shall consider the function

$$\frac{1}{\cos x} = \sum_{n=0}^{\infty} E_{2n} \frac{x^{2n}}{(2n)!}, \qquad |x| < \frac{\pi}{2}.$$

The numbers E_{2k} satisfy the recurrent relation

$$E_0 = 1, \quad E_0 - \binom{2n}{2} E_2 + \binom{2n}{4} E_4 - \ldots + (-1)^n \binom{2n}{2n} E_{2n} = 0$$

and the numbers $(-1)^k E_k$ are known as Euler numbers.

We start from the development of the function $\frac{1}{\cos x}$ in Fourier series

$$\frac{1}{\cos x} = 2 \sum_{n=1}^{\infty} \cos(n-1)\frac{\pi}{2} \cdot \cos nx.$$

This equality implies the following consequence. Replacing $\cos nx = \sum\limits_{k=-\infty}^{\infty} (-1)^k n^{2k} \frac{x^{2k}}{(2k)!}$, we obtain

$$\frac{1}{\cos x} = 2 \sum_{n=1}^{\infty} \cos(n-1)\frac{\pi}{2} \sum_{k=-\infty}^{\infty} (-1)^k n^{2k} \frac{x^{2k}}{(2k)!} =$$

$$= \sum_{k=-\infty}^{\infty} (-1)^k 2 \Big(\sum_{n=1}^{\infty} n^{2k} \cdot \cos(n-1)\frac{\pi}{2} \Big) \frac{x^{2k}}{(2k)!} = \sum_{k=-\infty}^{\infty} E_{2k} \frac{x^{2k}}{(2k)!},$$

where $E_{2k} = (-1)^k 2 \Big(\sum\limits_{n=1}^{\infty} n^{2k} \cdot \cos(n-1)\frac{\pi}{2} \Big)$. Hence we obtain the required natural representation of $\frac{1}{\cos x}$. Note that the numbers E_{-2k} can easily be calculated numerically, because

$$E_{-2k} = (-1)^k 2 \Big(1 - \frac{1}{3^{2k}} + \frac{1}{5^{2k}} - \frac{1}{7^{2k}} + \ldots \Big), \qquad k \in N.$$

The recurrent relation for $E_{-2}, E_{-4}, E_{-6}, \ldots$ is given by

$$E_{-2k} - \binom{2k+1}{2} E_{-2k-2} + \binom{2k+3}{4} E_{-2k-4} - \binom{2k+5}{6} E_{-2k-6} + \ldots = 0, \ k \in N$$

and it can be proved analogously as it is proved for the Bernoulli numbers B_k^* (Proposition 2.1).

If $\frac{1}{\cos x}$ is an ideal function, then for arbitrary α, the α-th number E_α should satisfy

$$E_\alpha = \frac{d^\alpha}{dx^\alpha} \Big(\frac{1}{\cos x} \Big) \Big|_{x=0},$$

and using the Fourier series it is easy to obtain

$$E_\alpha = 2\cos\frac{\alpha\pi}{2}\sum_{n=1}^{\infty} n^\alpha \cdot \cos(n-1)\frac{\pi}{2}.$$

Specially, $E_{2k+1} = 0$ for $k \in Z$.

Proposition 3.1. *$\frac{1}{\cos x}$ is an ideal function, such that*

$$\frac{1}{\cos x} = \sum_{j=-\infty}^{\infty} E_{\alpha+j}\frac{x^{\alpha+j}}{(\alpha+j)!}.$$

Proof.

$$\sum_{j=-\infty}^{\infty} E_{\alpha+j}\frac{x^{\alpha+j}}{(\alpha+j)!}$$

$$= \sum_{j=-\infty}^{\infty} 2\cos\frac{(\alpha+j)\pi}{2}\Big(\sum_{n=1}^{\infty} n^{\alpha+j}\cdot\cos(n-1)\frac{\pi}{2}\Big)\frac{x^{\alpha+j}}{(\alpha+j)!}$$

$$= \sum_{n=1}^{\infty}\cos(n-1)\frac{\pi}{2}\sum_{j=-\infty}^{\infty} 2\cos\frac{(\alpha+j)\pi}{2}n^{\alpha+j}\cdot\frac{x^{\alpha+j}}{(\alpha+j)!}$$

$$= \sum_{n=1}^{\infty}\cos(n-1)\frac{\pi}{2}\sum_{j=-\infty}^{\infty}\Big[e^{i\frac{(\alpha+j)\pi}{2}} + e^{-i\frac{(\alpha+j)\pi}{2}}\Big]n^{\alpha+j}\cdot\frac{x^{\alpha+j}}{(\alpha+j)!}$$

$$= \sum_{n=1}^{\infty}\cos(n-1)\frac{\pi}{2}\sum_{j=-\infty}^{\infty}\Big[\frac{(e^{i\pi/2}\cdot nx)^{\alpha+j}}{(\alpha+j)!} + \frac{(e^{-i\pi/2}\cdot nx)^{\alpha+j}}{(\alpha+j)!}\Big]$$

$$= \sum_{n=1}^{\infty}\cos(n-1)\frac{\pi}{2}(e^{inx}+e^{-inx}) = 2\sum_{n=1}^{\infty}\cos(n-1)\frac{\pi}{2}\cos nx = \frac{1}{\cos x}.$$

Corollary 3.1. *The function $\frac{1}{\cosh x}$ is an ideal function, such that*

$$\frac{1}{\cosh x} = \sum_{i=-\infty}^{\infty}\frac{(-1)^{\alpha+i}E_{\alpha+i}}{(\alpha+i)!}x^{\alpha+i}.$$

Proof. The function $\frac{1}{\cosh x}$ is an ideal function because $\frac{1}{\cosh x} = f(ix)$, where $f(x) = \frac{1}{\cos x}$.

4 Representations and natural representations of some other ideal functions.

In this section we shall consider some functions whose coefficients of the Taylor series contain Bernoulli numbers.

Proposition 4.1. *The function* $\frac{x}{e^x-1}$ *is an ideal function, such that*

$$\frac{x}{e^x-1} = \sum_{i=-\infty}^{\infty} (-1)^{i+\alpha} B^*_{\alpha+i} \frac{x^{\alpha+i}}{(\alpha+i)!}, \qquad \alpha \in C. \tag{4.1}$$

Proof. Using the identity (2.3) we obtain

$$\sum_{i=-\infty}^{\infty} (-1)^{i+\alpha} B^*_{\alpha+i} \frac{x^{\alpha+i}}{(\alpha+i)!} = \sum_{i=-\infty}^{\infty} (-1)^{i+\alpha} \Big[\sum_{n=1}^{\infty} -(\alpha+i)n^{\alpha+i-1}\Big] \frac{x^{\alpha+i}}{(\alpha+i)!}$$

$$= x\sum_{n=1}^{\infty}\sum_{i=-\infty}^{\infty} \frac{(-nx)^{\alpha+i-1}}{(\alpha+i-1)!} = x\sum_{n=1}^{\infty} e^{-nx} = x\frac{e^{-x}}{1-e^{-x}} = \frac{x}{e^x-1}.$$

Proposition 4.2. *The function* $\frac{xe^x}{e^x-1}$ *is an ideal function, such that*

$$\frac{xe^x}{e^x-1} = \sum_{i=-\infty}^{\infty} B^*_{\alpha+i} \frac{x^{\alpha+i}}{(\alpha+i)!}, \qquad \alpha \in C. \tag{4.2}$$

Proof. Namely, $\frac{xe^x}{e^x-1}$ is an ideal function because $\frac{xe^x}{e^x-1} = f(-x)$, where $f(x) = \frac{x}{e^x-1}$ is an ideal function.

Corollary 4.1. *For each* $\alpha \in C$*,*

$$\frac{d^\alpha}{dx^\alpha}\Big(\frac{xe^x}{e^x-1}\Big)\Big|_{x=0} = B^*_\alpha. \tag{4.3}$$

Remark. In Propositions 4.1 and 4.2 were obtained the natural representations for $\frac{x}{e^x-1}$ and $\frac{xe^x}{e^x-1}$:

$$\frac{x}{e^x-1} = \sum_{i=-\infty}^{\infty} (-1)^i B^*_i \frac{x^i}{i!}, \qquad \frac{xe^x}{e^x-1} = \sum_{i=-\infty}^{\infty} B^*_i \frac{x^i}{i!}.$$

Both functions $\frac{xe^x}{e^x-1}$ and $\frac{x}{e^x-1}$ with the corresponding representations are ideal and hence their difference

$$h(x) = \frac{xe^x}{e^x-1} - \frac{x}{e^x-1} = x + 2B^*_{-1}\frac{x^{-1}}{(-1)!} + 2B^*_{-3}\frac{x^{-3}}{(-3)!} + 2B^*_{-5}\frac{x^{-5}}{(-5)!} + \cdots \tag{4.4}$$

is also an ideal function.

This function h generates a family of ideal polynomials $\mathcal{P}$, using the following generator transformations:

i) if $P \in \mathcal{P}$, then $P^{(\alpha)} \in \mathcal{P}$ for each $\alpha \in Z$,

ii) if $P \in \mathcal{P}$, then $x^n P \in \mathcal{P}$ for each nonnegative integer n.

iii) if $P, Q \in \mathcal{P}$, then $\lambda P + \mu Q \in \mathcal{P}$ for each scalars λ, μ.

Proposition 4.3. *The function* $x \cdot \cot x$ *is an ideal function, such that*

$$x \cdot \cot x = \sum_{j=-\infty}^{\infty} 2^{\alpha+j} \cos\frac{(\alpha+j)\pi}{2} \cdot B^*_{\alpha+j} \frac{x^{\alpha+j}}{(\alpha+j)!}, \quad \alpha \in C. \tag{4.5}$$

Proof. A simple transformation shows that $x \cdot \cot x = \frac{f(2ix)+f(-2ix)}{2}$, where $f(x) = \frac{xe^x}{e^x-1} = \sum_{k=-\infty}^{\infty} B_k^* \frac{x^k}{k!}$.

Hence the function $x \cot x$ is an ideal function and its natural representation is given by

$$x \cdot \cot x = \sum_{k=-\infty}^{\infty} (-1)^k 2^{2k} \cdot B_{2k}^* \cdot \frac{x^{2k}}{(2k)!}. \tag{4.6}$$

According to Proposition 4.2, the general representation for arbitrary α is given by

$$x \cdot \cot x = \frac{1}{2} \sum_{j=-\infty}^{\infty} \Big(B_{\alpha+j}^* \frac{(2ix)^{\alpha+j}}{(\alpha+j)!} + B_{\alpha+j}^* \frac{(-2ix)^{\alpha+j}}{(\alpha+j)!} \Big) =$$

$$= \sum_{j=-\infty}^{\infty} 2^{\alpha+j} \cos \frac{(\alpha+j)\pi}{2} \cdot B_{\alpha+j}^* \frac{x^{\alpha+j}}{(\alpha+j)!}.$$

As a consequence we obtain the following representation for the Bernoulli numbers.

Corollary 4.2. *For each* $\alpha \in C$,

$$\frac{d^\alpha}{dx^\alpha}(x \cdot \cot x)\Big|_{x=0} = 2^\alpha \cos \frac{\alpha\pi}{2} \cdot B_\alpha^*. \tag{4.7}$$

Proposition 4.4. *The function* $\frac{x}{\sin x}$ *is an ideal function, such that*

$$\frac{x}{\sin x} = \sum_{i=-\infty}^{\infty} B_{\alpha+i}^* (2 - 2^{\alpha+i}) \cos \frac{(\alpha+i)\pi}{2} \cdot \frac{x^{\alpha+i}}{(\alpha+i)!}, \quad \alpha \in C. \tag{4.8}$$

Proof. We start from the identity $\frac{1}{\sin x} = \cot \frac{x}{2} - \cot x$ and thus $\frac{x}{\sin x}$ is an ideal function. Further, from

$$\frac{x}{\sin x} = 2\frac{x}{2} \cdot \cot \frac{x}{2} - x \cdot \cot x \tag{4.9}$$

and from (4.6) we obtain the natural representation of $\frac{x}{\sin x}$:

$$\frac{x}{\sin x} = \sum_{k=-\infty}^{\infty} (-1)^k (2 - 2^{2k}) B_{2k}^* \frac{x^{2k}}{(2k)!}.$$

Using the formulas (4.9) and (4.5), the general representation for arbitrary α is given by (4.8).

Corollary 4.3. *For each* $\alpha \in C$,

$$\frac{d^\alpha}{dx^\alpha}\Big(\frac{x}{\sin x}\Big)\Big|_{x=0} = (2 - 2^\alpha) \cos \frac{\alpha\pi}{2} \cdot B_\alpha^*. \tag{4.10}$$

Proposition 4.5. *The function* $\frac{x}{e^x+1}$ *is an ideal function such that*

$$\frac{x}{e^x+1} = \sum_{i=-\infty}^{\infty} (-1)^{i+\alpha}(1-2^{\alpha+i})B^*_{\alpha+i}\frac{x^{\alpha+i}}{(\alpha+i)!}.$$

Proof. Using the identity (2.2), we obtain

$$\sum_{i=-\infty}^{\infty} (-1)^{\alpha+i}(1-2^{\alpha+i})B^*_{\alpha+i}\frac{x^{\alpha+i}}{(\alpha+i)!} =$$

$$= \sum_{i=-\infty}^{\infty} (-1)^{\alpha+i}(1-2^{\alpha+i})\Big[-\sum_{n=1}^{\infty}(\alpha+i)n^{\alpha+i-1}\Big]\cdot\frac{x^{\alpha+i}}{(\alpha+i)!} =$$

$$= x\sum_{n=1}^{\infty}\sum_{i=-\infty}^{\infty}\frac{(-nx)^{\alpha+i-1}(-2^{\alpha+i}+1)}{(\alpha+i-1)!} =$$

$$= x\sum_{n=1}^{\infty}\sum_{i=-\infty}^{\infty}\frac{(-nx)^{\alpha+i-1}}{(\alpha+i-1)!} - 2x\sum_{n=1}^{\infty}\sum_{i=-\infty}^{\infty}\frac{(-2nx)^{\alpha+i-1}}{(\alpha+i-1)!} =$$

$$= x\sum_{n=1}^{\infty}e^{-nx} - 2x\sum_{n=1}^{\infty}e^{-2nx} = \frac{x}{e^x-1} - \frac{2x}{e^{2x}-1} = \frac{x}{e^x+1}.$$

Corollary 4.4. *The function* $\frac{xe^x}{e^x+1}$ *is an ideal function, such that*

$$\frac{xe^x}{e^x+1} = \sum_{i=-\infty}^{\infty}(2^{\alpha+i}-1)B^*_{\alpha+i}\frac{x^{\alpha+i}}{(\alpha+i)!}, \tag{4.11}$$

whence

$$\frac{d^\alpha}{dx^\alpha}\Big(\frac{xe^x}{e^x+1}\Big)\Big|_{x=0} = (2^\alpha-1)B^*_\alpha. \tag{4.12}$$

Proof. The function $f(x) = \frac{x}{e^x+1}$ is an ideal function, and hence $\frac{xe^x}{e^x+1} = -f(-x)$ is also an ideal function, i.e. the equalities (4.11) and (4.12) hold.

Proposition 4.6. *The function* $x\tanh x$ *is an ideal function such that*

$$x\tanh x = \sum_{i=-\infty}^{\infty}(2^{\alpha+i}-1)B^*_{\alpha+i}\cdot 2^{\alpha+i}\frac{x^{\alpha+i}}{(\alpha+i)!}.$$

Proof.

$$x\tanh x = x\frac{e^x-e^{-x}}{e^x+e^{-x}} = x\frac{e^{2x}-1}{e^{2x}+1} = -\frac{f(2x)+f(-2x)}{2}, \tag{4.13}$$

where $f(x) = \frac{x}{e^x+1}$. According to Proposition 4.5 by (4.13) we obtain that $x\tanh x$ is also an ideal function.

Since

$$f(x) = \sum_{i=-\infty}^{\infty} (-1)^i (1-2^i) B_i^* \frac{x^i}{i!},$$

we obtain

$$f(2x)+f(-2x) = \sum_{i=-\infty}^{\infty} (1-2^i) B_i^* \frac{2^i x^i}{i!} [(-1)^i+1] = 2 \sum_{i=-\infty}^{\infty} (1-2^{2i}) B_{2i}^* 2^{2i} \frac{x^{2i}}{(2i)!},$$

i.e.

$$x \tanh x = \sum_{i=-\infty}^{\infty} (2^{2i}-1) B_{2i}^* \cdot 2^{2i} \frac{x^{2i}}{(2i)!}.$$

According to Proposition 4.5, the general representation for arbitrary α is given by

$$x \tanh x = \sum_{i=-\infty}^{\infty} (2^{\alpha+i}-1) B_{\alpha+i}^* \cdot 2^{\alpha+i} \frac{x^{\alpha+i}}{(\alpha+i)!}.$$

Corollary 4.5. *For each $\alpha \in C$,*

$$\frac{d^\alpha}{dx^\alpha}(x \tanh x)\Big|_{x=0} = (2^\alpha - 1) B_\alpha^* \cdot 2^\alpha.$$

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