THE WEIGHTED ELEMENTARY SYMMETRIC MEAN

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ABSTRACT. We give a definition of the weighted elementary symmetric mean, and obtain a class of inequalities for the same mean.

1. Introduction

Definition 1.1. *Let* $a = (a_1, a_2, \cdots, a_n)$*, where* $a_i, 1 \leqslant i \leqslant n$ *be non-negative real numbers, and let* r *be an integer for* $1 \leqslant r \leqslant n$*, then*

$$e_r = e_r(a) = \sum_{0 \leqslant i_1, i_2, \cdots, i_n \leqslant 1}^{i_1+i_2+\cdots+i_n=r} \prod_{j=1}^{n} a_{i_j} \tag{1.1}$$

is called the r*-th elementary symmetric function of* a*, where the sum is over all* $\binom{n}{r}$ n*-tuples of non-negative integers with* $i_1 + i_2 + \cdots + i_n = r$*, and* $0 \leq i_1, i_2, \cdots, i_n \leq 1$*. In addition* $e_0 = 1$*. We note that*

$$p_r = p_r(a) = \frac{e_r}{\binom{n}{r}}, \tag{1.2}$$

$(r = 0, 1, \cdots, n)$ *is called the* r*-th elementary symmetric mean of* a*.*

Theorem 1.1. (*Newton* [1] *and Maclaurin* [2]) *Let* $a = (a_1, a_2, \cdots, a_n)$ *be a group of* n *non-negative real numbers, and* r *be an integer with* $1 \leqslant r \leqslant n$*, then*

$$p_r^2(a) \geqslant p_{r-1}(a) p_{r+1}(a) \tag{1.3}$$

and

$$[p_r(a)]^{1/r} \geqslant [p_{r+1}(a)]^{1/(r+1)} \tag{1.4}$$

with both equalities holding if and only if $a_1 = a_2 = \cdots = a_n$*.*

The inequality (1.4) can be deduced from (1.3), since

$$p_1^2 p_2^4 p_3^6 \cdots p_r^{2r} \geqslant (p_0 p_2)(p_1 \, p_3)^2 (p_2 p_4)^3 \cdots (p_{r-1} p_{r+1})^r$$

gives $p_r^{r+1} \geqslant p_{r+1}^r$, this is equivalent inequality (1.4).

In this article, the weighted r-th elementary symmetric function and the same mean of a for the positive weight number $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ be defined, the weighted formula of inequalities (1.3) and (1.4) be obtained.

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2. Main Results

Definition 2.1. *Let a be a group of n non-negative real numbers, λ be a group of n positive weighted numbers, and r be an integer for $1 \leqslant r \leqslant n-1$,then*

$$e_r(a,\lambda) = \sum_{0\leqslant i_1,i_2,\cdots,i_n\leqslant 1}^{i_1+i_2+\cdots+i_n=r} (\sum_{k=1}^{n} \lambda_k - \sum_{j=1}^{r} \lambda_{i_j}) \prod_{j=1}^{n} a_{i_j} \tag{2.1}$$

is called the r-th weighted elementary symmetric function of a for the positive weight λ, where the sum is over all $\binom{n}{r}$ n-tuples of non-negative integers with $i_1+i_2+\cdots+i_n = r$,and $0 \leqslant i_1,i_2,\cdots,i_n \leqslant 1$. In addition we have $e_0(a,\lambda) = \sum_{i=1}^{n}\lambda_i$. And the r-th weighted elementary symmetric mean of a for the positive weight λ is defined by

$$p_r(a,\lambda) = \frac{e_r(a,\lambda)}{\binom{n-1}{r} \sum_{i=1}^{n} \lambda_i} \tag{2.2}$$

$(r = 0, 1, \cdots, n-1)$.

To prove the following Theorem2.1, the following lemma [3] will be used.

Lemma 2.1. *if the real polynomials of n-order*

$$f(x) = \sum_{i=0}^{n} c_i x^i \tag{2.3}$$

which have only real roots, and $c_i = \binom{n}{i} d_i, (1 \leqslant i \leqslant n)$, then

$$d_i^2 - d_{i-1}d_{i+1} \geqslant 0 \quad (i = 1, 2, \cdots, n-1), \tag{2.4}$$

over and above $x_1 = x_2 = \cdots = x_n$, the equality (2.4) be rigorous. Where $x_i (1 \leqslant i \leqslant n)$ are the real roots of the polynomials (2.3).

Theorem 2.1. *Let $a = (a_1, a_2, \cdots, a_n)$ be a group of n non-negative real numbers, λ is a group of n positive weighted, and let r be an integer, $1 \leqslant r < n-1$, then*

$$p_r^2(a,\lambda) \geqslant p_{r-1}(a,\lambda) p_{r+1}(a,\lambda) \tag{2.5}$$

and

$$[p_r(a,\lambda)]^{1/r} \geqslant [p_{r+1}(a,\lambda)]^{1/(r+1)} \tag{2.6}$$

with both equalities holding if and only if $a_1 = a_2 = \cdots = a_n$.

Proof. Set a polynomials

$$f(x) = \sum_{i=1}^{n} \lambda_i \prod_{\substack{j=1 \\ j\neq i}}^{n} (a_j + x) = \sum_{i=0}^{n-1} c_i x^i \tag{2.7}$$

In addition $p_0(a,\lambda) = 1$, and if $1 \leqslant i \leqslant n-1$, then we have

$$c_i = \binom{n-1}{n-i-1} (\sum_{j=1}^{n} \lambda_j) p_{n-i-1}(\mathrm{a};\lambda) = \binom{n-1}{i} (\sum_{j=1}^{n} \lambda_j) p_{n-i-1}(\mathrm{a};\lambda)$$

Let $0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$, then we have

$$f(-\infty) \begin{cases} > 0, \text{if} \quad n \quad \text{be a even number,} \\ < 0, \text{if} \quad n \quad \text{be a odd number,} \end{cases}$$

$$f(-a_1)=\lambda_1\prod_{j=2}^{n}(a_j-a_1)\geqslant 0,$$

$$f(-a_2)=\lambda_2(a_1-a_2)\prod_{j=3}^{n}(a_j-a_2)\leqslant 0$$

$$f(-a_2)\leqslant 0, f(-x_3)\geqslant 0,\cdots,$$

$$f(-a_n)\begin{cases} >0, \text{if} \quad n \quad \text{be a even number,} \\ <0, \text{if} \quad n \quad \text{be a odd number,}\end{cases}$$

and $f(+\infty)>0$. That is polynomials (2.3) which have $(n-1)'$s roots in the real set $\mathbf{R}$. If $x_i(1\leqslant i<n)$ be the real roots of the polynomials (2.3), and let $x_1\leqslant x_2\leqslant\cdots\leqslant x_{n-1}$, then

$$-a_n\leqslant x_1\leqslant -a_{n-1}\leqslant x_2\leqslant\cdots\leqslant -a_2\leqslant x_{n-1}\leqslant -a_1$$

and

$$f(x)=\sum_{i=1}^{n}\lambda_i\prod_{\substack{j=1\\ j\neq i}}^{n}(a_j+x)=(\sum_{i=1}^{n}\lambda_i)\prod_{i=1}^{n-1}(x-x_j) \tag{2.8}$$

From Lemma2.1, we have

$$[(\sum_{j=1}^{n}\lambda_j)p_{n-i-1}(a;\lambda)]^2-[(\sum_{j=1}^{n}\lambda_j)p_{n-i-2}(a;\lambda)][(\sum_{j=1}^{n}\lambda_j)p_{n-i}(a;\lambda)]\geqslant 0$$

i.e.

$$p_{n-i-1}^2(a;\lambda)\geqslant p_{n-i-2}(a;\lambda)p_{n-i}(a;\lambda)(1\leqslant i\leqslant n-1).$$

That is the inequality (2.5), with equality holding if and only if $(n-1)'$s real roots $x_i(1\leqslant i<n)$ of the polynomials (2.3) be equal, or $x_1=x_2=\cdots=x_{n-1}$, and $a_2=a_3=\cdots=a_{n-1}$, therefore

$$f(x)=(\sum_{i=1}^{n}\lambda_i)(x-x_1)^{n-1} \tag{2.9}$$

and

$$f(x)=\sum_{i=1}^{n}\lambda_i\prod_{\substack{j=1\\ j\neq i}}^{n}(a_j+x)=(\sum_{i=0}^{n}\lambda_i)(a_2+x)\left(\frac{\lambda_1a_n+a_2\sum\limits_{j=2}^{n}\lambda_j+\lambda_na_1}{\sum\limits_{i=2}^{n}\lambda_i}+x\right) \tag{2.10}$$

From (2.9), (2.10) and $a_2=-x_1$, we have

$$a_2=\frac{\lambda_1a_n+a_2\sum\limits_{j=2}^{n}\lambda_j+\lambda_na_1}{\sum\limits_{i=1}^{n}\lambda_i}$$

that is

$$\lambda_1(a_2-a_n)+\lambda_n(a_2-a_1)=0. \tag{2.11}$$

Pay attention to arbitrary λ_1,λ_n, the equality (2.11) is true, therefore $a_1=a_n=a_2$. That be said the inequality (2.5) with equality holding if and only if $a_1=a_2=\cdots=a_n$.The proof of inequality (2.5) is completed.

We may discuss the inequality (2.6) in the same way as in the part 1 that proof of the inequality (2.5). □

Let $\lambda_1 = \lambda_2 = \cdots = \lambda_n$, then the inequalities (2.5) and (2.6) become the inequalities (1.3) and (1.4), respectively.

3. Concluding Remarks

R.E.Muirhead in [4] and A.E.Jolliffe in [5] respectively obtained the following two identities (3.1) and (3.2):

Theorem 3.1. *If p_r be the r-th elementary symmetric means of a , then the following identity holds*

$$p_r^2 - p_{r-1}p_{r+1} = \frac{1}{r(r+1)\binom{n}{r}\binom{n}{r+1}} \sum_{k=0}^{n-1} \binom{2k}{k} \frac{(k,k)}{k+1} \tag{3.1}$$

where

$$(k,k) = \sum (\prod_{j=1}^{r-k-1} a_j^2)(\prod_{j=r-k}^{r+k-1} a_j)(a_{r+k} - a_{r+k_1})^2$$

the summation extending over all such products obtainable from $a_1, a_2, \cdots, a_n$.

Theorem 3.2. *If p_r be the r-th elementary symmetric means of a , then the following identity holds*

$$\begin{aligned} &\Big[\frac{n!}{(r-1)!(n-r-1)!}\Big]^2 (p_r^2 - p_{r-1}p_{r+1}) \\ &= (n-1)\sum (a_1-a_2)^2 (c_{r-1}^{n-2})^2 + \frac{2!(n-3)}{(r-1)(n-r-1)} \sum (a_1-a_2)^2(a_3-a_4)^2(c_{r-2}^{n-4})^2 \\ &+ \frac{3!(n-5)}{(r-1)(r-2)(n-r-1)(n-r-2)} \sum (a_1-a_2)^2(a_3-a_4)^2(a_5-a_6)^2(c_{r-3}^{n-6})^2 + \cdots \end{aligned} \tag{3.2}$$

where $c_{r-1}^{n-2}$is the sum terms of the product $r-1$ numbers of $n-2$ number a_k which differs from a_1, a_2;similarly,c_{r-2}^{n-4}, $c_{r-3}^{n-6}, \cdots$ can be obtained, from these.

We conclude the paper by asking: please find an identity for the r-th weighted elementary symmetric mean similarly identities (3.1) and (3.2).

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