A SURVEY OF RECENT REVERSES FOR THE GENERALISED TRIANGLE INEQUALITY IN INNER PRODUCT SPACES

SEVER S. DRAGOMIR

Abstract. Recent results concerning reverses of the generalised triangle inequality in inner product spaces and applications are surveyed.

1. Introduction

The following reverse of the generalised triangle inequality

$$\cos\theta \sum_{k=1}^{n} |z_k| \le \left| \sum_{k=1}^{n} z_k \right|,$$

provided the complex numbers z_k, $k \in \{1, \dots, n\}$ satisfy the assumption

$$a - \theta \le \arg(z_k) \le a + \theta, \quad \text{for any} \quad k \in \{1, \dots, n\},$$

where $a \in \mathbb{R}$ and $\theta \in \left(0, \frac{\pi}{2}\right)$ was first discovered by M. Petrovich in 1917, [11] (see [10, p. 492]) and subsequently was rediscovered by other authors, including J. Karamata [6, p. 300 – 301], H.S. Wilf [12], and in an equivalent form by M. Marden [8].

In 1966, J.B. Diaz and F.T. Metcalf [1] proved the following reverse of the triangle inequality:

Theorem 1. *Let a be a unit vector in the inner product space $(H; \langle\cdot,\cdot\rangle)$ over the real or complex number field $\mathbb{K}$. Suppose that the vectors $x_i \in H \setminus \{0\}$, $i \in \{1, \dots, n\}$ satisfy*

$$0 \le r \le \frac{\operatorname{Re}\langle x_i, a\rangle}{\|x_i\|}, \qquad i \in \{1, \dots, n\}. \tag{1.1}$$

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Then

$$r\sum_{i=1}^{n}\|x_i\| \le \left\|\sum_{i=1}^{n}x_i\right\|, \tag{1.2}$$

where equality holds if and only if

$$\sum_{i=1}^{n}x_i = r\left(\sum_{i=1}^{n}\|x_i\|\right)a. \tag{1.3}$$

A generalisation of this result for orthonormal families is incorporated in the following result [1].

Theorem 2. *Let $a_1,\dots,a_n$ be orthonormal vectors in H. Suppose the vectors $x_1,\dots,x_n\in H\backslash\{0\}$ satisfy*

$$0\le r_k\le\frac{\operatorname{Re}\langle x_i,a_k\rangle}{\|x_i\|},\qquad i\in\{1,\dots,n\},\ k\in\{1,\dots,m\}. \tag{1.4}$$

Then

$$\left(\sum_{k=1}^{m}r_k^2\right)^{\frac{1}{2}}\sum_{i=1}^{n}\|x_i\|\le\left\|\sum_{i=1}^{n}x_i\right\|, \tag{1.5}$$

where equality holds if and only if

$$\sum_{i=1}^{n}x_i=\left(\sum_{i=1}^{n}\|x_i\|\right)\sum_{k=1}^{m}r_ka_k. \tag{1.6}$$

Similar results valid for semi-inner products may be found in [7] and [9].

For other classical inequalities related to the triangle inequality, see Chapter XVII of the book [10] and the references therein.

2. Some Inequalities of Diaz-Metcalf Type

2.1. **The Case of One Vector.** The following result with a natural geometrical meaning holds [3]:

Theorem 3. *Let a be a unit vector in the inner product space $(H;\langle\cdot,\cdot\rangle)$ and $\rho\in(0,1)$. If $x_i\in H$, $i\in\{1,\dots,n\}$ are such that*

$$\|x_i-a\|\le\rho\ \text{ for each }\ i\in\{1,\dots,n\}, \tag{2.1}$$

then we have the inequality

$$\sqrt{1-\rho^2}\sum_{i=1}^{n}\|x_i\|\le\left\|\sum_{i=1}^{n}x_i\right\|, \tag{2.2}$$

with equality if and only if

$$\sum_{i=1}^{n} x_i = \sqrt{1-\rho^2}\left(\sum_{i=1}^{n}\|x_i\|\right) a. \tag{2.3}$$

Proof. From (2.1) we have

$$\|x_i\|^2 - 2\operatorname{Re}\langle x_i, a\rangle + 1 \le \rho^2,$$

giving

$$\|x_i\|^2 + 1 - \rho^2 \le 2\operatorname{Re}\langle x_i, a\rangle, \tag{2.4}$$

for each $i \in \{1, \ldots, n\}$.

Dividing by $\sqrt{1-\rho^2} > 0$, we deduce

$$\frac{\|x_i\|^2}{\sqrt{1-\rho^2}} + \sqrt{1-\rho^2} \le \frac{2\operatorname{Re}\langle x_i, a\rangle}{\sqrt{1-\rho^2}}, \tag{2.5}$$

for each $i \in \{1, \ldots, n\}$.

On the other hand, by the elementary inequality

$$\frac{p}{\alpha} + q\alpha \ge 2\sqrt{pq}, \quad p, q \ge 0,\ \alpha > 0 \tag{2.6}$$

we have

$$2\|x_i\| \le \frac{\|x_i\|^2}{\sqrt{1-\rho^2}} + \sqrt{1-\rho^2} \tag{2.7}$$

and thus, by (2.5) and (2.7), we deduce

$$\frac{\operatorname{Re}\langle x_i, a\rangle}{\|x_i\|} \ge \sqrt{1-\rho^2},$$

for each $i \in \{1, \ldots, n\}$. Applying Theorem 1 for $r = \sqrt{1-\rho^2}$, we deduce the desired inequality (2.2). ∎

The following results may be stated as well.

Theorem 4. *Let a be a unit vector in the inner product space $(H; \langle\cdot,\cdot\rangle)$ and $M \ge m > 0$. If $x_i \in H$, $i \in \{1, \ldots, n\}$ are such that either*

$$\operatorname{Re}\langle Ma - x_i, x_i - ma\rangle \ge 0 \tag{2.8}$$

or, equivalently,

$$\left\| x_i - \frac{M+m}{2}\cdot a\right\| \le \frac{1}{2}(M-m) \tag{2.9}$$

holds for each $i \in \{1, \ldots, n\}$, then we have the inequality

$$\frac{2\sqrt{mM}}{m+M}\sum_{i=1}^{n}\|x_i\| \le \left\|\sum_{i=1}^{n} x_i\right\|, \tag{2.10}$$

or, equivalently,

$$(0 \leq) \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| \leq \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} \left\| \sum_{i=1}^{n} x_i \right\|. \tag{2.11}$$

The equality holds in (2.10) (or in (2.11)) if and only if

$$\sum_{i=1}^{n} x_i = \frac{2\sqrt{mM}}{m+M} \left(\sum_{i=1}^{n} \|x_i\| \right) a. \tag{2.12}$$

Proof. Firstly, we remark that if $x, z, Z \in H$, then the following statements are equivalent:

(i) $\operatorname{Re} \langle Z - x, x - z \rangle \geq 0$;
(ii) $\left\| x - \frac{Z+z}{2} \right\| \leq \frac{1}{2} \|Z - z\|$.

Using this fact, one may simply realize that (2.8) and (2.9) are equivalent.

Now, from (2.8), we get

$$\|x_i\|^2 + mM \leq (M+m) \operatorname{Re} \langle x_i, a \rangle,$$

for any $i \in \{1, \dots, n\}$. Dividing this inequality by $\sqrt{mM} > 0$, we deduce the following inequality that will be used in the sequel

$$\frac{\|x_i\|^2}{\sqrt{mM}} + \sqrt{mM} \leq \frac{M+m}{\sqrt{mM}} \operatorname{Re} \langle x_i, a \rangle, \tag{2.13}$$

for each $i \in \{1, \dots, n\}$.

Using the inequality (2.6) from Theorem 3, we also have

$$2 \|x_i\| \leq \frac{\|x_i\|^2}{\sqrt{mM}} + \sqrt{mM}, \tag{2.14}$$

for each $i \in \{1, \dots, n\}$.

Utilizing (2.13) and (2.14), we may conclude with the following inequality

$$\|x_i\| \leq \frac{M+m}{\sqrt{mM}} \operatorname{Re} \langle x_i, a \rangle,$$

which is equivalent to

$$\frac{2\sqrt{mM}}{m+M} \leq \frac{\operatorname{Re} \langle x_i, a \rangle}{\|x_i\|} \tag{2.15}$$

for any $i \in \{1, \dots, n\}$.

Finally, on applying the Diaz-Metcalf result in Theorem 1 for $r = \frac{2\sqrt{mM}}{m+M}$, we deduce the desired conclusion.

The equivalence between (2.10) and (2.11) follows by simple calculation and we omit the details. ∎

2.2. **The Case of m Vectors.** In a similar manner to the one used in the proof of Theorem 3 and by the use of the Diaz-Metcalf inequality incorporated in Theorem 2, we can also prove the following result [3] :

Theorem 5. *Let $a_1,\dots,a_n$ be orthonormal vectors in H. Suppose the vectors $x_1,\dots,x_n \in H\backslash\{0\}$ satisfy*

$$\|x_i - a_k\| \le \rho_k \quad \text{for each} \quad i\in\{1,\dots,n\},\ k\in\{1,\dots,m\}, \tag{2.16}$$

where $\rho_k \in (0,1)$, $k\in\{1,\dots,m\}$. Then we have the following reverse of the triangle inequality

$$\left(m-\sum_{k=1}^{m}\rho_k^2\right)^{1/2}\sum_{i=1}^{n}\|x_i\| \le \left\|\sum_{i=1}^{n}x_i\right\|. \tag{2.17}$$

The equality holds in (2.17) if and only if

$$\sum_{i=1}^{n}x_i = \left(\sum_{i=1}^{n}\|x_i\|\right)\sum_{k=1}^{m}\left(1-\rho_k^2\right)^{1/2}a_k. \tag{2.18}$$

Finally, by the use of Theorem 2 and a similar technique to that employed in the proof of Theorem 4, we may state the following result [3]:

Theorem 6. *Let $a_1,\dots,a_n$ be orthonormal vectors in H. Suppose the vectors $x_1,\dots,x_n \in H\backslash\{0\}$ satisfy*

$$\operatorname{Re}\langle M_k a_k - x_i, x_i - \mu_k a_k\rangle \ge 0, \tag{2.19}$$

or, equivalently,

$$\left\|x_i - \frac{M_k+\mu_k}{2}a_k\right\| \le \frac{1}{2}\left(M_k-\mu_k\right), \tag{2.20}$$

for any $i\in\{1,\dots,n\}$ and $k\in\{1,\dots,m\}$, where $M_k \ge \mu_k > 0$ for each $k\in\{1,\dots,m\}$.

Then we have the inequality

$$2\left(\sum_{k=1}^{m}\frac{\mu_k M_k}{\left(\mu_k+M_k\right)^2}\right)^{\frac{1}{2}}\sum_{i=1}^{n}\|x_i\| \le \left\|\sum_{i=1}^{n}x_i\right\|. \tag{2.21}$$

The equality holds in (2.21) iff

$$\sum_{i=1}^{n}x_i = 2\left(\sum_{i=1}^{n}\|x_i\|\right)\sum_{k=1}^{m}\frac{\sqrt{\mu_k M_k}}{\mu_k+M_k}a_k. \tag{2.22}$$

3. Additive Reverses for the Triangle Inequality

3.1. **The Case of One Vector.**
In this section we establish some additive reverses of the generalised triangle inequality in real or complex inner product spaces.

The following result holds [3]:

Theorem 7. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ and e, $x_i \in H$, $i \in \{1,\dots,n\}$ with $\|e\| = 1$. If $k_i \geq 0$, $i \in \{1,\dots,n\}$, are such that*

$$\|x_i\| - \operatorname{Re}\langle e, x_i\rangle \leq k_i \quad \text{for each } \ i \in \{1,\dots,n\}, \tag{3.1}$$

then we have the inequality

$$(0 \leq) \sum_{i=1}^n \|x_i\| - \left\|\sum_{i=1}^n x_i\right\| \leq \sum_{i=1}^n k_i. \tag{3.2}$$

The equality holds in (3.2) if and only if

$$\sum_{i=1}^n \|x_i\| \geq \sum_{i=1}^n k_i \tag{3.3}$$

and

$$\sum_{i=1}^n x_i = \left(\sum_{i=1}^n \|x_i\| - \sum_{i=1}^n k_i\right) e. \tag{3.4}$$

Proof. If we sum in (3.1) over i from 1 to n, then we get

$$\sum_{i=1}^n \|x_i\| \leq \operatorname{Re}\left\langle e, \sum_{i=1}^n x_i\right\rangle + \sum_{i=1}^n k_i. \tag{3.5}$$

By Schwarz's inequality for e and $\sum_{i=1}^n x_i$, we have

$$\begin{aligned} \operatorname{Re}\left\langle e, \sum_{i=1}^n x_i\right\rangle &\leq \left|\operatorname{Re}\left\langle e, \sum_{i=1}^n x_i\right\rangle\right| \\ &\leq \left|\left\langle e, \sum_{i=1}^n x_i\right\rangle\right| \leq \|e\| \left\|\sum_{i=1}^n x_i\right\| = \left\|\sum_{i=1}^n x_i\right\|. \end{aligned} \tag{3.6}$$

Making use of (3.5) and (3.6), we deduce the desired inequality (3.1).

If (3.3) and (3.4) hold, then

$$\left\|\sum_{i=1}^n x_i\right\| = \left|\sum_{i=1}^n \|x_i\| - \sum_{i=1}^n k_i\right| \|e\| = \sum_{i=1}^n \|x_i\| - \sum_{i=1}^n k_i,$$

and the equality in the second part of (3.2) holds true.

Conversely, if the equality holds in (3.2), then, obviously (3.3) is valid and we need only to prove (3.4).

Now, if the equality holds in (3.2) then it must hold in (3.1) for each $i \in \{1, \ldots, n\}$ and also must hold in any of the inequalities in (3.6).

It is well known that in Schwarz's inequality $|\langle u, v \rangle| \leq \|u\| \|v\|$ $(u, v \in H)$ the case of equality holds iff there exists a $\lambda \in \mathbb{K}$ such that $u = \lambda v$. We note that in the weaker inequality $\operatorname{Re} \langle u, v \rangle \leq \|u\| \|v\|$ the case of equality holds iff $\lambda \geq 0$ and $u = \lambda v$.

Consequently, the equality holds in all inequalities (3.6) simultaneously iff there exists a $\mu \geq 0$ with

$$\mu e = \sum_{i=1}^{n} x_i. \tag{3.7}$$

If we sum the equalities in (3.1) over i from 1 to n, then we deduce

$$\sum_{i=1}^{n} \|x_i\| - \operatorname{Re} \left\langle e, \sum_{i=1}^{n} x_i \right\rangle = \sum_{i=1}^{n} k_i. \tag{3.8}$$

Replacing $\sum_{i=1}^{n} \|x_i\|$ from (3.7) into (3.8), we deduce

$$\sum_{i=1}^{n} \|x_i\| - \mu \|e\|^2 = \sum_{i=1}^{n} k_i,$$

from where we get $\mu = \sum_{i=1}^{n} \|x_i\| - \sum_{i=1}^{n} k_i$. Using (3.7), we deduce (3.4) and the theorem is proved. ∎

3.2. **The Case of m Vectors.** If we turn our attention to the case of orthogonal families, then we may state the following result as well [3].

Theorem 8. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $\{e_k\}_{k \in \{1,\ldots,m\}}$ a family of orthonormal vectors in H, $x_i \in H$, $M_{i,k} \geq 0$ for $i \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$ such that*

$$\|x_i\| - \operatorname{Re} \langle e_k, x_i \rangle \leq M_{ik} \tag{3.9}$$

for each $i \in \{1, \ldots, n\}$, $k \in \{1, \ldots, m\}$. Then we have the inequality

$$\sum_{i=1}^{n} \|x_i\| \leq \frac{1}{\sqrt{m}} \left\| \sum_{i=1}^{n} x_i \right\| + \frac{1}{m} \sum_{i=1}^{n} \sum_{k=1}^{m} M_{ik}. \tag{3.10}$$

The equality holds true in (3.10) if and only if

$$\sum_{i=1}^{n} \|x_i\| \geq \frac{1}{m} \sum_{i=1}^{n} \sum_{k=1}^{m} M_{ik} \tag{3.11}$$

and

$$\sum_{i=1}^{n} x_i = \left(\sum_{i=1}^{n} \|x_i\| - \frac{1}{m} \sum_{i=1}^{n} \sum_{k=1}^{m} M_{ik} \right) \sum_{k=1}^{m} e_k. \tag{3.12}$$

Proof. If we sum over i from 1 to n in (3.9), then we obtain

$$\sum_{i=1}^{n} \|x_i\| \le \operatorname{Re} \left\langle e, \sum_{i=1}^{n} x_i \right\rangle + \sum_{i=1}^{n} M_{ik},$$

for each $k \in \{1, \dots, m\}$. Summing these inequalities over k from 1 to m, we deduce

$$\sum_{i=1}^{n} \|x_i\| \le \frac{1}{m} \operatorname{Re} \left\langle \sum_{k=1}^{m} e_k, \sum_{i=1}^{n} x_i \right\rangle + \frac{1}{m} \sum_{i=1}^{n} \sum_{k=1}^{m} M_{ik}. \tag{3.13}$$

By Schwarz's inequality for $\sum_{k=1}^{m} e_k$ and $\sum_{i=1}^{n} x_i$ we have

$$\begin{aligned} \operatorname{Re} \left\langle \sum_{k=1}^{m} e_k, \sum_{i=1}^{n} x_i \right\rangle &\le \left| \operatorname{Re} \left\langle \sum_{k=1}^{m} e_k, \sum_{i=1}^{n} x_i \right\rangle \right| \\ &\le \left| \left\langle \sum_{k=1}^{m} e_k, \sum_{i=1}^{n} x_i \right\rangle \right| \\ &\le \left\| \sum_{k=1}^{m} e_k \right\| \left\| \sum_{i=1}^{n} x_i \right\| \\ &= \sqrt{m} \left\| \sum_{i=1}^{n} x_i \right\|, \end{aligned} \tag{3.14}$$

since, obviously,

$$\left\| \sum_{k=1}^{m} e_k \right\| = \sqrt{\left\| \sum_{k=1}^{m} e_k \right\|^2} = \sqrt{\sum_{k=1}^{m} \|e_k\|^2} = \sqrt{m}.$$

Making use of (3.13) and (3.14), we deduce the desired inequality (3.10).

If (3.11) and (3.12) hold, then

$$\begin{aligned}\frac{1}{\sqrt{m}}\left\|\sum_{i=1}^{n} x_i\right\| &= \left|\sum_{i=1}^{n}\|x_i\| - \frac{1}{m}\sum_{i=1}^{n}\sum_{k=1}^{m} M_{ik}\right|\left\|\sum_{k=1}^{m} e_k\right\| \\ &= \frac{\sqrt{m}}{\sqrt{m}}\left(\sum_{i=1}^{n}\|x_i\| - \frac{1}{m}\sum_{i=1}^{n}\sum_{k=1}^{m} M_{ik}\right) \\ &= \sum_{i=1}^{n}\|x_i\| - \frac{1}{m}\sum_{i=1}^{n}\sum_{k=1}^{m} M_{ik},\end{aligned}$$

and the equality in (3.10) holds true.

Conversely, if the equality holds in (3.10), then, obviously (3.11) is valid.

Now if the equality holds in (3.10), then it must hold in (3.9) for each $i \in \{1, \dots, n\}$ and $k \in \{1, \dots, m\}$ and also must hold in any of the inequalities in (3.14).

It is well known that in Schwarz's inequality $\operatorname{Re}\langle u, v\rangle \le \|u\| \|v\|$, the equality occurs iff $u = \lambda v$ with $\lambda \ge 0$, consequently, the equality holds in all inequalities (3.14) simultaneously iff there exists a $\mu \ge 0$ with

$$\mu \sum_{k=1}^{m} e_k = \sum_{i=1}^{n} x_i. \tag{3.15}$$

If we sum the equality in (3.9) over i from 1 to n and k from 1 to m, then we deduce

$$m \sum_{i=1}^{n} \|x_i\| - \operatorname{Re}\left\langle \sum_{k=1}^{m} e_k, \sum_{i=1}^{n} x_i \right\rangle = \sum_{i=1}^{n}\sum_{k=1}^{m} M_{ik}. \tag{3.16}$$

Replacing $\sum_{i=1}^{n} x_i$ from (3.15) into (3.16), we deduce

$$m \sum_{i=1}^{n} \|x_i\| - \mu \sum_{k=1}^{m} \|e_k\|^2 = \sum_{i=1}^{n}\sum_{k=1}^{m} M_{ik}$$

giving

$$\mu = \sum_{i=1}^{n} \|x_i\| - \frac{1}{m}\sum_{i=1}^{n}\sum_{k=1}^{m} M_{ik}.$$

Using (3.15), we deduce (3.12) and the theorem is proved. ∎

4. Further Additive Reverses

4.1. **The Case of Small Balls.**

In this section we point out different additive reverses of the generalised triangle inequality under simpler conditions for the vectors involved.

The following result holds [3]:

Theorem 9. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $e, x_i \in H$, $i \in \{1, \dots, n\}$ with $\|e\| = 1$. If $\rho \in (0, 1)$ and x_i, $i \in \{1, \dots, n\}$ are such that*

$$\|x_i - e\| \leq \rho \quad \text{for each} \quad i \in \{1, \dots, n\}, \tag{4.1}$$

then we have the inequality

$$\begin{aligned}(0 \leq) \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| &\leq \frac{\rho^2}{\sqrt{1-\rho^2}\left(1+\sqrt{1-\rho^2}\right)} \operatorname{Re} \left\langle \sum_{i=1}^{n} x_i, e \right\rangle \\ &\left(\leq \frac{\rho^2}{\sqrt{1-\rho^2}\left(1+\sqrt{1-\rho^2}\right)} \left\| \sum_{i=1}^{n} x_i \right\| \right).\end{aligned} \tag{4.2}$$

The equality holds in (4.2) if and only if

$$\sum_{i=1}^{n} \|x_i\| \geq \frac{\rho^2}{\sqrt{1-\rho^2}\left(1+\sqrt{1-\rho^2}\right)} \operatorname{Re} \left\langle \sum_{i=1}^{n} x_i, e \right\rangle \tag{4.3}$$

and

$$\sum_{i=1}^{n} x_i = \left(\sum_{i=1}^{n} \|x_i\| - \frac{\rho^2}{\sqrt{1-\rho^2}\left(1+\sqrt{1-\rho^2}\right)} \operatorname{Re} \left\langle \sum_{i=1}^{n} x_i, e \right\rangle \right) e. \tag{4.4}$$

Proof. We know, from the proof of Theorem 7, that, if (4.1) is fulfilled, then we have the inequality

$$\|x_i\| \leq \frac{1}{\sqrt{1-\rho^2}} \operatorname{Re} \langle x_i, e \rangle$$

for each $i \in \{1, \ldots, n\}$, implying

$$\begin{aligned} (4.5) \quad \|x_i\| - \operatorname{Re}\langle x_i, e\rangle &\leq \left(\frac{1}{\sqrt{1-\rho^2}} - 1\right) \operatorname{Re}\langle x_i, e\rangle \\ &= \frac{\rho^2}{\sqrt{1-\rho^2}\left(1+\sqrt{1-\rho^2}\right)} \operatorname{Re}\langle x_i, e\rangle \end{aligned}$$

for each $i \in \{1, \ldots, n\}$.

Now, making use of Theorem 3, for

$$k_i := \frac{\rho^2}{\sqrt{1-\rho^2}\left(1+\sqrt{1-\rho^2}\right)} \operatorname{Re}\langle x_i, e\rangle, \quad i \in \{1, \ldots, n\},$$

we easily deduce the conclusion of the theorem.

We omit the details. ∎

We may state the following result as well [3]:

Theorem 10. *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space and $e \in H$, $M \geq m > 0$. If $x_i \in H$, $i \in \{1, \ldots, n\}$ are such that either*

$$(4.6) \qquad \operatorname{Re}\langle Me - x_i, x_i - me\rangle \geq 0,$$

or, equivalently,

$$(4.7) \qquad \left\|x_i - \frac{M+m}{2}e\right\| \leq \frac{1}{2}(M-m)$$

holds for each $i \in \{1, \ldots, n\}$, then we have the inequality

$$\begin{aligned} (4.8) \quad (0 \leq) \sum_{i=1}^n \|x_i\| - \left\|\sum_{i=1}^n x_i\right\| &\leq \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{mM}} \operatorname{Re}\left\langle \sum_{i=1}^n x_i, e\right\rangle \\ &\left(\leq \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{mM}} \left\|\sum_{i=1}^n x_i\right\|\right). \end{aligned}$$

The equality holds in (4.8) if and only if

$$(4.9) \qquad \sum_{i=1}^n \|x_i\| \geq \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{mM}} \operatorname{Re}\left\langle \sum_{i=1}^n x_i, e\right\rangle$$

and

$$(4.10) \qquad \sum_{i=1}^n x_i = \left(\sum_{i=1}^n \|x_i\| - \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{mM}} \operatorname{Re}\left\langle \sum_{i=1}^n x_i, e\right\rangle\right) e.$$

Proof. We know, from the proof of Theorem 4, that if (4.6) is fulfilled, then we have the inequality

$$\|x_i\| \leq \frac{M+m}{2\sqrt{mM}} \operatorname{Re}\langle x_i, e\rangle$$

for each $i \in \{1, \ldots, n\}$. This is equivalent to

$$\|x_i\| - \operatorname{Re}\langle x_i, e\rangle \leq \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{mM}} \operatorname{Re}\langle x_i, e\rangle$$

for each $i \in \{1, \ldots, n\}$.

Now, making use of Theorem 7, we deduce the conclusion of the theorem. We omit the details. ∎

Remark 1. *If one uses Theorem 8 instead of Theorem 7 above, then one can state the corresponding generalisation for families of orthonormal vectors of the inequalities (4.2) and (4.8) respectively. We do not provide them here.*

4.2. **The Case of Arbitrary Balls.** Now, on utilising a slightly different approach, we may point out the following result [3]:

Theorem 11. *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ and e, $x_i \in H$, $i \in \{1, \ldots, n\}$ with $\|e\| = 1$. If $r_i > 0$, $i \in \{1, \ldots, n\}$ are such that*

$$\|x_i - e\| \leq r_i \quad \textit{for each} \quad i \in \{1, \ldots, n\}, \tag{4.11}$$

then we have the inequality

$$0 \leq \sum_{i=1}^{n} \|x_i\| - \left\|\sum_{i=1}^{n} x_i\right\| \leq \frac{1}{2}\sum_{i=1}^{n} r_i^2. \tag{4.12}$$

The equality holds in (4.12) if and only if

$$\sum_{i=1}^{n} \|x_i\| \geq \frac{1}{2}\sum_{i=1}^{n} r_i^2 \tag{4.13}$$

and

$$\sum_{i=1}^{n} x_i = \left(\sum_{i=1}^{n} \|x_i\| - \frac{1}{2}\sum_{i=1}^{n} r_i^2\right) e. \tag{4.14}$$

Proof. The condition (4.11) is clearly equivalent to

$$\|x_i\|^2 + 1 \leq \operatorname{Re}\langle x_i, e\rangle + r_i^2 \tag{4.15}$$

for each $i \in \{1, \ldots, n\}$.

Using the elementary inequality

$$2\|x_i\| \leq \|x_i\|^2 + 1, \tag{4.16}$$

for each $i \in \{1, \dots, n\}$, then, by (4.15) and (4.16), we deduce

$$2\|x_i\| \leq 2\operatorname{Re}\langle x_i, e\rangle + r_i^2,$$

giving

$$\|x_i\| - \operatorname{Re}\langle x_i, e\rangle \leq \frac{1}{2} r_i^2 \tag{4.17}$$

for each $i \in \{1, \dots, n\}$.

Now, utilising Theorem 7 for $k_i = \frac{1}{2} r_i^2$, $i \in \{1, \dots, n\}$, we deduce the desired result. We omit the details. ∎

Finally, we may state and prove the following result as well [3].

Theorem 12. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{K}$ and e, $x_i \in H$, $i \in \{1, \dots, n\}$ with $\|e\| = 1$. If $M_i \geq m_i > 0$, $i \in \{1, \dots, n\}$, are such that*

$$\left\| x_i - \frac{M_i + m_i}{2} e \right\| \leq \frac{1}{2}(M_i - m_i), \tag{4.18}$$

or, equivalently,

$$\operatorname{Re}\langle M_i e - x, x - m_i e\rangle \geq 0 \tag{4.19}$$

for each $i \in \{1, \dots, n\}$, then we have the inequality

$$(0 \leq) \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| \leq \frac{1}{4} \sum_{i=1}^{n} \frac{(M_i - m_i)^2}{M_i + m_i}. \tag{4.20}$$

The equality holds in (4.20) if and only if

$$\sum_{i=1}^{n} \|x_i\| \geq \frac{1}{4} \sum_{i=1}^{n} \frac{(M_i - m_i)^2}{M_i + m_i} \tag{4.21}$$

and

$$\sum_{i=1}^{n} x_i = \left(\sum_{i=1}^{n} \|x_i\| - \frac{1}{4} \sum_{i=1}^{n} \frac{(M_i - m_i)^2}{M_i + m_i} \right) e. \tag{4.22}$$

Proof. The condition (4.18) is equivalent to:

$$\|x_i\|^2 + \left(\frac{M_i + m_i}{2}\right)^2 \leq 2\operatorname{Re}\left\langle x_i, \frac{M_i + m_i}{2} e \right\rangle + \frac{1}{4}(M_i - m_i)^2$$

and since

$$2\left(\frac{M_i + m_i}{2}\right)\|x_i\| \leq \|x_i\|^2 + \left(\frac{M_i + m_i}{2}\right)^2,$$

then we get

$$2\left(\frac{M_i+m_i}{2}\right)\|x_i\| \le 2\cdot\frac{M_i+m_i}{2}\operatorname{Re}\langle x_i,e\rangle+\frac{1}{4}\left(M_i-m_i\right)^2,$$

or, equivalently,

$$\|x_i\|-\operatorname{Re}\langle x_i,e\rangle \le \frac{1}{4}\cdot\frac{\left(M_i-m_i\right)^2}{M_i+m_i}$$

for each $i\in\{1,\dots,n\}$.

Now, making use of Theorem 7 for $k_i:=\frac{1}{4}\cdot\frac{(M_i-m_i)^2}{M_i+m_i}$, $i\in\{1,\dots,n\}$, we deduce the desired result. ∎

Remark 2. *If one uses Theorem 8 instead of Theorem 7 above, then one can state the corresponding generalisation for families of orthonormal vectors of the inequalities in (4.12) and (4.20) respectively. We omit the details.*

5. Reverses of Schwarz Inequality

In this section we outline a procedure showing how some of the above results for triangle inequality may be employed to obtain reverses for the celebrated Schwarz inequality.

For $a\in H$, $\|a\|=1$ and $r\in(0,1)$ define the closed ball

$$\overline{D}(a,r):=\{x\in H,\|x-a\|\le r\}.$$

The following reverse of the Schwarz inequality holds [3]:

Proposition 1. *If $x,y\in\overline{D}(a,r)$ with $a\in H$, $\|a\|=1$ and $r\in(0,1)$, then we have the inequality*

$$(0\le)\,\frac{\|x\|\,\|y\|-\operatorname{Re}\langle x,y\rangle}{\left(\|x\|+\|y\|\right)^2}\le\frac{1}{2}r^2. \tag{5.1}$$

The constant $\frac{1}{2}$ in (5.1) is best possible in the sense that it cannot be replaced by a smaller quantity.

Proof. Using Theorem 3 for $x_1=x,x_2=y,\rho=r$, we have

$$\sqrt{1-r^2}\left(\|x\|+\|y\|\right)\le\|x+y\|. \tag{5.2}$$

Taking the square in (5.2) we deduce

$$\left(1-r^2\right)\left(\|x\|^2+2\|x\|\,\|y\|+\|y\|^2\right)\le\|x\|^2+2\operatorname{Re}\langle x,y\rangle+\|y\|^2$$

which is clearly equivalent to (5.1).

Now, assume that (5.1) holds with a constant $C>0$ instead of $\frac{1}{2}$, *i.e.*,

$$\frac{\|x\|\,\|y\|-\operatorname{Re}\langle x,y\rangle}{\left(\|x\|+\|y\|\right)^2}\le Cr^2 \tag{5.3}$$

provided $x, y \in \overline{D}(a, r)$ with $a \in H$, $\|a\| = 1$ and $r \in (0, 1)$.

Let $e \in H$ with $\|e\| = 1$ and $e \perp a$. Define $x = a + re, y = a - re$. Then

$$\|x\| = \sqrt{1 + r^2} = \|y\|, \quad \operatorname{Re}\langle x, y\rangle = 1 - r^2$$

and thus, from (5.3), we have

$$\frac{1 + r^2 - (1 - r^2)}{\left(2\sqrt{1 + r^2}\right)^2} \leq Cr^2$$

giving

$$\frac{1}{2} \leq \left(1 + r^2\right) C$$

for any $r \in (0, 1)$. If in this inequality we let $r \to 0+$, then we get $C \geq \frac{1}{2}$ and the proposition is proved. ∎

In a similar way, by the use of Theorem 4, we may prove the following reverse of the Schwarz inequality as well [3]:

Proposition 2. *If $a \in H$, $\|a\| = 1$, $M \geq m > 0$ and $x, y \in H$ are so that either*

$$\operatorname{Re}\langle Ma - x, x - ma\rangle, \operatorname{Re}\langle Ma - y, y - ma\rangle \geq 0$$

or, equivalently,

$$\left\|x - \frac{m + M}{2}a\right\|, \left\|y - \frac{m + M}{2}a\right\| \leq \frac{1}{2}(M - m)$$

hold, then

$$(0 \leq) \frac{\|x\|\|y\| - \operatorname{Re}\langle x, y\rangle}{(\|x\| + \|y\|)^2} \leq \frac{1}{2}\left(\frac{M - m}{M + m}\right)^2.$$

The constant $\frac{1}{2}$ cannot be replaced by a smaller quantity.

Remark 3. *On utilising Theorem 5 and Theorem 6, we may deduce some similar reverses of Schwarz inequality provided $x, y \in \cap_{k=1}^{m}\overline{D}(a_k, \rho_k)$, assumed not to be empty, where $a_1, ..., a_n$ are orthonormal vectors in H and $\rho_k \in (0, 1)$ for $k \in \{1, ..., m\}$. We omit the details.*

Remark 4. *For various different reverses of Schwarz inequality in inner product spaces, see the recent survey* [2].

6. Quadratic Reverses of the Triangle Inequality

6.1. The General Case.

The following lemma holds [4]:

Lemma 1. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $x_i \in H$, $i \in \{1,\dots,n\}$ and $k_{ij} > 0$ for $1 \leq i < j \leq n$ such that*

$$0 \leq \|x_i\| \|x_j\| - \operatorname{Re}\langle x_i, x_j\rangle \leq k_{ij} \tag{6.1}$$

for $1 \leq i < j \leq n$. Then we have the following quadratic reverse of the triangle inequality

$$\left(\sum_{i=1}^{n} \|x_i\|\right)^2 \leq \left\|\sum_{i=1}^{n} x_i\right\|^2 + 2\sum_{1\leq i<j\leq n} k_{ij}. \tag{6.2}$$

The case of equality holds in (6.2) if and only if it holds in (6.1) for each i, j with $1 \leq i < j \leq n$.

Proof. We observe that the following identity holds:

$$\begin{aligned} &\left(\sum_{i=1}^{n} \|x_i\|\right)^2 - \left\|\sum_{i=1}^{n} x_i\right\|^2 \\ &= \sum_{i,j=1}^{n} \|x_i\| \|x_j\| - \left\langle \sum_{i=1}^{n} x_i, \sum_{j=1}^{n} x_j \right\rangle \\ &= \sum_{i,j=1}^{n} \|x_i\| \|x_j\| - \sum_{i,j=1}^{n} \operatorname{Re}\langle x_i, x_j\rangle \\ &= \sum_{i,j=1}^{n} \left[\|x_i\| \|x_j\| - \operatorname{Re}\langle x_i, x_j\rangle\right] \\ &= \sum_{1\leq i<j\leq n} \left[\|x_i\| \|x_j\| - \operatorname{Re}\langle x_i, x_j\rangle\right] \\ &+ \sum_{1\leq j<i\leq n} \left[\|x_i\| \|x_j\| - \operatorname{Re}\langle x_i, x_j\rangle\right] \\ &= 2\sum_{1\leq i<j\leq n} \left[\|x_i\| \|x_j\| - \operatorname{Re}\langle x_i, x_j\rangle\right]. \end{aligned} \tag{6.3}$$

Using the condition (6.1), we deduce that

$$\sum_{1\leq i<j\leq n} \left[\|x_i\| \|x_j\| - \operatorname{Re}\langle x_i, x_j\rangle\right] \leq \sum_{1\leq i<j\leq n} k_{ij},$$

and by (6.3), we get the desired inequality (6.2).

The case of equality is obvious by the identity (6.3) and we omit the details. ∎

Remark 5. *From (6.2) one may deduce the coarser inequality that might be useful in some applications:*

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n}\|x_i\| - \left\|\sum_{i=1}^{n} x_i\right\| \\ &\leq \sqrt{2}\left(\sum_{1\leq i<j\leq n} k_{ij}\right)^{\frac{1}{2}} \quad \left(\leq \sqrt{2}\sum_{1\leq i<j\leq n}\sqrt{k_{ij}}\right). \end{aligned}$$

Remark 6. *If the condition (6.1) is replaced with the following refinement of Schwarz's inequality:*

$$(0 \leq)\, \delta_{ij} \leq \|x_i\|\,\|x_j\| - \operatorname{Re}\langle x_i, x_j\rangle \quad \textit{for } 1 \leq i < j \leq n, \tag{6.4}$$

then the following refinement of the quadratic generalised triangle inequality is valid:

$$\left(\sum_{i=1}^{n}\|x_i\|\right)^2 \geq \left\|\sum_{i=1}^{n} x_i\right\|^2 + 2\sum_{1\leq i<j\leq n}\delta_{ij} \quad \left(\geq \left\|\sum_{i=1}^{n} x_i\right\|^2\right). \tag{6.5}$$

The equality holds in the first part of (6.5) iff the case of equality holds in (6.4) for each $1 \leq i < j \leq n$.

The following result holds [4].

Proposition 3. *Let $(H;\langle\cdot,\cdot\rangle)$ be as above, $x_i \in H$, $i \in \{1,\ldots,n\}$ and $r > 0$ such that*

$$\|x_i - x_j\| \leq r \tag{6.6}$$

for $1 \leq i < j \leq n$. Then

$$\left(\sum_{i=1}^{n}\|x_i\|\right)^2 \leq \left\|\sum_{i=1}^{n} x_i\right\|^2 + \frac{n(n-1)}{2}r^2. \tag{6.7}$$

The case of equality holds in (6.7) if and only if

$$\|x_i\|\,\|x_j\| - \operatorname{Re}\langle x_i, x_j\rangle = \frac{1}{2}r^2 \tag{6.8}$$

for each i, j with $1 \leq i < j \leq n$.

Proof. The inequality (6.6) is obviously equivalent to

$$\|x_i\|^2 + \|x_j\|^2 \leq 2\operatorname{Re}\langle x_i, x_j\rangle + r^2$$

for $1 \leq i < j \leq n$. Since

$$2\|x_i\|\|x_j\| \leq \|x_i\|^2 + \|x_j\|^2, \quad 1 \leq i < j \leq n;$$

hence

$$\|x_i\|\|x_j\| - \operatorname{Re}\langle x_i, x_j\rangle \leq \frac{1}{2}r^2 \tag{6.9}$$

for any i, j with $1 \leq i < j \leq n$.

Applying Lemma 1 for $k_{ij} := \frac{1}{2}r^2$ and taking into account that

$$\sum_{1 \leq i < j \leq n} k_{ij} = \frac{n(n-1)}{4}r^2,$$

we deduce the desired inequality (6.7). The case of equality is also obvious by the above lemma and we omit the details. ∎

6.2. **Inequalities in Terms of the Forward Difference.** In the same spirit, and if some information about the forward difference $\Delta x_k := x_{k+1} - x_k$ $(1 \leq k \leq n-1)$ are available, then the following simple quadratic reverse of the generalised triangle inequality may be stated [4].

Corollary 1. *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space and $x_i \in H$, $i \in \{1, \dots, n\}$. Then we have the inequality*

$$\left(\sum_{i=1}^{n}\|x_i\|\right)^2 \leq \left\|\sum_{i=1}^{n} x_i\right\|^2 + \frac{n(n-1)}{2}\sum_{k=1}^{n-1}\|\Delta x_k\|. \tag{6.10}$$

The constant $\frac{1}{2}$ is best possible in the sense that it cannot be replaced in general by a smaller quantity.

Proof. Let $1 \leq i < j \leq n$. Then, obviously,

$$\|x_j - x_i\| = \left\|\sum_{k=i}^{j-1}\Delta x_k\right\| \leq \sum_{k=i}^{j-1}\|\Delta x_k\| \leq \sum_{k=1}^{n-1}\|\Delta x_k\|.$$

Applying Proposition 3 for $r := \sum_{k=1}^{n-1}\|\Delta x_k\|$, we deduce the desired result (6.10).

To prove the sharpness of the constant $\frac{1}{2}$, assume that the inequality (6.10) holds with a constant $c > 0$, i.e.,

$$\left(\sum_{i=1}^{n}\|x_i\|\right)^2 \leq \left\|\sum_{i=1}^{n} x_i\right\|^2 + cn(n-1)\sum_{k=1}^{n-1}\|\Delta x_k\| \tag{6.11}$$

for $n \geq 2$, $x_i \in H$, $i \in \{1, \dots, n\}$.

If we choose in (6.11), $n = 2$, $x_1 = -\frac{1}{2}e$, $x_2 = \frac{1}{2}e$, $e \in H$, $\|e\| = 1$, then we get $1 \leq 2c$, giving $c \geq \frac{1}{2}$. ∎

The following result providing a reverse of the quadratic generalised triangle inequality in terms of the sup-norm of the forward differences also holds [4].

Proposition 4. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be an inner product space and* $x_i \in H$, $i \in \{1,\dots,n\}$. *Then we have the inequality*

$$\left(\sum_{i=1}^{n}\|x_i\|\right)^2 \le \left\|\sum_{i=1}^{n}x_i\right\|^2 + \frac{n^2\left(n^2-1\right)}{12}\max_{1\le k\le n-1}\|\Delta x_k\|^2 . \tag{6.12}$$

The constant $\frac{1}{12}$ *is best possible in (6.12).*

Proof. As above, we have that

$$\|x_j - x_i\| \le \sum_{k=i}^{j-1}\|\Delta x_k\| \le (j-i)\max_{1\le k\le n-1}\|\Delta x_k\|,$$

for $1 \le i < j \le n$.

Squaring the above inequality, we get

$$\|x_j\|^2 + \|x_i\|^2 \le 2\operatorname{Re}\langle x_i, x_j\rangle + (j-i)^2\max_{1\le k\le n-1}\|\Delta x_k\|^2$$

for any i,j with $1 \le i < j \le n$, and since

$$2\|x_i\|\|x_j\| \le \|x_j\|^2 + \|x_i\|^2,$$

hence

$$0 \le \|x_i\|\|x_j\| - \operatorname{Re}\langle x_i, x_j\rangle \le \frac{1}{2}(j-i)^2\max_{1\le k\le n-1}\|\Delta x_k\|^2 \tag{6.13}$$

for any i,j with $1 \le i < j \le n$.

Applying Lemma 1 for $k_{ij} := \frac{1}{2}(j-i)^2\max_{1\le k\le n-1}\|\Delta x_k\|^2$, we can state that

$$\left(\sum_{i=1}^{n}\|x_i\|\right)^2 \le \left\|\sum_{i=1}^{n}x_i\right\|^2 + \sum_{1\le i<j\le n}(j-i)^2\max_{1\le k\le n-1}\|\Delta x_k\|^2 .$$

However,

$$\begin{aligned}\sum_{1\le i<j\le n}(j-i)^2 &= \frac{1}{2}\sum_{i,j=1}^{n}(j-i)^2 = n\sum_{k=1}^{n}k^2 - \left(\sum_{k=1}^{n}k\right)^2 \\ &= \frac{n^2\left(n^2-1\right)}{12}\end{aligned}$$

giving the desired inequality.

To prove the sharpness of the constant, assume that (6.12) holds with a constant $D > 0$, i.e.,

$$\left(\sum_{i=1}^{n}\|x_i\|\right)^2 \leq \left\|\sum_{i=1}^{n}x_i\right\|^2 + Dn^2\left(n^2-1\right)\max_{1\leq k\leq n-1}\|\Delta x_k\|^2 \tag{6.14}$$

for $n \geq 2$, $x_i \in H$, $i \in \{1,\ldots,n\}$.

If in (6.14) we choose $n = 2$, $x_1 = -\frac{1}{2}e$, $x_2 = \frac{1}{2}e$, $e \in H$, $\|e\| = 1$, then we get $1 \leq 12D$ giving $D \geq \frac{1}{12}$. ∎

The following result may be stated as well [4].

Proposition 5. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space and $x_i \in H$, $i \in \{1,\ldots,n\}$. Then we have the inequality:*

$$\left(\sum_{i=1}^{n}\|x_i\|\right)^2 \leq \left\|\sum_{i=1}^{n}x_i\right\|^2 + \sum_{1\leq i<j\leq n}(j-i)^{\frac{2}{q}}\left(\sum_{k=1}^{n-1}\|\Delta x_k\|^p\right)^{\frac{2}{p}}, \tag{6.15}$$

where $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$.

The constant $E = 1$ in front of the double sum cannot generally be replaced by a smaller constant.

Proof. Using Hölder's inequality, we have

$$\begin{aligned}\|x_j - x_i\| &\leq \sum_{k=i}^{j-1}\|\Delta x_k\| \leq (j-i)^{\frac{1}{q}}\left(\sum_{k=i}^{j-1}\|\Delta x_k\|^p\right)^{\frac{1}{p}}\\ &\leq (j-i)^{\frac{1}{q}}\left(\sum_{k=1}^{n-1}\|\Delta x_k\|^p\right)^{\frac{1}{p}},\end{aligned}$$

for $1 \leq i < j \leq n$.

Squaring the previous inequality, we get

$$\|x_j\|^2 + \|x_i\|^2 \leq 2\operatorname{Re}\langle x_i, x_j\rangle + (j-i)^{\frac{2}{q}}\left(\sum_{k=1}^{n-1}\|\Delta x_k\|^p\right)^{\frac{2}{p}},$$

for $1 \leq i < j \leq n$.

Utilising the same argument from the proof of Proposition 4, we deduce the desired inequality (6.15).

Now assume that (6.15) holds with a constant $E > 0$, i.e.,

$$\left(\sum_{i=1}^{n}\|x_i\|\right)^2 \leq \left\|\sum_{i=1}^{n}x_i\right\|^2 + E\sum_{1\leq i<j\leq n}(j-i)^{\frac{2}{q}}\left(\sum_{k=1}^{n-1}\|\Delta x_k\|^p\right)^{\frac{2}{p}},$$

for $n \geq 2$ and $x_i \in H$, $i \in \{1,\ldots,n\}$, $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$.

For $n = 2$, $x_1 = -\frac{1}{2}e$, $x_2 = \frac{1}{2}e$, $\|e\| = 1$, we get $1 \leq E$, showing the fact that the inequality (6.15) is sharp. ∎

The particular case $p = q = 2$ is of interest [4].

Corollary 2. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space and $x_i \in H$, $i \in \{1, \dots, n\}$. Then we have the inequality:*

$$\left(\sum_{i=1}^{n} \|x_i\| \right)^2 \leq \left\| \sum_{i=1}^{n} x_i \right\|^2 + \frac{(n^2 - 1)\, n}{6} \sum_{k=1}^{n-1} \|\Delta x_k\|^2 . \tag{6.16}$$

The constant $\frac{1}{6}$ is best possible in (6.16).

Proof. For $p = q = 2$, Proposition 5 provides the inequality

$$\left(\sum_{i=1}^{n} \|x_i\| \right)^2 \leq \left\| \sum_{i=1}^{n} x_i \right\|^2 + \sum_{1 \leq i < j \leq n} (j - i) \sum_{k=1}^{n-1} \|\Delta x_k\|^2 ,$$

and since

$$\begin{aligned}
&\sum_{1 \leq i < j \leq n} (j - i) \\
&= 1 + (1 + 2) + (1 + 2 + 3) + \cdots + (1 + 2 + \cdots + n - 1) \\
&= \sum_{k=1}^{n-1} (1 + 2 + \cdots + k) = \sum_{k=1}^{n-1} \frac{k\,(k+1)}{2} \\
&= \frac{1}{2} \left[\frac{(n-1)\, n\, (2n-1)}{6} + \frac{n\,(n-1)}{2} \right] \\
&= \frac{n\,(n^2 - 1)}{6},
\end{aligned}$$

hence the inequality (6.15) is proved. The best constant may be shown in the same way as above but we omit the details. ∎

6.3. **A Different Quadratic Inequality.** Finally, we may state and prove the following different result [4].

Theorem 13. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space, $y_i \in H$, $i \in \{1, \dots, n\}$ and $M \geq m > 0$ are such that either*

$$\operatorname{Re} \langle M y_j - y_i, y_i - m y_j \rangle \geq 0 \quad \text{for } 1 \leq i < j \leq n, \tag{6.17}$$

or, equivalently,

$$\left\| y_i - \frac{M + m}{2} y_j \right\| \leq \frac{1}{2} (M - m) \|y_j\| \quad \text{for } 1 \leq i < j \leq n. \tag{6.18}$$

Then we have the inequality

$$\left(\sum_{i=1}^{n}\|y_i\|\right)^2 \le \left\|\sum_{i=1}^{n} y_i\right\|^2 + \frac{1}{2}\cdot\frac{(M-m)^2}{M+m}\sum_{k=1}^{n-1} k\,\|y_{k+1}\|^2. \tag{6.19}$$

The case of equality holds in (6.19) if and only if

$$\|y_i\|\,\|y_j\| - \operatorname{Re}\langle y_i, y_j\rangle = \frac{1}{4}\cdot\frac{(M-m)^2}{M+m}\|y_j\|^2 \tag{6.20}$$

for each i, j with $1 \le i < j \le n$.

Proof. Taking the square in (6.18), we get

$$\begin{aligned}
&\|y_i\|^2 + \frac{(M-m)^2}{M+m}\|y_j\|^2 \\
\le\ & 2\operatorname{Re}\left\langle y_i, \frac{M+m}{2} y_j\right\rangle + \frac{1}{n}(M-m)^2\|y_j\|^2
\end{aligned}$$

for $1 \le i < j \le n$, and since, obviously,

$$2\left(\frac{M+m}{2}\right)\|y_i\|\,\|y_j\| \le \|y_i\|^2 + \frac{(M-m)^2}{M+m}\|y_j\|^2,$$

hence

$$\begin{aligned}
&2\left(\frac{M+m}{2}\right)\|y_i\|\,\|y_j\| \\
\le\ & 2\operatorname{Re}\left\langle y_i, \frac{M+m}{2} y_j\right\rangle + \frac{1}{n}(M-m)^2\|y_j\|^2,
\end{aligned}$$

giving the much simpler inequality

$$\|y_i\|\,\|y_j\| - \operatorname{Re}\langle y_i, y_j\rangle \le \frac{1}{4}\cdot\frac{(M-m)^2}{M+m}\|y_j\|^2, \tag{6.21}$$

for $1 \le i < j \le n$.

Applying Lemma 1 for $k_{ij} := \frac{1}{4}\cdot\frac{(M-m)^2}{M+m}\|y_j\|^2$, we deduce

$$\left(\sum_{i=1}^{n}\|y_i\|\right)^2 \le \left\|\sum_{i=1}^{n} y_i\right\|^2 + \frac{1}{2}\frac{(M-m)^2}{M+m}\sum_{1\le i<j\le n}\|y_j\|^2 \tag{6.22}$$

with equality if and only if (6.21) holds for each i, j with $1 \le i < j \le n$.

Since

$$\begin{aligned}\sum_{1\le i<j\le n}\|y_j\|^2 &= \sum_{1<j\le n}\|y_j\|^2+\sum_{2<j\le n}\|y_j\|^2+\cdots+\sum_{n-1<j\le n}\|y_j\|^2\\ &=\sum_{j=2}^{n}\|y_j\|^2+\sum_{j=3}^{n}\|y_j\|^2+\cdots+\sum_{j=n-1}^{n}\|y_j\|^2+\|y_n\|^2\\ &=\sum_{j=2}^{n}(j-1)\|y_j\|^2=\sum_{k=1}^{n-1}k\|y_{k+1}\|^2,\end{aligned}$$

hence the inequality (6.19) is obtained. ∎

7. Further Quadratic Refinements

7.1. **The General Case.** The following lemma is of interest in itself as well [4].

Lemma 2. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $x_i\in H$, $i\in\{1,\dots,n\}$ and $k\ge 1$ with the property that:*

$$\|x_i\|\,\|x_j\|\le k\operatorname{Re}\langle x_i,x_j\rangle, \tag{7.1}$$

for each i,j with $1\le i<j\le n$. Then

$$\left(\sum_{i=1}^{n}\|x_i\|\right)^2+(k-1)\sum_{i=1}^{n}\|x_i\|^2\le k\left\|\sum_{i=1}^{n}x_i\right\|^2. \tag{7.2}$$

The equality holds in (7.2) if and only if it holds in (7.1) for each i,j with $1\le i<j\le n$.

Proof. Firstly, let us observe that the following identity holds true:

$$\begin{aligned}&\left(\sum_{i=1}^{n}\|x_i\|\right)^2-k\left\|\sum_{i=1}^{n}x_i\right\|^2 \\ &=\sum_{i,j=1}^{n}\|x_i\|\,\|x_j\|-k\left\langle\sum_{i=1}^{n}x_i,\sum_{j=1}^{n}x_j\right\rangle\\ &=\sum_{i,j=1}^{n}\left[\|x_i\|\,\|x_j\|-k\operatorname{Re}\langle x_i,x_j\rangle\right]\\ &=2\sum_{1\le i<j\le n}\left[\|x_i\|\,\|x_j\|-k\operatorname{Re}\langle x_i,x_j\rangle\right]+(1-k)\sum_{i=1}^{n}\|x_i\|^2,\end{aligned} \tag{7.3}$$

since, obviously, $\operatorname{Re}\langle x_i,x_j\rangle=\operatorname{Re}\langle x_j,x_i\rangle$ for any $i,j\in\{1,\dots,n\}$.

Using the assumption (7.1), we obtain

$$\sum_{1\leq i<j\leq n}\left[\|x_i\|\,\|x_j\| - k\operatorname{Re}\langle x_i, x_j\rangle\right] \leq 0$$

and thus, from (7.3), we deduce the desired inequality (7.2).

The case of equality is obvious by the identity (7.3) and we omit the details. ∎

Remark 7. *The inequality (7.2) provides the following reverse of the quadratic generalised triangle inequality:*

$$0 \leq \left(\sum_{i=1}^{n}\|x_i\|\right)^2 - \sum_{i=1}^{n}\|x_i\|^2 \leq k\left[\left\|\sum_{i=1}^{n}x_i\right\|^2 - \sum_{i=1}^{n}\|x_i\|^2\right]. \tag{7.4}$$

Remark 8. *Since* $k = 1$ *and* $\sum_{i=1}^{n}\|x_i\|^2 \geq 0$, *hence by (7.2) one may deduce the following reverse of the triangle inequality*

$$\sum_{i=1}^{n}\|x_i\| \leq \sqrt{k}\left\|\sum_{i=1}^{n}x_i\right\|, \tag{7.5}$$

provided (7.1) holds true for $1 \leq i < j \leq n$.

The following corollary providing a better bound for $\sum_{i=1}^{n}\|x_i\|$, holds [4].

Corollary 3. *With the assumptions in Lemma 2, one has the inequality:*

$$\sum_{i=1}^{n}\|x_i\| \leq \sqrt{\frac{nk}{n+k-1}}\left\|\sum_{i=1}^{n}x_i\right\|. \tag{7.6}$$

Proof. Using the Cauchy-Bunyakovsky-Schwarz inequality

$$n\sum_{i=1}^{n}\|x_i\|^2 \geq \left(\sum_{i=1}^{n}\|x_i\|\right)^2$$

we get

$$(k-1)\sum_{i=1}^{n}\|x_i\|^2 + \left(\sum_{i=1}^{n}\|x_i\|\right)^2 \geq \left(\frac{k-1}{n}+1\right)\left(\sum_{i=1}^{n}\|x_i\|\right)^2. \tag{7.7}$$

Consequently, by (7.7) and (7.2) we deduce

$$k\left\|\sum_{i=1}^{n}x_i\right\|^2 \geq \frac{n+k-1}{n}\left(\sum_{i=1}^{n}\|x_i\|\right)^2$$

giving the desired inequality (7.6). ∎

7.2. **Assymetric Assumptions.** The following result may be stated as well [4].

Theorem 14. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space and $x_i\in H\backslash\{0\}$, $i\in\{1,\dots,n\}$, $\rho\in(0,1)$, such that*

$$\left\|x_i-\frac{x_j}{\|x_j\|}\right\|\leq\rho\quad\text{for } 1\leq i<j\leq n. \tag{7.8}$$

Then we have the inequality

$$\sqrt{1-\rho^2}\left(\sum_{i=1}^{n}\|x_i\|\right)^2+\left(1-\sqrt{1-\rho^2}\right)\sum_{i=1}^{n}\|x_i\|^2 \leq \left\|\sum_{i=1}^{n}x_i\right\|^2. \tag{7.9}$$

The case of equality holds in (7.9) iff

$$\|x_i\|\,\|x_j\|=\frac{1}{\sqrt{1-\rho^2}}\operatorname{Re}\langle x_i,x_j\rangle \tag{7.10}$$

for any $1\leq i<j\leq n$.

Proof. The condition (7.1) is obviously equivalent to

$$\|x_i\|^2+1-\rho^2\leq 2\operatorname{Re}\left\langle x_i,\frac{x_j}{\|x_j\|}\right\rangle$$

for each $1\leq i<j\leq n$.

Dividing by $\sqrt{1-\rho^2}>0$, we deduce

$$\frac{\|x_i\|^2}{\sqrt{1-\rho^2}}+\sqrt{1-\rho^2}\leq\frac{2}{\sqrt{1-\rho^2}}\operatorname{Re}\left\langle x_i,\frac{x_j}{\|x_j\|}\right\rangle, \tag{7.11}$$

for $1\leq i<j\leq n$.

On the other hand, by the elementary inequality

$$\frac{p}{\alpha}+q\alpha\geq 2\sqrt{pq},\quad p,q\geq 0,\ \alpha>0 \tag{7.12}$$

we have

$$2\|x_i\|\leq\frac{\|x_i\|^2}{\sqrt{1-\rho^2}}+\sqrt{1-\rho^2}. \tag{7.13}$$

Making use of (7.11) and (7.13), we deduce that

$$\|x_i\|\,\|x_j\|\leq\frac{1}{\sqrt{1-\rho^2}}\operatorname{Re}\langle x_i,x_j\rangle$$

for $1\leq i<j\leq n$.

Now, applying Lemma 1 for $k=\frac{1}{\sqrt{1-\rho^2}}$, we deduce the desired result. ∎

Remark 9. *If we assume that* $\|x_i\|=1$, $i\in\{1,\dots,n\}$, *satisfying the simpler condition*

$$\|x_j-x_i\|\le\rho \quad \text{for } 1\le i<j\le n, \tag{7.14}$$

then, from (7.9), we deduce the following lower bound for $\|\sum_{i=1}^n x_i\|$, *namely*

$$\left[n+n(n-1)\sqrt{1-\rho^2}\right]^{\frac{1}{2}}\le\left\|\sum_{i=1}^n x_i\right\|. \tag{7.15}$$

The equality holds in (7.15) iff $\sqrt{1-\rho^2}=\operatorname{Re}\langle x_i,x_j\rangle$ *for* $1\le i<j\le n$.

Remark 10. *Under the hypothesis of Proposition 5, we have the coarser but simpler reverse of the triangle inequality*

$$\sqrt[4]{1-\rho^2}\sum_{i=1}^n\|x_i\|\le\left\|\sum_{i=1}^n x_i\right\|. \tag{7.16}$$

Also, applying Corollary 3 for $k=\frac{1}{\sqrt{1-\rho^2}}$, *we can state that*

$$\sum_{i=1}^n\|x_i\|\le\sqrt{\frac{n}{n\sqrt{1-\rho^2}+1-\sqrt{1-\rho^2}}}\left\|\sum_{i=1}^n x_i\right\|, \tag{7.17}$$

provided $x_i\in H$ *satisfy (7.8) for* $1\le i<j\le n$.

In the same manner, we can state and prove the following reverse of the quadratic generalised triangle inequality [4].

Theorem 15. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be an inner product space over the real or complex number field* $\mathbb{K}$, $x_i\in H$, $i\in\{1,\dots,n\}$ *and* $M\ge m>0$ *such that either*

$$\operatorname{Re}\langle Mx_j-x_i,x_i-mx_j\rangle\ge0 \quad \text{for } 1\le i<j\le n, \tag{7.18}$$

or, equivalently,

$$\left\|x_i-\frac{M+m}{2}x_j\right\|\le\frac{1}{2}(M-m)\|x_j\| \quad \text{for } 1\le i<j\le n \tag{7.19}$$

hold. Then

$$\frac{2\sqrt{mM}}{M+m}\left(\sum_{i=1}^{n}\|x_i\|\right)^2+\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{M+m}\sum_{i=1}^{n}\|x_i\|^2 \leq \left\|\sum_{i=1}^{n}x_i\right\|^2. \tag{7.20}$$

The case of equality holds in (7.20) if and only if

$$\|x_i\|\,\|x_j\|=\frac{M+m}{2\sqrt{mM}}\operatorname{Re}\langle x_i,x_j\rangle \quad \text{for } 1\leq i<j\leq n. \tag{7.21}$$

Proof. From (7.18), observe that

$$\|x_i\|^2+Mm\,\|x_j\|^2\leq (M+m)\operatorname{Re}\langle x_i,x_j\rangle, \tag{7.22}$$

for $1\leq i<j\leq n$. Dividing (7.22) by $\sqrt{mM}>0$, we deduce

$$\frac{\|x_i\|^2}{\sqrt{mM}}+\sqrt{mM}\,\|x_j\|^2\leq\frac{M+m}{\sqrt{mM}}\operatorname{Re}\langle x_i,x_j\rangle,$$

and since, obviously

$$2\,\|x_i\|\,\|x_j\|\leq\frac{\|x_i\|^2}{\sqrt{mM}}+\sqrt{mM}\,\|x_j\|^2$$

hence

$$\|x_i\|\,\|x_j\|\leq\frac{M+m}{2\sqrt{mM}}\operatorname{Re}\langle x_i,x_j\rangle, \quad \text{for } 1\leq i<j\leq n.$$

Applying Lemma 2 for $k=\frac{M+m}{2\sqrt{mM}}\geq 1$, we deduce the desired result. ∎

Remark 11. *We also must note that a simpler but coarser inequality that can be obtained from (7.20) is*

$$\left(\frac{2\sqrt{mM}}{M+m}\right)^{\frac{1}{2}}\sum_{i=1}^{n}\|x_i\|\leq\left\|\sum_{i=1}^{n}x_i\right\|,$$

provided (7.18) holds true.

Finally, a different result related to the generalised triangle inequality is incorporated in the following theorem [4].

Theorem 16. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$, $\eta>0$ and $x_i\in H$, $i\in\{1,\dots,n\}$ with the property that*

$$\|x_j-x_i\|\leq\eta<\|x_j\| \quad \text{for each } \ i,j\in\{1,\dots,n\}. \tag{7.23}$$

Then we have the following reverse of the triangle inequality

$$\frac{\sum_{i=1}^{n}\sqrt{\|x_i\|^2-\eta^2}}{\|\sum_{i=1}^{n}x_i\|}\le\frac{\|\sum_{i=1}^{n}x_i\|}{\sum_{i=1}^{n}\|x_i\|}. \tag{7.24}$$

The equality holds in (7.24) iff

$$\|x_i\|\sqrt{\|x_j\|^2-\eta^2}=\operatorname{Re}\langle x_i,x_j\rangle \quad \textit{for each} \quad i,j\in\{1,\dots,n\}. \tag{7.25}$$

Proof. From (7.23), we have

$$\|x_i\|^2+\|x_j\|^2-\eta^2\le 2\operatorname{Re}\langle x_i,x_j\rangle, \quad i,j\in\{1,\dots,n\}.$$

On the other hand,

$$2\|x_i\|\sqrt{\|x_j\|^2-\eta^2}\le\|x_i\|^2+\|x_j\|^2-\eta^2, \quad i,j\in\{1,\dots,n\}$$

and thus

$$\|x_i\|\sqrt{\|x_j\|^2-\eta^2}\le\operatorname{Re}\langle x_i,x_j\rangle, \quad i,j\in\{1,\dots,n\}.$$

Summing over $i,j\in\{1,\dots,n\}$, we deduce the desired inequality (7.24).

The case of equality is also obvious from the above, and we omit the details. ∎

8. Reverses for Complex Spaces

8.1. The Case of One Vector.

The following result holds [5].

Theorem 17. *Let $(H;\langle\cdot,\cdot\rangle)$ be a complex inner product space. Suppose that the vectors $x_k\in H$, $k\in\{1,\dots,n\}$ satisfy the condition*

$$0\le r_1\|x_k\|\le\operatorname{Re}\langle x_k,e\rangle, \quad 0\le r_2\|x_k\|\le\operatorname{Im}\langle x_k,e\rangle \tag{8.1}$$

for each $k\in\{1,\dots,n\}$, where $e\in H$ is such that $\|e\|=1$ and $r_1,r_2\ge 0$. Then we have the inequality

$$\sqrt{r_1^2+r_2^2}\sum_{k=1}^{n}\|x_k\|\le\left\|\sum_{k=1}^{n}x_k\right\|, \tag{8.2}$$

where equality holds if and only if

$$\sum_{k=1}^{n}x_k=(r_1+ir_2)\left(\sum_{k=1}^{n}\|x_k\|\right)e. \tag{8.3}$$

Proof. In view of the Schwarz inequality in the complex inner product space $(H; \langle\cdot,\cdot\rangle)$, we have

$$\left\|\sum_{k=1}^{n} x_k\right\|^2 = \left\|\sum_{k=1}^{n} x_k\right\|^2 \|e\|^2 \geq \left|\left\langle \sum_{k=1}^{n} x_k, e\right\rangle\right|^2 \tag{8.4}$$
$$= \left|\left\langle \sum_{k=1}^{n} x_k, e\right\rangle\right|^2$$
$$= \left|\sum_{k=1}^{n} \operatorname{Re}\langle x_k, e\rangle + i\left(\sum_{k=1}^{n} \operatorname{Im}\langle x_k, e\rangle\right)\right|^2$$
$$= \left(\sum_{k=1}^{n} \operatorname{Re}\langle x_k, e\rangle\right)^2 + \left(\sum_{k=1}^{n} \operatorname{Im}\langle x_k, e\rangle\right)^2 .$$

Now, by hypothesis (8.1)

$$\left(\sum_{k=1}^{n} \operatorname{Re}\langle x_k, e\rangle\right)^2 \geq r_1^2 \left(\sum_{k=1}^{n} \|x_k\|\right)^2 \tag{8.5}$$

and

$$\left(\sum_{k=1}^{n} \operatorname{Im}\langle x_k, e\rangle\right)^2 \geq r_2^2 \left(\sum_{k=1}^{n} \|x_k\|\right)^2 . \tag{8.6}$$

If we add (8.5) and (8.6) and use (8.4), then we deduce the desired inequality (8.2).

Now, if (8.3) holds, then

$$\left\|\sum_{k=1}^{n} x_k\right\| = |r_1 + ir_2| \left(\sum_{k=1}^{n} \|x_k\|\right) \|e\| = \sqrt{r_1^2 + r_2^2} \sum_{k=1}^{n} \|x_k\|$$

and the case of equality is valid in (8.2).

Before we prove the reverse implication, let us observe that for $x \in H$ and $e \in H$, $\|e\| = 1$, the following identity is true

$$\|x - \langle x, e\rangle e\|^2 = \|x\|^2 - |\langle x, e\rangle|^2 ,$$

therefore $\|x\| = |\langle x, e\rangle|$ if and only if $x = \langle x, e\rangle e$.

If we assume that equality holds in (8.2), then the case of equality must hold in all the inequalities required in the argument used to prove the inequality (8.2), and we may state that

$$\left\|\sum_{k=1}^{n} x_k\right\| = \left|\left\langle \sum_{k=1}^{n} x_k, e\right\rangle\right| , \tag{8.7}$$

and

$$r_1 \|x_k\| = \operatorname{Re}\langle x_k, e\rangle, \quad r_2 \|x_k\| = \operatorname{Im}\langle x_k, e\rangle \tag{8.8}$$

for each $k \in \{1, \dots, n\}$.

From (8.7) we deduce

$$\sum_{k=1}^{n} x_k = \left\langle \sum_{k=1}^{n} x_k, e \right\rangle e \tag{8.9}$$

and from (8.8), by multiplying the second equation with i and summing both equations over k from 1 to n, we deduce

$$(r_1 + i r_2) \sum_{k=1}^{n} \|x_k\| = \left\langle \sum_{k=1}^{n} x_k, e \right\rangle. \tag{8.10}$$

Finally, by (8.10) and (8.9), we get the desired equality (8.3). ∎

The following corollary is of interest [5].

Corollary 4. *Let e a unit vector in the complex inner product space $(H; \langle\cdot,\cdot\rangle)$ and $\rho_1, \rho_2 \in (0,1)$. If $x_k \in H$, $k \in \{1, \dots, n\}$ are such that*

$$\|x_k - e\| \le \rho_1, \quad \|x_k - ie\| \le \rho_2 \quad \text{for each} \ \ k \in \{1, \dots, n\}, \tag{8.11}$$

then we have the inequality

$$\sqrt{2 - \rho_1^2 - \rho_2^2} \sum_{k=1}^{n} \|x_k\| \le \left\| \sum_{k=1}^{n} x_k \right\|, \tag{8.12}$$

with equality if and only if

$$\sum_{k=1}^{n} x_k = \left(\sqrt{1 - \rho_1^2} + i \sqrt{1 - \rho_2^2} \right) \left(\sum_{k=1}^{n} \|x_k\| \right) e. \tag{8.13}$$

Proof. From the first inequality in (8.11) we deduce

$$0 \le \sqrt{1 - \rho_1^2} \|x_k\| \le \operatorname{Re}\langle x_k, e\rangle \tag{8.14}$$

for each $k \in \{1, \dots, n\}$.

From the second inequality in (8.11) we deduce

$$0 \le \sqrt{1 - \rho_2^2} \|x_k\| \le \operatorname{Re}\langle x_k, ie\rangle$$

for each $k \in \{1, \dots, n\}$. Since

$$\operatorname{Re}\langle x_k, ie\rangle = \operatorname{Im}\langle x_k, e\rangle,$$

hence

$$0 \le \sqrt{1 - \rho_2^2} \|x_k\| \le \operatorname{Im}\langle x_k, e\rangle \tag{8.15}$$

for each $k \in \{1, \ldots, n\}$.

Now, observe from (8.14) and (8.15), that the condition (8.1) of Theorem 17 is satisfied for $r_1 = \sqrt{1-\rho_1^2}$, $r_2 = \sqrt{1-\rho_2^2} \in (0,1)$, and thus the corollary is proved. ∎

The following corollary may be stated as well [5].

Corollary 5. *Let e be a unit vector in the complex inner product space $(H; \langle \cdot, \cdot \rangle)$ and $M_1 \geq m_1 > 0$, $M_2 \geq m_2 > 0$. If $x_k \in H$, $k \in \{1, \ldots, n\}$ are such that either*

$$\begin{aligned} \operatorname{Re} \langle M_1 e - x_k, x_k - m_1 e \rangle &\geq 0, \\ \operatorname{Re} \langle M_2 ie - x_k, x_k - m_2 ie \rangle &\geq 0 \end{aligned} \tag{8.16}$$

or, equivalently,

$$\begin{aligned} \left\| x_k - \frac{M_1 + m_1}{2} e \right\| &\leq \frac{1}{2} (M_1 - m_1), \\ \left\| x_k - \frac{M_2 + m_2}{2} ie \right\| &\leq \frac{1}{2} (M_2 - m_2), \end{aligned} \tag{8.17}$$

for each $k \in \{1, \ldots, n\}$, then we have the inequality

$$2 \left[\frac{m_1 M_1}{(M_1 + m_1)^2} + \frac{m_2 M_2}{(M_2 + m_2)^2} \right]^{1/2} \sum_{k=1}^{n} \|x_k\| \leq \left\| \sum_{k=1}^{n} x_k \right\|. \tag{8.18}$$

The equality holds in (8.18) if and only if

$$\sum_{k=1}^{n} x_k = 2 \left(\frac{\sqrt{m_1 M_1}}{M_1 + m_1} + i \frac{\sqrt{m_2 M_2}}{M_2 + m_2} \right) \left(\sum_{k=1}^{n} \|x_k\| \right) e. \tag{8.19}$$

Proof. From the first inequality in (8.16)

$$0 \leq \frac{2\sqrt{m_1 M_1}}{M_1 + m_1} \|x_k\| \leq \operatorname{Re} \langle x_k, e \rangle \tag{8.20}$$

for each $k \in \{1, \ldots, n\}$.

Now, the proof follows the same path as the one of Corollary 4 and we omit the details. ∎

8.2. **The Case of m Orthonormal Vectors.** In [1], the authors have proved the following reverse of the generalised triangle inequality in terms of orthonormal vectors [5].

Theorem 18. *Let $e_1, \ldots, e_m$ be orthonormal vectors in $(H; \langle \cdot, \cdot \rangle)$, i.e., we recall that $\langle e_i, e_j \rangle = 0$ if $i \neq j$ and $\|e_i\| = 1$, $i, j \in \{1, \ldots, m\}$. Suppose that the vectors $x_1, \ldots, x_n \in H$ satisfy*

$$0 \leq r_k \|x_j\| \leq \operatorname{Re} \langle x_j, e_k \rangle,$$

$j\in\{1,\dots,n\}$, $k\in\{1,\dots,m\}$. *Then*

$$\left(\sum_{k=1}^{m}r_k^2\right)^{\frac{1}{2}}\sum_{j=1}^{n}\|x_j\|\le\left\|\sum_{j=1}^{n}x_j\right\|, \tag{8.21}$$

where equality holds if and only if

$$\sum_{j=1}^{n}x_j=\left(\sum_{j=1}^{n}\|x_j\|\right)\sum_{k=1}^{m}r_ke_k. \tag{8.22}$$

If the space $(H;\langle\cdot,\cdot\rangle)$ is complex and more information is available for the imaginary part, then the following result may be stated as well [5].

Theorem 19. *Let $e_1,\dots,e_m\in H$ be an orthonormal family of vectors in the complex inner product space H. If the vectors $x_1,\dots,x_n\in H$ satisfy the conditions*

$$0\le r_k\|x_j\|\le\operatorname{Re}\langle x_j,e_k\rangle,\qquad 0\le\rho_k\|x_j\|\le\operatorname{Im}\langle x_j,e_k\rangle \tag{8.23}$$

for each $j\in\{1,\dots,n\}$ and $k\in\{1,\dots,m\}$, then we have the following reverse of the generalised triangle inequality;

$$\left[\sum_{k=1}^{m}\left(r_k^2+\rho_k^2\right)\right]^{\frac{1}{2}}\sum_{j=1}^{n}\|x_j\|\le\left\|\sum_{j=1}^{n}x_j\right\|. \tag{8.24}$$

The equality holds in (8.24) if and only if

$$\sum_{j=1}^{n}x_j=\left(\sum_{j=1}^{n}\|x_j\|\right)\sum_{k=1}^{m}\left(r_k+i\rho_k\right)e_k. \tag{8.25}$$

Proof. Before we prove the theorem, let us recall that, if $x\in H$ and $e_1,\dots,e_m$ are orthogonal vectors, then the following identity holds true:

$$\left\|x-\sum_{k=1}^{m}\langle x,e_k\rangle e_k\right\|^2=\|x\|^2-\sum_{k=1}^{n}|\langle x,e_k\rangle|^2. \tag{8.26}$$

As a consequence of this identity, we note the *Bessel inequality*

$$\sum_{k=1}^{m}|\langle x,e_k\rangle|^2\le\|x\|^2, x\in H. \tag{8.27}$$

The case of equality holds in (8.27) if and only if (see (8.26))

$$x=\sum_{k=1}^{m}\langle x,e_k\rangle e_k. \tag{8.28}$$

Applying Bessel's inequality for $x = \sum_{j=1}^{n} x_j$, we have

$$\left\| \sum_{j=1}^{n} x_j \right\|^2 \geq \sum_{k=1}^{m} \left| \left\langle \sum_{j=1}^{n} x_j, e_k \right\rangle \right|^2 = \sum_{k=1}^{m} \left| \sum_{j=1}^{n} \langle x_j, e_k \rangle \right|^2 \tag{8.29}$$

$$= \sum_{k=1}^{m} \left| \left(\sum_{j=1}^{n} \operatorname{Re} \langle x_j, e_k \rangle \right) + i \left(\sum_{j=1}^{n} \operatorname{Im} \langle x_j, e_k \rangle \right) \right|^2$$

$$= \sum_{k=1}^{m} \left[\left(\sum_{j=1}^{n} \operatorname{Re} \langle x_j, e_k \rangle \right)^2 + \left(\sum_{j=1}^{n} \operatorname{Im} \langle x_j, e_k \rangle \right)^2 \right].$$

Now, by the hypothesis (8.23) we have

$$\left(\sum_{j=1}^{n} \operatorname{Re} \langle x_j, e_k \rangle \right)^2 \geq r_k^2 \left(\sum_{j=1}^{n} \|x_j\| \right)^2 \tag{8.30}$$

and

$$\left(\sum_{j=1}^{n} \operatorname{Im} \langle x_j, e_k \rangle \right)^2 \geq \rho_k^2 \left(\sum_{j=1}^{n} \|x_j\| \right)^2. \tag{8.31}$$

Further, on making use of (8.29) – (8.31), we deduce

$$\left\| \sum_{j=1}^{n} x_j \right\|^2 \geq \sum_{k=1}^{m} \left[r_k^2 \left(\sum_{j=1}^{n} \|x_j\| \right)^2 + \rho_k^2 \left(\sum_{j=1}^{n} \|x_j\| \right)^2 \right]$$

$$= \left(\sum_{j=1}^{n} \|x_j\| \right)^2 \sum_{k=1}^{m} \left(r_k^2 + \rho_k^2 \right),$$

which is clearly equivalent to (8.24).

Now, if (8.25) holds, then

$$\left\| \sum_{j=1}^{n} x_j \right\|^2 = \left(\sum_{j=1}^{n} \|x_j\| \right)^2 \left\| \sum_{k=1}^{m} (r_k + i\rho_k) e_k \right\|^2$$

$$= \left(\sum_{j=1}^{n} \|x_j\| \right)^2 \sum_{k=1}^{m} |r_k + i\rho_k|^2$$

$$= \left(\sum_{j=1}^{n} \|x_j\| \right)^2 \sum_{k=1}^{m} \left(r_k^2 + \rho_k^2 \right),$$

and the case of equality holds in (8.24).

Conversely, if the equality holds in (8.24), then it must hold in all the inequalities used to prove (8.24) and therefore we must have

$$\left\| \sum_{j=1}^{n} x_j \right\|^2 = \sum_{k=1}^{m} \left| \sum_{j=1}^{n} \langle x_j, e_k \rangle \right|^2 \tag{8.32}$$

and

$$r_k \|x_j\| = \operatorname{Re} \langle x_j, e_k \rangle, \qquad \rho_k \|x_j\| = \operatorname{Im} \langle x_j, e_k \rangle \tag{8.33}$$

for each $j \in \{1, \dots, n\}$ and $k \in \{1, \dots, m\}$.

Using the identity (8.26), we deduce from (8.32) that

$$\sum_{j=1}^{n} x_j = \sum_{k=1}^{m} \left\langle \sum_{j=1}^{n} x_j, e_k \right\rangle e_k. \tag{8.34}$$

Multiplying the second equality in (8.33) with the imaginary unit i and summing the equality over j from 1 to n, we deduce

$$(r_k + i\rho_k) \sum_{j=1}^{n} \|x_j\| = \left\langle \sum_{j=1}^{n} x_j, e_k \right\rangle \tag{8.35}$$

for each $k \in \{1, \dots, n\}$.

Finally, utilising (8.34) and (8.35), we deduce (8.25) and the theorem is proved. ∎

The following corollaries are of interest [5].

Corollary 6. *Let $e_1, \dots, e_m$ be orthonormal vectors in the complex inner product space $(H; \langle \cdot, \cdot \rangle)$ and $\rho_k, \eta_k \in (0, 1)$, $k \in \{1, \dots, n\}$. If $x_1, \dots, x_n \in H$ are such that*

$$\|x_j - e_k\| \le \rho_k, \qquad \|x_j - ie_k\| \le \eta_k$$

for each $j \in \{1, \dots, n\}$ and $k \in \{1, \dots, m\}$, then we have the inequality

$$\left[\sum_{k=1}^{m} \left(2 - \rho_k^2 - \eta_k^2\right) \right]^{\frac{1}{2}} \sum_{j=1}^{n} \|x_j\| \le \left\| \sum_{j=1}^{n} x_j \right\|. \tag{8.36}$$

The case of equality holds in (8.36) if and only if

$$\sum_{j=1}^{n} x_j = \left(\sum_{j=1}^{n} \|x_j\| \right) \sum_{k=1}^{m} \left(\sqrt{1 - \rho_k^2} + i\sqrt{1 - \eta_k^2} \right) e_k. \tag{8.37}$$

The proof employs Theorem 19 and is similar to the one from Corollary 4. We omit the details.

Corollary 7. *Let* $e_1, \ldots, e_m$ *be as in Corollary 6 and* $M_k \geq m_k > 0$, $N_k \geq n_k > 0$, $k \in \{1, \ldots, m\}$. *If* $x_1, \ldots, x_n \in H$ *are such that either*

$$\operatorname{Re}\langle M_k e_k - x_j, x_j - m_k e_k\rangle \geq 0, \quad \operatorname{Re}\langle N_k i e_k - x_j, x_j - n_k i e_k\rangle \geq 0$$

or, equivalently,

$$\left\| x_j - \frac{M_k + m_k}{2} e_k \right\| \leq \frac{1}{2}(M_k - m_k),$$
$$\left\| x_j - \frac{N_k + n_k}{2} i e_k \right\| \leq \frac{1}{2}(N_k - n_k)$$

for each $j \in \{1, \ldots, n\}$ *and* $k \in \{1, \ldots, m\}$, *then we have the inequality*

$$2\left\{\sum_{k=1}^{m}\left[\frac{m_k M_k}{(M_k + m_k)^2} + \frac{n_k N_k}{(N_k + n_k)^2}\right]\right\}^{\frac{1}{2}} \sum_{j=1}^{n}\|x_j\| \leq \left\|\sum_{j=1}^{n} x_j\right\|. \tag{8.38}$$

The case of equality holds in (8.38) if and only if

$$\sum_{j=1}^{n} x_j = 2\left(\sum_{j=1}^{n}\|x_j\|\right)\sum_{k=1}^{m}\left(\frac{\sqrt{m_k M_k}}{M_k + m_k} + i\frac{\sqrt{n_k N_k}}{N_k + n_k}\right)e_k. \tag{8.39}$$

The proof employs Theorem 19 and is similar to the one in Corollary 5. We omit the details.

9. Applications for Vector-Valued Integral Inequalities

Let $(H; \langle\cdot,\cdot\rangle)$ be a Hilbert space over the real or complex number field, $[a, b]$ a compact interval in $\mathbb{R}$ and $\eta : [a, b] \to [0, \infty)$ a Lebesgue integrable function on $[a, b]$ with the property that $\int_a^b \eta(t)\,dt = 1$. If, by $L_\eta([a, b]; H)$ we denote the Hilbert space of all Bochner measurable functions $f : [a, b] \to H$ with the property that $\int_a^b \eta(t)\|f(t)\|^2\,dt < \infty$, then the norm $\|\cdot\|_\eta$ of this space is generated by the inner product $\langle\cdot,\cdot\rangle_\eta : H \times H \to \mathbb{K}$ defined by

$$\langle f, g\rangle_\eta := \int_a^b \eta(t)\langle f(t), g(t)\rangle\,dt.$$

The following proposition providing a reverse of the integral generalised triangle inequality may be stated [3].

Proposition 6. *Let* $(H; \langle\cdot,\cdot\rangle)$ *be a Hilbert space and* $\eta : [a, b] \to [0, \infty)$ *as above. If* $g \in L_\eta([a, b]; H)$ *is so that* $\int_a^b \eta(t)\|g(t)\|^2\,dt = 1$ *and* $f_i \in L_\eta([a, b]; H)$, $i \in \{1, \ldots, n\}$, $\rho \in (0, 1)$ *are so that*

$$\|f_i(t) - g(t)\| \leq \rho \tag{9.1}$$

for a.e. $t \in [a,b]$ *and each* $i \in \{1,\dots,n\}$, *then we have the inequality*

$$\sqrt{1-\rho^2}\sum_{i=1}^{n}\left(\int_a^b \eta(t)\left\|f_i(t)\right\|^2 dt\right)^{1/2} \leq \left(\int_a^b \eta(t)\left\|\sum_{i=1}^{n} f_i(t)\right\|^2 dt\right)^{1/2}. \tag{9.2}$$

The case of equality holds in (9.2) if and only if

$$\sum_{i=1}^{n} f_i(t) = \sqrt{1-\rho^2}\sum_{i=1}^{n}\left(\int_a^b \eta(t)\left\|f_i(t)\right\|^2 dt\right)^{1/2}\cdot g(t)$$

for a.e. $t \in [a,b]$.

Proof. Observe, by (9.2), that

$$\begin{aligned}\left\|f_i - g\right\|_\eta &= \left(\int_a^b \eta(t)\left\|f_i(t)-g(t)\right\|^2 dt\right)^{1/2} \\ &\leq \left(\int_a^b \eta(t)\rho^2 dt\right)^{1/2} = \rho\end{aligned}$$

for each $i \in \{1,\dots,n\}$. Applying Theorem 3 for the Hilbert space $L_\eta([a,b];H)$, we deduce the desired result. ∎

The following result may be stated as well [3].

Proposition 7. *Let* H,η,g *be as in Proposition 6. If* $f_i \in L_\eta([a,b];H)$, $i \in \{1,\dots,n\}$ *and* $M \geq m > 0$ *are so that either*

$$\operatorname{Re}\langle Mg(t)-f_i(t), f_i(t)-mg(t)\rangle \geq 0$$

or, equivalently,

$$\left\|f_i(t)-\frac{m+M}{2}g(t)\right\| \leq \frac{1}{2}(M-m)$$

for a.e. $t \in [a,b]$ *and each* $i \in \{1,\dots,n\}$, *then we have the inequality*

$$\frac{2\sqrt{mM}}{m+M}\sum_{i=1}^{n}\left(\int_a^b \eta(t)\left\|f_i(t)\right\|^2 dt\right)^{1/2} \leq \left(\int_a^b \eta(t)\left\|\sum_{i=1}^{n} f_i(t)\right\|^2 dt\right)^{1/2}. \tag{9.3}$$

The equality holds in (9.3) if and only if

$$\sum_{i=1}^{n} f_i(t) = \frac{2\sqrt{mM}}{m+M} \sum_{i=1}^{n} \left(\int_a^b \eta(t) \|f_i(t)\|^2 dt \right)^{1/2} \cdot g(t),$$

for a.e. $t \in [a,b]$.

The following proposition providing a reverse of the integral generalised triangle inequality may be stated [4].

Proposition 8. *Let* $(H; \langle \cdot, \cdot \rangle)$ *be a Hilbert space and* $\eta : [a,b] \to [0,\infty)$ *as above. If* $g \in L_\eta([a,b]; H)$ *is so that* $\int_a^b \eta(t) \|g(t)\|^2 dt = 1$ *and* $f_i \in L_\eta([a,b]; H), i \in \{1,\dots,n\}$, *and* $M \geq m > 0$ *are so that either*

$$\operatorname{Re} \langle M f_j(t) - f_i(t), f_i(t) - m f_j(t) \rangle \geq 0 \tag{9.4}$$

or, equivalently,

$$\left\| f_i(t) - \frac{m+M}{2} f_j(t) \right\| \leq \frac{1}{2} (M-m) \|f_j(t)\|$$

for a.e. $t \in [a,b]$ *and* $1 \leq i < j \leq n$, *then we have the inequality*

$$\begin{aligned} &\left[\sum_{i=1}^{n} \left(\int_a^b \eta(t) \|f_i(t)\|^2 dt \right)^{1/2} \right]^2 \\ \leq\ & \int_a^b \eta(t) \left\| \sum_{i=1}^{n} f_i(t) \right\|^2 dt \\ &+ \frac{1}{2} \cdot \frac{(M-m)^2}{m+M} \int_a^b \eta(t) \left(\sum_{k=1}^{n-1} k \|f_{k+1}(t)\|^2 \right) dt. \end{aligned} \tag{9.5}$$

The case of equality holds in (9.5) if and only if

$$\begin{aligned} & \left(\int_a^b \eta(t) \|f_i(t)\|^2 dt \right)^{1/2} \left(\int_a^b \eta(t) \|f_j(t)\|^2 dt \right)^{1/2} \\ & - \int_a^b \eta(t) \operatorname{Re} \langle f_i(t), f_j(t) \rangle dt \\ =\ & \frac{1}{4} \cdot \frac{(M-m)^2}{m+M} \int_a^b \eta(t) \|f_j(t)\|^2 dt \end{aligned}$$

for each i,j *with* $1 \leq i < j \leq n$.

Proof. We observe that

$$\begin{aligned}
&\operatorname{Re}\langle Mf_j - f_i, f_i - mf_j\rangle_\eta \\
&= \int_a^b \eta(t) \operatorname{Re}\langle Mf_j(t) - f_i(t), f_i(t) - mf_j(t)\rangle\, dt \geq 0
\end{aligned}$$

for any i, j with $1 \leq i < j \leq n$.

Applying Theorem 13 for the Hilbert space $L_\eta([a,b]; H)$ and for $y_i = f_i, i \in \{1, \dots, n\}$, we deduce the desired result. ∎

Another integral inequality incorporated in the following proposition holds [4]:

Proposition 9. *With the assumptions of Proposition 8, we have*

$$\begin{aligned}
&\frac{2\sqrt{mM}}{m+M}\left[\sum_{i=1}^n \left(\int_a^b \eta(t)\|f_i(t)\|^2 dt\right)^{1/2}\right]^2 \\
&+\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{m+M}\sum_{i=1}^n \int_a^b \eta(t)\|f_i(t)\|^2 dt \\
&\leq \int_a^b \eta(t)\left\|\sum_{i=1}^n f_i(t)\right\|^2 dt.
\end{aligned} \tag{9.6}$$

The case of equality holds in (9.6) if and only if

$$\begin{aligned}
&\left(\int_a^b \eta(t)\|f_i(t)\|^2 dt\right)^{1/2}\left(\int_a^b \eta(t)\|f_j(t)\|^2 dt\right)^{1/2} \\
&= \frac{M+m}{2\sqrt{mM}}\int_a^b \eta(t)\operatorname{Re}\langle f_i(t), f_j(t)\rangle\, dt
\end{aligned}$$

for any i, j *with* $1 \leq i < j \leq n$.

The proof is obvious by Theorem 15 and we omit the details.orms obtained above, but we do not mention them here.

10. Applications for Complex Numbers

The following reverse of the generalised triangle inequality with a clear geometric meaning may be stated [5].

Proposition 10. *Let* $z_1, \dots, z_n$ *be complex numbers with the property that*

$$0 \leq \varphi_1 \leq \arg(z_k) \leq \varphi_2 < \frac{\pi}{2} \tag{10.1}$$

for each $k \in \{1, \ldots, n\}$. *Then we have the inequality*

$$\sqrt{\sin^2 \varphi_1 + \cos^2 \varphi_2} \sum_{k=1}^{n} |z_k| \leq \left| \sum_{k=1}^{n} z_k \right|. \tag{10.2}$$

The equality holds in (10.2) if and only if

$$\sum_{k=1}^{n} z_k = (\cos \varphi_2 + i \sin \varphi_1) \sum_{k=1}^{n} |z_k|. \tag{10.3}$$

Proof. Let $z_k = a_k + ib_k$. We may assume that $b_k \geq 0$, $a_k > 0$, $k \in \{1, \ldots, n\}$, since, by (10.1), $\frac{b_k}{a_k} = \tan [\arg (z_k)] \in \left[0, \frac{\pi}{2}\right)$, $k \in \{1, \ldots, n\}$. By (10.1), we obviously have

$$0 \leq \tan^2 \varphi_1 \leq \frac{b_k^2}{a_k^2} \leq \tan^2 \varphi_2, \qquad k \in \{1, \ldots, n\}$$

from where we get

$$\frac{b_k^2 + a_k^2}{a_k^2} \leq \frac{1}{\cos^2 \varphi_2}, \qquad k \in \{1, \ldots, n\}, \ \varphi_2 \in \left(0, \frac{\pi}{2}\right)$$

and

$$\frac{a_k^2 + b_k^2}{a_k^2} \leq \frac{1 + \tan^2 \varphi_1}{\tan^2 \varphi_1} = \frac{1}{\sin^2 \varphi_1}, \qquad k \in \{1, \ldots, n\}, \ \varphi_1 \in \left(0, \frac{\pi}{2}\right)$$

giving the inequalities

$$|z_k| \cos \varphi_2 \leq \operatorname{Re}(z_k), \quad |z_k| \sin \varphi_1 \leq \operatorname{Im}(z_k)$$

for each $k \in \{1, \ldots, n\}$.

Now, applying Theorem 17 for the complex inner product $\mathbb{C}$ endowed with the inner product $\langle z, w \rangle = z \cdot \bar{w}$ for $x_k = z_k$, $r_1 = \cos \varphi_2$, $r_2 = \sin \varphi_1$ and $e = 1$, we deduce the desired inequality (10.2). The case of equality is also obvious by Theorem 17 and the proposition is proven. ∎

Another result that has an obvious geometrical interpretation is the following one.

Proposition 11. *Let* $e \in \mathbb{C}$ *with* $|z| = 1$ *and* $\rho_1, \rho_2 \in (0,1)$. *If* $z_k \in \mathbb{C}$, $k \in \{1, \ldots, n\}$ *are such that*

$$|z_k - c| \leq \rho_1, \quad |z_k - ic| \leq \rho_2 \quad \textit{for each} \quad k \in \{1, \ldots, n\}, \tag{10.4}$$

then we have the inequality

$$\sqrt{2 - \rho_1^2 - \rho_2^2} \sum_{k=1}^{n} |z_k| \leq \left| \sum_{k=1}^{n} z_k \right|, \tag{10.5}$$

with equality if and only if

$$\sum_{k=1}^{n} z_k = \left(\sqrt{1-\rho_1^2} + i\sqrt{1-\rho_2^2}\right)\left(\sum_{k=1}^{n} |z_k|\right) e. \tag{10.6}$$

The proof is obvious by Corollary 4 applied for $H = \mathbb{C}$.

Remark 12. *If we choose $e = 1$, and for $\rho_1, \rho_2 \in (0,1)$ we define $\bar{D}(1,\rho_1) := \{z \in \mathbb{C} | \, |z-1| \leq \rho_1\}$, $\bar{D}(i,\rho_2) := \{z \in \mathbb{C} | \, |z-i| \leq \rho_2\}$, then obviously the intersection*

$$S_{\rho_1,\rho_2} := \bar{D}(1,\rho_1) \cap \bar{D}(i,\rho_2)$$

is nonempty if and only if $\rho_1 + \rho_2 > \sqrt{2}$.

If $z_k \in S_{\rho_1,\rho_2}$ for $k \in \{1,\dots,n\}$, then (10.5) holds true. The equality holds in (10.5) if and only if

$$\sum_{k=1}^{n} z_k = \left(\sqrt{1-\rho_1^2} + i\sqrt{1-\rho_2^2}\right)\sum_{k=1}^{n} |z_k|.$$

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School of Computer Science & Mathematics, Victoria University, PO Box 14428,, Melbourne City MC, Victoria 8001, Australia

E-mail address: Sever.Dragomir@vu.edu.au

URL: http://rgmia.vu.edu.au/SSDragomirWeb.html