GENERALIZED SYMMETRIC DIVERGENCE MEASURES AND INEQUALITIES

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ABSTRACT. In this paper, we have studied the following two *divergence measures of type* s:

$$\mathcal{V}_s(P||Q) = \begin{cases} J_s(P||Q) = [s(s-1)]^{-1} \left[\sum_{i=1}^{n} \left(p_i^s q_i^{1-s} + p_i^{1-s} q_i^s \right) - 2 \right], & s \neq 0, 1 \\ J(P||Q) = \sum_{i=1}^{n} (p_i - q_i) \ln \left(\frac{p_i}{q_i} \right), & s = 0, 1 \end{cases}$$

and

$$\mathcal{W}_s(P||Q) = \begin{cases} IT_s(P||Q) = [s(s-1)]^{-1} \left[\sum_{i=1}^{n} \left(\frac{p_i^{1-s} + q_i^{1-s}}{2} \right) \left(\frac{p_i + q_i}{2} \right)^s - 1 \right], & s \neq 0, 1 \\ I(P||Q) = \frac{1}{2} \left[\sum_{i=1}^{n} p_i \ln \left(\frac{2p_i}{p_i + q_i} \right) + \sum_{i=1}^{n} q_i \ln \left(\frac{2q_i}{p_i + q_i} \right) \right], & s = 0 \\ T(P||Q) = \sum_{i=1}^{n} \left(\frac{p_i + q_i}{2} \right) \ln \left(\frac{p_i + q_i}{2\sqrt{p_i q_i}} \right), & s = 1 \end{cases}$$

The first measure generalizes the well known *J-divergence* due to Jeffreys [16] and Kullback and Leibler [17]. The second measure gives a unified generalization of *Jensen-Shannon divergence* due to Sibson [21] and Burbea and Rao [2, 3], and *arithmetic-geometric mean divergence* due to Taneja [26]. These two measures contain in particular some well known divergences such as: *Hellinger's discrimination*, *triangular discrimination* and *symmetric chi-square divergence*. In this paper we have studied the properties of the above two measures and derived some inequalities among them.

1. INTRODUCTION

Let

$$\Gamma_n = \left\{ P = (p_1, p_2, ..., p_n) \middle| p_i > 0, \sum_{i=1}^{n} p_i = 1 \right\}, \ n \geqslant 2,$$

be the set of all complete finite discrete probability distributions. For all $P, Q \in \Gamma_n$, the following measures are well known in the literature on information theory and statistics:

- **Hellinger Discrimination** (Hellinger [15])

$$h(P||Q) = 1 - B(P||Q) = \frac{1}{2} \sum_{i=1}^{n} (\sqrt{p_i} - \sqrt{q_i})^2 , \tag{1.1}$$

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where

$$B(P||Q) = \sum_{i=1}^{n} \sqrt{p_i q_i}. \tag{1.2}$$

is the well-known Bhattacharyya [1] *coefficient*.

- **Triangular Discrimination**

$$\Delta(P||Q) = 2\left[1 - W(P||Q)\right] = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{p_i + q_i}, \tag{1.3}$$

where

$$W(P||Q) = \sum_{i=1}^{n} \frac{2p_i q_i}{p_i + q_i}, \tag{1.4}$$

is the well-known *harmonic mean divergence.*

- **Symmetric Chi-square Divergence** (Dragomir et al. [14])

$$\Psi(P||Q) = \chi^2(P||Q) + \chi^2(Q||P) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2 (p_i + q_i)}{p_i q_i}, \tag{1.5}$$

where

$$\chi^2(P||Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{q_i} = \sum_{i=1}^{n} \frac{p_i^2}{q_i} - 1, \tag{1.6}$$

is the well-known χ^2-*divergence* (Pearson [20])

- **J-Divergence** (Jeffreys [16]; Kullback-Leibler [17])

$$J(P||Q) = \sum_{i=1}^{n} (p_i - q_i) \ln(\frac{p_i}{q_i}). \tag{1.7}$$

- **Jensen-Shannon Divergence** (Sibson [21]; Burbea and Rao [2, 3])

$$I(P||Q) = \frac{1}{2}\left[\sum_{i=1}^{n} p_i \ln\left(\frac{2p_i}{p_i + q_i}\right) + \sum_{i=1}^{n} q_i \ln\left(\frac{2q_i}{p_i + q_i}\right)\right]. \tag{1.8}$$

- **Arithmetic-Geometric Divergence** (Taneja [26])

$$T(P||Q) = \sum_{i=1}^{n} \left(\frac{p_i + q_i}{2}\right) \ln\left(\frac{p_i + q_i}{2\sqrt{p_i q_i}}\right). \tag{1.9}$$

After simplification, we can write

$$J(P||Q) = 4\left[I(P||Q) + T(P||Q)\right]. \tag{1.10}$$

The measures $J(P||Q)$, $I(P||Q)$ and $T(P||Q)$ can also be written as

$$J(P||Q) = K(P||Q) + K(Q||P), \tag{1.11}$$

$$I(P||Q) = \frac{1}{2}\left[K\left(P||\frac{P+Q}{2}\right) + K\left(Q||\frac{P+Q}{2}\right)\right] \tag{1.12}$$

and

$$T(P||Q) = \frac{1}{2}\left[K\left(\frac{P+Q}{2}||P\right) + K\left(\frac{P+Q}{2}||Q\right)\right], \tag{1.13}$$

where

$$K(P||Q) = \sum_{i=1}^{n} p_i \log\left(\frac{p_i}{q_i}\right), \tag{1.14}$$

is the well known Kullback-Leibler [17] *relative information*.

The measure (1.9) is also known by *Jensen difference divergence measure* (Burbea and Rao [2, 3]). The measure (1.10) is new in the literature and is studied for the first time by Taneja [26] and is called *arithmetic and geometric mean divergence measure.* For simplicity, the three measures appearing in (1.8), (1.9) and (1.10) we shall call, *JS-divergence*, *J-divergence* and the *AG-divergence* respectively. More details on these divergence measures can be seen in on line book by Taneja [27].

We call the measures given in (1.1), (1.3), (1.5), (1.7), (1.9) and (1.10) by *symmetric divergence measures*, since they are symmetric with respect to the probability distributions P and Q. While the measures (1.6) and (1.14) are not symmetric with respect to probability distributions.

2. Generalizations of Symmetric Divergence Measures

In this section, we shall present new generalizations of the symmetric divergence measures given in Section 1. Before that, first we shall present a well known generalization of Kullback-Leibler's *relative information*.

- **Relative Information of Type s**

$$\Phi_s(P||Q) = \begin{cases} K_s(P||Q) = [s(s-1)]^{-1}\left[\sum_{i=1}^{n} p_i^s q_i^{1-s} - 1\right], & s \neq 0,1 \\ K(Q||P) = \sum_{i=1}^{n} q_i \ln\left(\frac{q_i}{p_i}\right), & s = 0 \\ K(P||Q) = \sum_{i=1}^{n} p_i \ln\left(\frac{p_i}{q_i}\right), & s = 1 \end{cases}, \tag{2.1}$$

for all $s \in \mathbb{R}$.

The measure (2.1) is due to Cressie and Read [7]. For more studies on this measure refer to Taneja [28] and Taneja and Kumar [32, 33] and reference therein.

The measure 2.1 admits the following particular cases:

(i) $\Phi_{-1}(P||Q) = \frac{1}{2}\chi^2(Q||P)$.
(ii) $\Phi_0(P||Q) = K(Q||P)$.
(iii) $\Phi_{1/2}(P||Q) = 4\left[1 - B(P||Q)\right] = 4h(P||Q)$.
(iv) $\Phi_1(P||Q) = K(P||Q)$.
(v) $\Phi_2(P||Q) = \frac{1}{2}\chi^2(P||Q)$.

Here, we observe that $\Phi_2(P||Q) = \Phi_{-1}(Q||P)$ and $\Phi_1(P||Q) = \Phi_0(Q||P)$.

2.1. J-Divergence of Type s. Replace $K(P||Q)$ by $\Phi_s(P||Q)$ in the relation (1.11), we get

$$\begin{aligned} \mathcal{V}_s(P||Q) &= \Phi_s(P||Q) + \Phi_s(Q||P) \\ &= \begin{cases} J_s(P||Q) = [s(s-1)]^{-1}\left[\sum\limits_{i=1}^{n}\left(p_i^s q_i^{1-s} + p_i^{1-s} q_i^s\right) - 2\right], & s \neq 0,1 \\ \\ J(P||Q) = \sum\limits_{i=1}^{n} (p_i - q_i)\ln\left(\frac{p_i}{q_i}\right), & s = 0,1 \end{cases}. \end{aligned} \tag{2.2}$$

The expression (2.2) admits the following particular cases:

(i) $\mathcal{V}_{-1}(P||Q) = \mathcal{V}_2(P||Q) = \frac{1}{2}\Psi(P||Q)$.
(ii $\mathcal{V}_0(P||Q) = \mathcal{V}_1(P||Q) = J(P||Q)$.
(iii) $\mathcal{V}_{1/2}(P||Q) = 8\,h(P||Q)$.

Remark 2.1. *The expression (2.2) is the modified form of the measure already known in the literature:*

$$\mathcal{V}_s^1(P||Q) = \begin{cases} J_s(P||Q) = (s-1)^{-1}\left[\sum\limits_{i=1}^{n}\left(p_i^s q_i^{1-s} + p_i^{1-s} q_i^s\right) - 2\right], & s \neq 1, s > 0 \\ \\ J(P||Q) = \sum\limits_{i=1}^{n} (p_i - q_i)\ln\left(\frac{p_i}{q_i}\right), & s = 1 \end{cases}. \tag{2.3}$$

For the propertied of the measure (2.3) refer to Burbea and Rao [2, 3]*, Taneja* [25, 26, 27]*, etc. For the axiomatic characterization of this measure refer to Rathie and Sheng* [22] *and Taneja* [24]*. The measures (2.2) considered here differs in constant and it permits in considering negative values of the parameter s.*

2.2. Unified AG and JS – Divergence of Type s. Replace $K(P||Q)$ by $\Phi_s(P||Q)$ in the relation (1.13) interestingly we have a unified generalization of the *AG and JS – divergence* given by

$$\mathcal{W}_s(P||Q) = \frac{1}{2}\left[\Phi_s\left(\frac{P+Q}{2}||P\right) + \Phi_s\left(\frac{P+Q}{2}||Q\right)\right] \tag{2.4}$$

$$= \begin{cases} IT_s(P||Q) = [s(s-1)]^{-1}\left[\sum_{i=1}^{n}\left(\frac{p_i^{1-s}+q_i^{1-s}}{2}\right)\left(\frac{p_i+q_i}{2}\right)^s - 1\right], & s \neq 0,1 \\ I(P||Q) = \frac{1}{2}\left[\sum_{i=1}^{n} p_i \ln\left(\frac{2p_i}{p_i+q_i}\right) + \sum_{i=1}^{n} q_i \ln\left(\frac{2q_i}{p_i+q_i}\right)\right], & s = 0 \\ T(P||Q) = \sum_{i=1}^{n}\left(\frac{p_i+q_i}{2}\right)\ln\left(\frac{p_i+q_i}{2\sqrt{p_iq_i}}\right), & s = 1 \end{cases}.$$

The measure (2.4) admits the following particular cases:

(i) $\mathcal{W}_{-1}(P||Q) = \frac{1}{4}\Delta(P||Q)$.
(ii) $\mathcal{W}_0(P||Q) = I(P||Q)$.
(iii) $\mathcal{W}_{1/2}(P||Q) = 4\,d(P||Q)$.
(iv) $\mathcal{W}_1(P||Q) = T(P||Q)$.
(v) $\mathcal{W}_2(P||Q) = \frac{1}{16}\Psi(P||Q)$.

The measure $d(P||Q)$ given in part (iii) is not studied elsewhere and given by

$$d(P||Q) = 1 - \sum_{i=1}^{n}\left(\frac{\sqrt{p_i}+\sqrt{q_i}}{2}\right)\left(\sqrt{\frac{p_i+q_i}{2}}\right). \tag{2.5}$$

A relation of the measure (2.5) with *Hellinger's discrimination* is given in the last section. Connections of the measure (2.5) with mean divergence measures can be seen in Taneja [31].

We can also write

$$\mathcal{W}_{1-s}(P||Q) = \frac{1}{2}\left[\Phi_s\left(P||\frac{P+Q}{2}\right) + \Phi_s\left(Q||\frac{P+Q}{2}\right)\right]. \tag{2.6}$$

Thus we have two *symmetric divergences of type s* given by (2.2) and (2.4) generalizing the six *symmetric divergence measures* given in Section 1. In this paper our aim is to study the *symmetric divergences of type s* and to find inequalities among them. These studies we shall do by making use of the properties of Csiszár's *f-divergence*.

3. Csiszár's $f-$Divergence and Its Properties

Given a function $f : (0,\infty) \to \mathbb{R}$, the *f-divergence* measure introduced by Csiszár's [5] is given by

$$C_f(P||Q) = \sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right), \tag{3.1}$$

for all $P, Q \in \Gamma_n$.

The following theorem is well known in the literature [5, 6].

Theorem 3.1. *If the function f is convex and normalized, i.e., $f(1) = 0$, then the f-divergence, $C_f(P||Q)$ is nonnegative and convex in the pair of probability distribution $(P, Q) \in \Gamma_n \times \Gamma_n$.*

The theorem given below give bounds on the measure (3.1).

Theorem 3.2. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be differentiable convex and normalized i.e., $f(1) = 0$. If $P, Q \in \Gamma_n$, are such that $0 < r \leqslant \frac{p_i}{q_i} \leqslant R < \infty$, $\forall i \in \{1, 2, ..., n\}$, for some r and R with $0 < r \leqslant 1 \leqslant R < \infty$, $r \neq R$, then we have*

$$0 \leqslant C_f(P||Q) \leqslant E_{C_f}(P||Q) \leqslant A_{C_f}(r, R), \tag{3.2}$$

and

$$0 \leqslant C_f(P||Q) \leqslant B_{C_f}(r, R) \leqslant A_{C_f}(r, R), \tag{3.3}$$

where

$$E_{C_f}(P||Q) = \sum_{i=1}^{n} (p_i - q_i) f'(\frac{p_i}{q_i}), \tag{3.4}$$

$$A_{C_f}(r, R) = \frac{1}{4}(R - r)\left(f'(R) - f'(r)\right) \tag{3.5}$$

and

$$B_{C_f}(r, R) = \frac{(R-1)f(r) + (1-r)f(R)}{R - r}. \tag{3.6}$$

The proof is based on the following lemma due to Dragomir [8].

Lemma 3.1. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}$ be a differentiable convex function on the interval I, $x_i \in \overset{o}{I}$ ($\overset{o}{I}$ is the interior of I), $\lambda_i \geqslant 0$ $(i = 1, 2, ..., n)$ with $\sum_{i=1}^{n} \lambda_i = 1$. If $m,\ M \in \overset{o}{I}$ and $m \leqslant x_i \leqslant M$, $\forall i = 1, 2, ..., n$, then we have the inequalities:*

$$\begin{aligned} 0 &\leqslant \sum_{i=1}^{n} \lambda_i f(x_i) - f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \\ &\leqslant \sum_{\imath=1}^{n} \lambda_i x_i f'(x_i) - \left(\sum_{i=1}^{n} \lambda_i x_i\right)\left(\sum_{i=1}^{n} \lambda_i f'(x_i)\right) \\ &\leqslant \frac{1}{4}(M - m)\left(f'(M) - f'(m)\right). \end{aligned} \tag{3.7}$$

As a consequence of above theorem we have the following corollary.

Corollary 3.1. *For all $a, b, \upsilon, \omega \in (0, \infty)$, the following inequalities hold:*

$$\begin{aligned} 0 &\leqslant \frac{\upsilon f(a) + \omega f(b)}{\upsilon + \omega} - f\left(\frac{\upsilon a + \omega b}{\upsilon + \omega}\right) \\ &\leqslant \frac{\upsilon a f'(a) + \omega b f'(b)}{\upsilon + \omega} - \left(\frac{\upsilon a + \omega b}{\upsilon + \omega}\right)\left(\frac{\upsilon f'(a) + \omega f'(b)}{\upsilon + \omega}\right) \\ &\leqslant \frac{1}{4}(b - a)\left(f'(b) - f'(a)\right). \end{aligned} \tag{3.8}$$

Proof. It follows from Lemma 3.1, by taking $\lambda_1 = \frac{\upsilon}{\upsilon+\omega}$, $\lambda_2 = \frac{\omega}{\upsilon+\omega}$, $\lambda_3 = ... = \lambda_n = 0$, $x_1 = a$, $x_2 = b$, $x_2 = ... = x_n = 0$. □

Proof. of the Theorem 3.2. For all $P, Q \in \Gamma_n$, take $x = \frac{p_i}{q_i}$ in (3.7), $\lambda_i = q_i$ and sum over all $i = 1, 2, ..., n$ we get the inequalities (3.2).

Again, take $\upsilon = R - x$, $\omega = x - r$, $a = r$ and $b = R$ in (3.8), we get

$$\begin{aligned} 0 &\leqslant \frac{(R-x)f(r)+(x-r)f(R)}{R-r} - f(x) \\ &\leqslant \frac{(R-x)(x-r)}{R-r}\left[f'(R)-f'(r)\right] \\ &\leqslant \frac{1}{4}(R-r)\left(f'(R)-f'(r)\right). \end{aligned} \tag{3.9}$$

From the first part of the inequalities (3.9), we get

$$f(x) \leqslant \frac{(R-x)f(r)+(x-r)f(R)}{R-r}. \tag{3.10}$$

For all $P, Q \in \Gamma_n$, take $x = \frac{p_i}{q_i}$ in (3.9) and (3.10), multiply by q_i and sum over all $i = 1, 2, ..., n$, we get

$$\begin{aligned} 0 &\leqslant B_{C_f}(r,R) - C_f(P||Q) \\ &\leqslant \frac{(R-1)(1-r)\left(f'(R)-f'(r)\right)}{R-r} \leqslant A_{C_f}(r,R) \end{aligned} \tag{3.11}$$

and

$$0 \leqslant C_f(P||Q) \leqslant B_{C_f}(r,R), \tag{3.12}$$

respectively.

The expression (3.12) completes the *l.h.s.* of the inequalities (3.3). In order to prove *r.h.s.* of the inequalities (3.3), let us take $x = 1$ in (3.9) and use the fact that $f(1) = 0$, we get

$$0 \leqslant B_{C_f}(r,R) \leqslant \frac{(R-1)(1-r)\left(f'(R)-f'(r)\right)}{R-r} \leqslant A_{C_f}(r,R). \tag{3.13}$$

From (3.13), we conclude the *r.h.s.* of the inequalities (3.3). □

Remark 3.1. *We observe that the inequalities (3.2) and (3.3) are the improvement over Dragomir's* [9, 10] *work. From the inequalities (3.3) and (3.13) we observe that there is better bound for $B_{C_f}(r,R)$ instead of $A_{C_f}(r,R)$. From the r.h.s. of the inequalities (3.13), we conclude the following inequality among r and R:*

$$(R-1)(1-r) \leqslant \frac{1}{4}(R-r)^2. \tag{3.14}$$

Theorem 3.3. (*Dragomir et al.* [11, 12]). (*i*) *Let $P, Q \in \Gamma_n$ be such that $0 < r \leqslant \frac{p_i}{q_i} \leqslant R < \infty$, $\forall i \in \{1, 2, ..., n\}$, for some r and R with $0 < r \leqslant 1 \leqslant R < \infty$, $r \neq R$. Let $f : [0, \infty) \to \mathbb{R}$ be a normalized mapping, i.e., $f(1) = 0$ such that f' is locally absolutely continuous on $[r, R]$ and there exists α, β satisfying*

$$\alpha \leqslant f''(x) \leqslant \beta, \ \forall x \in (r,R). \tag{3.15}$$

Then

$$\left|C_f(P||Q) - \frac{1}{2}E_{C_f}(P||Q)\right| \leqslant \frac{1}{8}(\beta-\alpha)\chi^2(P||Q) \tag{3.16}$$

and

$$\left|C_f(P||Q) - E^*_{C_f}(P||Q)\right| \leqslant \frac{1}{8}(\beta-\alpha)\chi^2(P||Q), \tag{3.17}$$

where $E_{C_f}(P||Q)$ is as given by (3.4), $\chi^2(P||Q)$ is as given by (1.6) and

$$E^*_{C_f}(P||Q) = 2\,E_{C_f}\left(\frac{P+Q}{2}||Q\right) = \sum_{i=1}^{n}(p_i-q_i)f'\left(\frac{p_i+q_i}{2q_i}\right). \tag{3.18}$$

(*ii*) *Additionally, if $f:[r,R]\to\mathbb{R}$ with f''' absolutely continuous on $[r,R]$ and $f'''\in L_\infty[r,R]$, then*

$$\left|C_f(P||Q) - \frac{1}{2}E_{C_f}(P||Q)\right| \leqslant \frac{1}{12}\left\|f'''\right\|_\infty |\chi|^3(P||Q) \tag{3.19}$$

and

$$\left|C_f(P||Q) - E^*_{C_f}(P||Q)\right| \leqslant \frac{1}{24}\left\|f'''\right\|_\infty |\chi|^3(P||Q), \tag{3.20}$$

where

$$|\chi|^3(P||Q) = \sum_{i=1}^{n}\frac{|p_i-q_i|^3}{q_i^2} \tag{3.21}$$

and

$$\left\|f'''\right\|_\infty = ess\sup_{x\in[r,R]}|f'''|. \tag{3.22}$$

Theorem 3.4. (*Dragomir et al.* [13]). *Suppose $f:[r,R]\to\mathbb{R}$ is differentiable and f' is of bounded variation, i.e., $\overset{R}{\underset{r}{V}}(f') = \int_r^R|f''(t)|dt < \infty$. Let the constants r, R satisfy the conditions:*

(i) $0 < r < 1 < R < \infty$;
(ii) $0 < r \leqslant \frac{p_i}{q_i} \leqslant R < \infty$, *for $i = 1,2,...,n$.*

Then

$$\left|C_f(P||Q) - \frac{1}{2}E_{C_f}(P||Q)\right| \leqslant \overset{R}{\underset{r}{V}}(f')V(P||Q) \tag{3.23}$$

and

$$\left|C_f(P||Q) - E^*_{C_f}(P||Q)\right| \leqslant \frac{1}{2}\overset{R}{\underset{r}{V}}(f')V(P||Q), \tag{3.24}$$

where

$$V(P||Q) = \sum_{i=1}^{n}|p_i-q_i|. \tag{3.25}$$

Remark 3.2. (*i*) *If the third order derivative of f exists and let us suppose that it is either positive or negative. Then the function f'' is either monotonically increasing or decreasing. In view of this we can write*

$$\beta - \alpha = k(f)\left[f''(R) - f''(r)\right], \tag{3.26}$$

where

$$k(f) = \begin{cases} -1, & \textit{if } f'' \textit{ is monotonically decresing} \\ 1, & \textit{if } f'' \textit{ is monotonically increasing} \end{cases}. \tag{3.27}$$

(*ii*) *Let the function $f(x)$ considered in the Theorem 3.4 be convex in $(0,\infty)$, then $f''(x) \geqslant 0$. This gives*

$$\begin{aligned} \overset{R}{\underset{r}{V}}(f') &= \int_r^R |f''(t)|dt \\ &= \int_r^R f''(t)dt = f'(R) - f'(r) \\ &= \frac{4}{R-r} A_{C_f}(r,R). \end{aligned} \tag{3.28}$$

Under these considerations, the bounds (3.23) and (3.24) can be re-written as

$$\left| C_f(P||Q) - \frac{1}{2} E_{C_f}(P||Q) \right| \leqslant \frac{4}{R-r} A_{C_f}(r,R) V(P||Q) \tag{3.29}$$

and

$$\left| C_f(P||Q) - E^*_{C_f}(P||Q) \right| \leqslant \frac{2}{R-r} A_{C_f}(r,R) V(P||Q). \tag{3.30}$$

Based on above remarks we can restate and combine the Theorems 3.3 and 3.4.

Theorem 3.5. *Let $P, Q \in \Gamma_n$ be such that $0 < r \leqslant \frac{p_i}{q_i} \leqslant R < \infty$, $\forall i \in \{1,2,...,n\}$, for some r and R with $0 < r < 1 < R < \infty$. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be differentiable convex, normalized, of bounded variation, and second derivative is monotonic with f''' absolutely continuous on $[r,R]$ and $f''' \in L_\infty[r,R]$, then*

$$\begin{aligned} &\left| C_f(P||Q) - \frac{1}{2} E_{C_f}(P||Q) \right| \\ &\leqslant \min \left\{ \frac{1}{8} k(f)\left[f''(R) - f''(r)\right] \chi^2(P||Q), \right. \\ &\qquad \left. \frac{1}{12} \left\| f''' \right\|_\infty |\chi|^3(P||Q), \;\; \left[f'(R) - f'(r)\right] V(P||Q) \right\}, \end{aligned} \tag{3.31}$$

and

$$\left|C_f(P||Q) - E^*_{C_f}(P||Q)\right| \tag{3.32}$$
$$\leqslant \min\left\{\frac{1}{8}k(f)\left[f''(R) - f''(r)\right]\chi^2(P||Q),\right.$$
$$\left.\frac{1}{24}\left\|f'''\right\|_\infty |\chi|^3(P||Q),\ \frac{1}{2}\left[f'(R) - f'(r)\right]V(P||Q)\right\},$$

where $k(f)$ is as given by (3.27).

Remark 3.3. *The measures (1.6), (3.21) and (3.25) are the particular cases of Vajda* [34] *$|\chi|^m-$divergence given by*

$$|\chi|^m(P||Q) = \sum_{i=1}^{n}\frac{|p_i - q_i|^m}{q_i^{m-1}},\ m \geqslant 1. \tag{3.33}$$

The above measure (3.33) [4] [11] *satisfies the following properties:*

$$|\chi|^m(P||Q) \leqslant \frac{(1-r)(R-1)}{(R-r)}\left[(1-r)^{m-1} + (R-1)^{m-1}\right] \tag{3.34}$$
$$\leqslant \left(\frac{R-r}{2}\right)^m,\ m \geqslant 1$$

and

$$\left(\frac{1-r^m}{1-r}\right)V(P||Q) \leqslant |\chi|^m(P||Q) \leqslant \left(\frac{R^m-1}{R-1}\right)V(P||Q),\ m \geqslant 1. \tag{3.35}$$

Take $m = 2, 3$ and 1, in (3.34), we get

$$\chi^2(P||Q) \leqslant (R-1)(1-r) \leqslant \frac{(R-r)^2}{4}, \tag{3.36}$$

$$|\chi|^3(P||Q) \leqslant \frac{1}{2}\frac{(R-1)(1-r)}{R-r}\left[(1-r)^2 + (R-1)^2\right] \leqslant \frac{1}{8}(R-r)^3 \tag{3.37}$$

and

$$V(P||Q) \leqslant \frac{2(R-1)(1-r)}{(R-r)} \leqslant \frac{1}{2}(R-r). \tag{3.38}$$

respectively.

In view of the last inequalities given in (3.36), (3.37) and (3.38), the bounds given in (3.31) and (3.32) can be written in terms of r, R as

$$\left|C_f(P||Q) - \frac{1}{2}E_{C_f}(P||Q)\right| \tag{3.39}$$
$$\leqslant \frac{(R-r)^2}{4}\min\left\{\frac{1}{8}k(f)\left[f''(R) - f''(r)\right],\right.$$
$$\left.\frac{R-r}{24}\left\|f'''\right\|_\infty,\ \frac{2\left[f'(R) - f'(r)\right]}{R-r}\right\}$$

and

$$\left|C_f(P||Q) - E^*_{C_f}(P||Q)\right| \tag{3.40}$$

$$\leqslant \frac{(R-r)^2}{4} \min\left\{\frac{1}{8}k(f)\left[f''(R)-f''(r)\right], \right.$$

$$\left. \frac{R-r}{48}\left\|f'''\right\|_\infty, \ \frac{f'(R)-f'(r)}{R-r}\right\}, \tag{3.41}$$

respectively.

We observe that the bounds (3.39) and (3.40) are based on the first, second and third order derivatives of the generating function.

Theorem 3.6. *Let $f_1, f_2 : I \subset \mathbb{R}_+ \to \mathbb{R}$ two generating mappings are normalized, i.e., $f_1(1) = f_2(1) = 0$ and satisfy the assumptions:*

(i) *f_1 and f_2 are twice differentiable on (r, R);*

(ii) *there exists the real constants m, M such that $m < M$ and*

$$m \leqslant \frac{f_1''(x)}{f_2''(x)} \leqslant M, \ f_2''(x) > 0, \ \forall x \in (r, R), \tag{3.42}$$

then we have

$$m \ C_{f_2}(P||Q) \leqslant C_{f_1}(P||Q) \leqslant M \ C_{f_2}(P||Q) \tag{3.43}$$

Proof. Let us consider two functions

$$\eta_m(x) = f_1(x) - m \ f_2(x), \tag{3.44}$$

and

$$\eta_M(x) = M \ f_2(x) - f_1(x), \tag{3.45}$$

where m and M are as given by (3.42)

Since $f_1(1) = f_2(1) = 0$, then $\eta_m(1) = \eta_M(1) = 0$. Also, the functions $f_1(x)$ and $f_2(x)$ are twice differentiable. Then in view of (3.42), we have

$$\eta_m''(x) = f_1''(x) - m \ f_2''(x) = f_2''(x)\left(\frac{f_1''(x)}{f_2''(x)} - m\right) \geqslant 0, \tag{3.46}$$

and

$$\eta_M''(x) = M \ f_2''(x) - f_1''(x) = f_2''(x)\left(M - \frac{f_1''(x)}{f_2''(x)}\right) \geqslant 0, \tag{3.47}$$

for all $x \in (r, R)$.

In view of (3.46) and (3.47), we can say that the functions $\eta_m(x)$ and $\eta_M(x)$ are convex on (r, R).

According to Theorem 3.1, we have

$$C_{\eta_m}(P||Q) = C_{f_1 - m f_2}(P||Q) = C_{f_1}(P||Q) - m \ C_{f_2}(P||Q) \geqslant 0, \tag{3.48}$$

and

$$C_{\eta_M}(P||Q) = C_{M f_2 - f_1}(P||Q) = M \ C_{f_2}(P||Q) - C_{f_1}(P||Q) \geqslant 0. \tag{3.49}$$

Combining (3.48) and (3.49) we get (3.43). □

For futher properties of the measure (3.1) based on the conditions of Theorem 3.6 refer to Taneja [31].

Remark 3.4. (i) *From now onwards, unless otherwise specified, it is understood that, if there are* r, R, *then* $0 < r \leqslant \frac{p_i}{q_i} \leqslant R < \infty$, $\forall i \in \{1, 2, ..., n\}$, *with* $0 < r < 1 < R < \infty$, $P = (p_1, p_2, ..., p_n) \in \Gamma_n$ *and* $Q = (q_1, q_2, ..., q_n) \in \Gamma_n$.

(ii) *In some particular cases studied below, we shall use the p-logarithmic power mean* [23] *given by*

$$L_p(a,b) = \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & p \neq -1, 0 \\ \frac{b-a}{\ln b - \ln a}, & p = -1 \\ \frac{1}{e}\left[\frac{b^b}{a^a}\right]^{\frac{1}{b-a}}, & p = 0 \end{cases}, \tag{3.50}$$

for all $p \in \mathbb{R}$, $a \neq b$. *In particular, we shall use the following notation*

$$L_p^p(a,b) = \begin{cases} \frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}, & p \neq -1, 0 \\ \frac{\ln b - \ln a}{b-a}, & p = -1 \\ 1, & p = 0 \end{cases}, \tag{3.51}$$

for all $p \in \mathbb{R}$, $a \neq b$.

4. Bounds on Generalized Divergence Measures

In this section we shall show that the generalized measures given in Section 2 are the particular cases of the Csiszár's *f-divergence*. Also, we shall give bounds on these measures similar to Theorems 3.1-3.5. The applications of Theorem 3.6 are given in Section 5.

4.1. Bounds on J-Divergence of Type s. Initially we shall give two important properties of *J-divergence of type s*.

Property 4.1. *The measure* $\mathcal{V}_s(P||Q)$ *is nonnegative and convex in the pair of probability distributions* $(P, Q) \in \Gamma_n \times \Gamma_n$ *for all* $s \in (-\infty, \infty)$.

Proof. For all $x > 0$ and $s \in (-\infty, \infty)$, let us consider in (2.1),

$$\phi_s(x) = \begin{cases} [s(s-1)]^{-1}\left[x^s + x^{1-s} - (1+x)\right], & s \neq 0, 1 \\ (x-1)\ln x, & s = 0, 1 \end{cases}, \tag{4.1}$$

then we have $C_f(P||Q) = \mathcal{V}_s(P||Q)$, where $\mathcal{V}_s(P||Q)$ is given by (2.2).

Moreover,

$$\phi_s'(x) = \begin{cases} [s(s-1)]^{-1}\left[s(x^{s-1} + x^{-s}) + x^{-s} - 1\right], & s \neq 0, 1 \\ 1 - x^{-1} + \ln x, & s = 0, 1 \end{cases}, \tag{4.2}$$

and

$$\phi_s''(x) = x^{s-2} + x^{-s-1}. \tag{4.3}$$

Thus we have $\phi_s''(x) > 0$ for all $x > 0$, and hence, $\phi_s(x)$ is convex for all $x > 0$. Also, we have $\phi_s(1) = 0$. In view of this we can say that *J-divergence of type s* is *nonnegative* and *convex* in the pair of probability distributions $(P, Q) \in \Gamma_n \times \Gamma_n$. □

Property 4.2. *The measure $\mathcal{V}_s(P||Q)$ is monotonically increasing in s for all $s \geqslant \frac{1}{2}$ and decreasing in $s \leqslant \frac{1}{2}$.*

In order to prove the above property, we shall make use the following lemma.

Lemma 4.1. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function and suppose that $f(1) = f'(1) = 0$, then*

$$f(x) \begin{cases} \geqslant 0, & \textit{if } f \textit{ is convex} \\ \leqslant 0, & \textit{if } f \textit{ is concave} \end{cases}. \tag{4.4}$$

Proof. It is well known that if the function f is convex, then we have the inequality

$$f'(x)(y - x) \leqslant f(y) - f(x) \leqslant f'(y)(y - x), \tag{4.5}$$

for all $x, y \in \mathbb{R}_+$. The above inequality is reversed if f is concave. Take $x = 1$ in the inequality (4.5) and use the fact that $f(1) = f'(1) = 0$ we get the required result. □

Proof. of the Property 4.2. Let use consider the first order derivative of the function $\phi_s(x)$ given in (4.1) with respect to s, we get

$$\begin{aligned} k_s(x) &= \frac{d}{ds}\left(\phi_s(x)\right) \\ &= [s(s-1)]^{-2}\left[s(s-1)(x^s - x^{1-s})\ln x \right. \\ &\qquad \left. +(1-2s)(x^s + x^{1-s} - (x+1))\right], \ s \neq 0, 1. \end{aligned} \tag{4.6}$$

Now, calculating the first and second order derivative of the function $k_s(x)$ with respect to x, we get

$$\begin{aligned} k_s'(x) = \frac{1}{s^2(1-s)^2}&\left[s^2(x^{-s} - x^{s-1}) + (1-2s)(x^{-s} - 1)\right. \\ &\left. +s(s-1)\left(sx^{s-1} + (s-1)x^{-s}\right)\ln x\right], \ s \neq 0, 1 \end{aligned}$$

and

$$k_s''(x) = (x^{s-2} - x^{-s-1})\ln x.$$

For all $x > 0$, we can easily check that

$$k_s''(x) \begin{cases} \geqslant 0, & s \geqslant \frac{1}{2} \\ \leqslant 0, & s \leqslant \frac{1}{2} \end{cases}. \tag{4.7}$$

Since $k_s(1) = k_s'(1) = 0$, then using Lemma 4.1 along with (4.7), we have

$$k_s(x) \begin{cases} \geqslant 0, & s \geqslant \frac{1}{2} \\ \leqslant 0, & s \leqslant \frac{1}{2} \end{cases}. \tag{4.8}$$

Thus from (4.8), we conclude that the function $\phi_s(x)$ is monotonically increasing in s for all $s \geqslant \frac{1}{2}$ and monotonically decreasing in s for all $s \leqslant \frac{1}{2}$. This completes the proof of the property. $\square$

By taking $s = \frac{1}{2}$, 1 and 2, and applying the Property 4.2, one gets

$$h(P||Q) \leqslant \frac{1}{8} J(P||Q) \leqslant \frac{1}{16}\Psi(P||Q). \tag{4.9}$$

Theorem 4.1. *The following bounds hold:*

$$\mathcal{V}_s(P||Q) \leqslant E_{\mathcal{V}_s}(P||Q) \leqslant A_{\mathcal{V}_s}(r,R), \tag{4.10}$$

$$\mathcal{V}_s(P||Q) \leqslant B_{\mathcal{V}_s}(r,R) \leqslant A_{\mathcal{V}_s}(r,R), \tag{4.11}$$

$$\begin{aligned} &\left|\mathcal{V}_s(P||Q) - \frac{1}{2}E_{\mathcal{V}_s}(P||Q)\right| \\ &\leqslant \min\left\{\frac{1}{8}\delta_{\mathcal{V}_s}(r,R)\chi^2(P||Q),\ \frac{1}{12}\left\|\phi_s'''\right\|_\infty |\chi|^3(P||Q),\ \bigvee_r^R(\phi')V(P||Q)\right\}, \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} &\left|\mathcal{V}_s(P||Q) - E^*_{\mathcal{V}_s}(P||Q)\right| \\ &\leqslant \min\left\{\frac{1}{8}\delta_{\mathcal{V}_s}(r,R)\chi^2(P||Q),\ \frac{1}{24}\left\|\phi_s'''\right\|_\infty |\chi|^3(P||Q),\ \frac{1}{2}\bigvee_r^R(\phi')V(P||Q)\right\}, \end{aligned} \tag{4.13}$$

where

$$E_{\mathcal{V}_s}(P||Q) = \begin{cases} \sum\limits_{i=1}^n (p_i - q_i)\left[(s-1)^{-1}\left(\frac{p_i}{q_i}\right)^{s-1} - s^{-1}\left(\frac{p_i}{q_i}\right)^{-s}\right], & s \neq 1 \\ \\ J(P||Q) + \chi^2(Q||P), & s = 1 \end{cases}, \tag{4.14}$$

$$E^*_{\mathcal{V}_s}(P||Q) = \begin{cases} \sum\limits_{i=1}^n (p_i - q_i)\left[(s-1)^{-1}\left(\frac{p_i+q_i}{2q_i}\right)^{s-1} - s^{-1}\left(\frac{p_i+q_i}{2q_i}\right)^{-s}\right], & s \neq 0,1 \\ \\ \Delta(P||Q) + 2\,J\left(\frac{P+Q}{2}||Q\right), & s = 0,1 \end{cases}, \tag{4.15}$$

$$A_{\mathcal{V}_s}(r,R) = \frac{1}{4}(R-r)^2\left\{L_{s-2}^{s-2}(r,R) + L_{-s-1}^{-s-1}(r,R)\right\}, \tag{4.16}$$

$$B_{\mathcal{V}_s}(r,R) = \begin{cases} [s(s-1)]^{-1}\left[\frac{(1-r)(R^s+R^{1-s})+(R-1)(r^s+r^{1-s})}{(R-r)} - 2\right], & s \neq 0,1 \\ \\ (1-r)(R-1)L_{-1}^{-1}(r,R), & s = 0,1 \end{cases}, \tag{4.17}$$

$$\delta_{\mathcal{V}_s}(r,R) = (R-r)\left[(2-s)L_{s-3}^{s-3}(r,R) + (1+s)L_{-s-2}^{-s-2}(r,R)\right],\ -1 \leqslant s \leqslant 2, \tag{4.18}$$

$$\left\|\phi_s'''\right\|_\infty = (2-s)r^{s-3} + (s+1)r^{-s-2},\ -1 \leqslant s \leqslant 2, \tag{4.19}$$

and

$$\overset{R}{\underset{r}{V}}(\phi') = \frac{4}{R-r} A_{\mathcal{V}_s}(r,R). \tag{4.20}$$

Proof. By making some calculations and applying the Theorem 3.2 we get the inequalities (4.10) and (4.11). Let us prove now the inequalities (4.12) and (4.13). The third order derivative of the function $\phi_s(x)$ is given by

$$\phi_s'''(x) = -\left[(2-s)x^{s-3} + (s+1)x^{-s-2}\right],\ x \in (0,\infty). \tag{4.21}$$

This gives

$$\phi_s'''(x) \leqslant 0,\ \forall\ -1 \leqslant s \leqslant 2. \tag{4.22}$$

From (4.22), we can say that the function $\phi''(x)$ is monotonically decreasing in $x \in (0,\infty)$, and hence, for all $x \in [r,R]$, we have

$$\begin{aligned} \delta_{\mathcal{V}_s}(r,R) &= \phi''(r) - \phi''(R) \\ &= (R-r)\left[(2-s)L_{s-3}^{s-3}(r,R) + (1+s)L_{-s-2}^{-s-2}(r,R)\right],\ -1 \leqslant s \leqslant 2. \end{aligned} \tag{4.23}$$

From (4.21), we have

$$|\phi_s'''(x)| = (2-s)x^{s-3} + (s+1)x^{-s-2},\ -1 \leqslant s \leqslant 2.$$

This gives

$$\begin{aligned} |\phi'''(x)|' &= -\left[(s-2)(s-3)x^{s-4} + (s+1)(s+2)x^{-3-s}\right] \\ &\leqslant 0,\ -1 \leqslant s \leqslant 2. \end{aligned} \tag{4.24}$$

In view of (4.24), we can say that the function $|\phi_s'''(x)|$ is monotonically decreasing in $x \in (0,\infty)$ for $-1 \leqslant s \leqslant 2$, and hence, for all $x \in [r,R]$, we have

$$\|\phi_s'''\|_\infty = \sup_{x\in[r,R]} |\phi_s'''(x)| = (2-s)r^{s-3} + (s+1)r^{-s-2},\ -1 \leqslant s \leqslant 2. \tag{4.25}$$

By applying Theorem 3.5 along with the expressions (4.23) and (4.25) for the measure (2.2) we get the first two parts of the inequalities (4.12) and (4.13). The last part of the inequalities (4.12) and (4.13) are obtained by using (4.20) and Theorem 3.5. □

In particular when $s = \frac{1}{2}$, 1 and 2, we get the results studies in Taneja [29, 30].

4.2. Bounds on AG and JS – Divergence of Type s. Initially we shall give two important properties of *AG and JS - divergences of type s*.

Property 4.3. *The measure $\mathcal{W}_s(P||Q)$ is nonnegative and convex in the pair of probability distributions $(P,Q) \in \Gamma_n \times \Gamma_n$ for all $s \in (-\infty,\infty)$.*

Proof. For all $x > 0$ and $s \in (-\infty,\infty)$, let us consider in (2.1)

$$\psi_s(x) = \begin{cases} [s(s-1)]^{-1}\left[\left(\frac{x^{1-s}+1}{2}\right)\left(\frac{x+1}{2}\right)^s - \left(\frac{x+1}{2}\right)\right], & s \neq 0,1 \\[2ex] \frac{x}{2}\ln x - \left(\frac{x+1}{2}\right)\ln\left(\frac{x+1}{2}\right), & s = 0 \\[2ex] \left(\frac{x+1}{2}\right)\ln\left(\frac{x+1}{2\sqrt{x}}\right), & s = 1 \end{cases}, \tag{4.26}$$

then we have $C_f(P||Q) = \mathcal{W}_s(P||Q)$, where $\mathcal{W}_s(P||Q)$ is as given by (2.4).

Moreover,

$$\psi_s'(x) = \begin{cases} (s-1)^{-1}\left[\frac{1}{s}\left[\left(\frac{x+1}{2x}\right)^s - 1\right] - \frac{x^{-s}-1}{4}\left(\frac{x+1}{2}\right)^{s-1}\right], & s \neq 0,1 \\ -\frac{1}{2}\ln\left(\frac{x+1}{2x}\right), & s = 0 \\ 1 - x^{-1} - \ln x - 2\ln\left(\frac{2}{x+1}\right), & s = 1 \end{cases} \tag{4.27}$$

and

$$\psi_s''(x) = \left(\frac{x^{-s-1}+1}{8}\right)\left(\frac{x+1}{2}\right)^{s-2}. \tag{4.28}$$

Thus we have $\psi_s''(x) > 0$ for all $x > 0$, and hence, $\psi_s(x)$ is convex for all $x > 0$. Also, we have $\psi_s(1) = 0$. In view of this we can say that *AG and JS – divergences of type s* is *nonnegative* and *convex* in the pair of probability distributions $(P,Q) \in \Gamma_n \times \Gamma_n$. □

Property 4.4. *The measure $\mathcal{W}_s(P||Q)$ is monotonically increasing in s for all $s \geqslant -1$.*

Proof. Let us consider the first order derivative of (4.26) with respect to s.

$$\begin{aligned} m_s(x) &= \frac{d}{ds}\left(\psi_s(x)\right) \\ &= -\left[s(s-1)\right]^{-2}\left(\frac{x+1}{2}\right)^s\Big[(2s-1)(x^{1-s}+x+2) \\ &\qquad -s(s-1)(x^{1-s}+1)\ln\left(\frac{x+1}{2}\right)\Big], \ s \neq 0,1. \end{aligned} \tag{4.29}$$

Now, calculating the first and second order derivatives of (4.29) with respect to x, we get

$$m_s'(x) = \frac{1-2s}{s^2(1-s)^2}\left[x^s + x^{1-s} - (x+1)\right] + \frac{1}{s(s-1)}(x^s - x^{1-s})\ln x, \ s \neq 0,1$$

and

$$m_s''(x) = \frac{1}{2x^2(x+1)^2}\left(\frac{x+1}{2}\right)^s\left[x^{1-s}\ln\left(\frac{x+1}{2x}\right) + x^2\ln\left(\frac{x+1}{2}\right)\right], \tag{4.30}$$

respectively.

Since $(x-1)^2 \geqslant 0$ for any x, this give us

$$\ln\left(\frac{x+1}{2x}\right) \geqslant \ln\left(\frac{2}{x+1}\right). \tag{4.31}$$

Now for all $0 < x \leqslant 1$ and for any $s \geqslant -1$, we have $x^{1-s} \geqslant x^2$. This together with (4.31) gives

$$m_s''(x) \geqslant 0, \text{ for all } 0 < x \leqslant 1 \text{ and } s \geqslant -1. \tag{4.32}$$

Reorganizing (4.30), we can write

$$m_s''(x) = \frac{x^{1-s}}{2x^2(x+1)^2}\left(\frac{x+1}{2}\right)^s\left[(x^{1+s}+1)\ln\left(\frac{x+1}{2}\right) - \ln x\right].$$

Again for all $x \geqslant 1$ and $s \geqslant -1$, we have $x^{1+s}+1 \geqslant 2$. This gives

$$(x^{1+s}+1)\ln\left(\frac{x+1}{2}\right) \geqslant 2\ln\left(\frac{x+1}{2}\right) \geqslant \ln x, \tag{4.33}$$

where we have used the fact that $(x+1)^2 \geqslant 4x$ for any x.

In view of (4.33), we have

$$m_s''(x) \geqslant 0, \text{ for all } x \geqslant 1 \text{ and } s \geqslant -1. \tag{4.34}$$

Combining (4.32) and (4.34), we have

$$m_s''(x) \geqslant 0, \text{ for all } x > 0 \text{ and } s \geqslant -1. \tag{4.35}$$

Since $m_s(1) = m_s'(1) = 0$, then (4.35) together with Lemma 4.1 complete the required proof. □

By taking $s = -1,\ 0,\ \frac{1}{2}$, 1 and 2, and applying Property 4.4, one gets

$$\frac{1}{4}\Delta(P||Q) \leqslant I(P||Q) \leqslant 4\,d(P||Q) \leqslant T(P||Q) \leqslant \frac{1}{16}\Psi(P||Q). \tag{4.36}$$

Theorem 4.2. *The following bounds hold:*

$$0 \leqslant \mathcal{W}_s(P||Q) \leqslant E_{\mathcal{W}_s}(P||Q) \leqslant A_{\mathcal{W}_s}(r,R), \tag{4.37}$$

$$0 \leqslant \mathcal{W}_s(P||Q) \leqslant B_{\mathcal{W}_s}(r,R) \leqslant A_{\mathcal{W}_s}(r,R), \tag{4.38}$$

$$\begin{aligned}&\left|\mathcal{W}_s(P||Q) - \frac{1}{2}E_{\mathcal{W}_s}(P||Q)\right| \\ &\leqslant \min\left\{\frac{1}{8}\delta_{\mathcal{W}_s}(r,R)\chi^2(P||Q),\ \frac{1}{12}\left\|\psi_s'''\right\|_\infty |\chi|^3(P||Q),\ \overset{R}{\underset{r}{V}}(\psi')V(P||Q)\right\},\end{aligned} \tag{4.39}$$

and

$$\begin{aligned}&\left|\mathcal{W}_s(P||Q) - E^*_{\mathcal{W}_s}(P||Q)\right| \\ &\leqslant \min\left\{\frac{1}{8}\delta_{\mathcal{W}_s}(r,R)\chi^2(P||Q),\ \frac{1}{24}\left\|\psi_s'''\right\|_\infty |\chi|^3(P||Q),\ \frac{1}{2}\overset{R}{\underset{r}{V}}(\psi')V(P||Q)\right\},\end{aligned} \tag{4.40}$$

where

$$\begin{aligned}&E_{\mathcal{W}_s}(P||Q) \\ &= \begin{cases}\frac{1}{2}\sum_{i=1}^n (p_i - q_i)\left\{(s-1)^{-1}\left(\frac{p_i^{1-s}+q_i^{1-s}}{2}\right)\left(\frac{p_i+q_i}{2}\right)^{s-1} - s^{-1}\left(\frac{p_i+q_i}{2p_i}\right)^s\right\}, & s \neq 0,1 \\ J\left(\frac{P+Q}{2}||P\right), & s = 0 \\ \frac{1}{4}\left[\chi^2(Q||P) - J(P||Q)\right] + J\left(\frac{P+Q}{2}||Q\right), & s = 1\end{cases},\end{aligned} \tag{4.41}$$

$$E^*_{\mathcal{W}_s}(P||Q) = \begin{cases} \left(\frac{1}{2}\right)^{s+1}\sum\limits_{i=1}^{n}(p_i-q_i)\Big\{(s-1)^{-1}\Big[\left(\frac{p_i+3q_i}{p_i+q_i}\right)^{s-1} \\ \qquad +\left(\frac{p_i+3q_i}{2q_i}\right)^{s-1}\Big]-s^{-1}\left(\frac{p_i+3q_i}{2q_i}\right)^{s}\Big\}, & s\neq 0,1 \\ 2J\left(\frac{P+Q}{2}||\frac{P+3Q}{4}\right), & s=0 \\ \frac{1}{4}\Delta(P||Q)-\frac{1}{2}J\left(\frac{P+Q}{2}||Q\right)+2J\left(\frac{P+3Q}{4}||Q\right), & s=1 \end{cases}, \tag{4.42}$$

$$A_{\mathcal{W}_s}(r,R)=\frac{(R-r)^2}{16}\left[\frac{1}{rR}L^{s-1}_{s-1}\left(\frac{r+1}{2r},\frac{R+1}{2R}\right)\right. \\ \left.-\frac{1}{2rR}L^{s-2}_{s-2}\left(\frac{r+1}{2r},\frac{R+1}{2r}\right)+\frac{1}{2}L^{s-2}_{s-2}\left(\frac{r+1}{2},\frac{R+1}{2}\right)\right], \tag{4.43}$$

$$B_{\mathcal{V}_s}(r,R) = \begin{cases} [s(s-1)]^{-1}\Big\{\frac{1}{R-r}\Big[(1-r)\left(\frac{R^{1-s}+1}{2}\right)\left(\frac{R+1}{2}\right)^{s} \\ \qquad +(R-1)\left(\frac{r^{1-s}+1}{2}\right)\left(\frac{r+1}{2}\right)^{s}\Big]-1\Big\}, & s\neq 0,1 \\ \frac{1}{2(R-r)}\Big\{(1-r)\left[R\ln R-(1+R)\ln\left(\frac{R+1}{2}\right)\right] \\ \qquad +(R-1)\left[r\ln r-(r+1)\ln\left(\frac{r+1}{2}\right)\right]\Big\}, & s=0 \\ \frac{1}{2}\left\{(1-rR)L^{-1}_{-1}(r+1,R+1)+\ln\left[\frac{(r+1)(R+1)}{8}\right]\right\}, & s=1 \end{cases}, \tag{4.44}$$

$$\delta_{\mathcal{W}_s}(r,R)=\left(\frac{r^{-s-1}+1}{8}\right)\left(\frac{r+1}{2}\right)^{s-2} \\ -\left(\frac{R^{-s-1}+1}{8}\right)\left(\frac{R+1}{2}\right)^{s-2},\ -1\leqslant s\leqslant 2, \tag{4.45}$$

$$\|\psi'''_s\|_\infty=\frac{1}{2(r^3+1)}\left(\frac{r+1}{2}\right)^{s}\times \\ \times\left[3r^{-s-1}+(s+1)r^{-s-2}+(2-s)\right],\ -1\leqslant s\leqslant 2, \tag{4.46}$$

and

$$\overset{R}{\underset{r}{V}}(\psi')=\frac{4}{R-r}A_{\mathcal{W}_s}(r,R). \tag{4.47}$$

Proof. By making some calculations and applying Theorem 3.2 we get the inequalities (4.37) and (4.38). Now, we shall prove the inequalities (4.39) and (4.40). The third order derivative of the function $\psi_s(x)$ is given by

$$\psi_s'''(x) = -\frac{1}{2(x+1)^3}\left(\frac{x+1}{2}\right)^s \times \times \left[3x^{-s-1} + (s+1)x^{-2-s} + (2-s)\right],\ x \in (0,\infty). \tag{4.48}$$

This gives

$$\psi_s'''(x) \leqslant 0,\ -1 \leqslant s \leqslant 2. \tag{4.49}$$

From (4.49), we can say that the function $\phi''(x)$ is monotonically decreasing in $x \in (0,\infty)$, and hence, for all $x \in [r,R]$, we have

$$\begin{aligned} \delta_{\mathcal{W}_s}(r,R) &= \psi''(r) - \psi''(R) \\ &= \left(\frac{r^{-s-1}+1}{8}\right)\left(\frac{r+1}{2}\right)^{s-2} \\ &\quad - \left(\frac{R^{-s-1}+1}{8}\right)\left(\frac{R+1}{2}\right)^{s-2},\ -1 \leqslant s \leqslant 2. \end{aligned} \tag{4.50}$$

Again, from (4.48), we have

$$|\psi_s'''(x)| = \frac{1}{2(x+1)^3}\left(\frac{x+1}{2}\right)^s \times \times \left[3x^{-s-1} + (s+1)x^{-2-s} + (2-s)\right],\ x \in (0,\infty),\ -1 \leqslant s \leqslant 2. \tag{4.51}$$

This gives

$$\begin{aligned} |\psi_s'''(x)| &= -\frac{x^{1-s}}{2(x+1)^4} \times \\ &\quad \times \left\{x^{1-s}\left[12x^2 + 8(s+1)x + (s+1)(s+2)\right] + x^4(s-2)(s-3)\right\} \\ &\leqslant 0,\ -1 \leqslant s \leqslant 2. \end{aligned} \tag{4.52}$$

In view of (4.52), we can say that the function $|\psi_s'''|$ is monotonically decreasing in $x \in (0,\infty)$ for $-1 \leqslant s \leqslant 2$, and hence, for all $x \in [r,R]$, we have

$$\begin{aligned} \|\psi_s'''\|_\infty &= \sup_{x\in[r,R]} |\psi_s'''(x)| \\ &= \frac{1}{2(r+1)^3}\left(\frac{r+1}{2}\right)^s \left[3r^{-s-1} + (s+1)r^{-2-s} + (2-s)\right],\ -1 \leqslant s \leqslant 2. \end{aligned} \tag{4.53}$$

By applying the Theorem 3.5 along with the expressions (4.50) and (4.53) for the measure (2.4) we get the first two parts of the bounds (4.39) and (4.40). The last part of the bounds (4.39) and (4.40) follows in view of (4.47) and Theorem 3.5. □

In particular when $s = -1$, 0, 1 and 2, we get the results studied in Taneja [29, 30].

5. Relations Among Generalized Relative Divergence Measures

In this section, we shall apply the Theorem 3.6 to obtain inequalities among the measures (2.2) and (2.4).

Let us consider

$$g_{(\psi_s,\phi_t)}(x)=\frac{\psi_s''(x)}{\phi_t''(x)}=\frac{x^{-1-s}+1}{8(x^{-t-1}+x^{t-2})}\left(\frac{x+1}{2}\right)^{s-2},\ x\in(0,\infty) \tag{5.1}$$

where $\psi_s''(x)$ and $\phi_t''(x)$ are as given by (4.28) and (4.3) respectively.

From (5.1) one has

$$\begin{aligned}
g'_{(\psi_s,\phi_t)}(x) &= \frac{1}{8x(x+1)(x^{-t-1}+x^{t-2})^2}\left(\frac{x+1}{2}\right)^{s-2}\times \\
&\quad\times\Big\{x^{-t-1}\left[x^{-s-1}\left((t-2)x+(t-s)\right)+(t+s-1)x+(t+1)\right] \\
&\qquad -x^{t-2}\left[x^{-s-1}\left((t+1)x+(t+s-1)\right)+(t-s)x+(t-2)\right]\Big\} \\
&= \frac{1}{8x(x+1)(x^{-t-1}+x^{t-2})^2}\left(\frac{x+1}{2}\right)^{s-2}\Big[(t-s)(x^{-s-t-2}-x^{t-1}) \\
&\quad+(t-2)(x^{-s-t-1}-x^{t-2}) \\
&\qquad+(t+s-1)(x^{-t}-x^{t-s-3})+(t+1)(x^{-t-1}-x^{t-s-2})\Big]. \\
&= \frac{1}{8x(x+1)(x^{-t-1}+x^{t-2})^2}\left(\frac{x+1}{2}\right)^{s-2}\Big\{(t-s)\left[x^{t-1}(x^{-2t-s-1}-1)\right] \\
&\quad+(t-2)\left[x^{t-2}(x^{-s-2t+1}-1)\right]+(t+s-1)\left[x^{t-s-3}(x^{s-2t+3}-1)\right] \\
&\qquad+(t+1)\left[x^{t-s-2}(x^{s-2t+1}-1)\right]\Big\}.
\end{aligned} \tag{5.2}$$

From the above expression we observe that it is difficult to know the nature of the expression (5.2) with respect to the parameters s and t. Here below we shall study the above expression for some particular cases of the parameters s and t.

5.1. Inequalities Among $\mathcal{W}_s(P||Q)$ and $J(P||Q)$.

Take $t=1$ in (5.1) and (5.2), we get

$$g_{(\psi_s,\phi_1)}(x)=\frac{\psi_s''(x)}{\phi_1''(x)}=\frac{x^{-1-s}+1}{8(x^{-2}+x^{-1})}\left(\frac{x+1}{2}\right)^{s-2},\ x\in(0,\infty) \tag{5.3}$$

and

$$g'_{(\psi_s,\phi_1)}(x)=-\frac{x}{8(x+1)^2}\left(\frac{x+1}{2}\right)^{s-2}\left[2(x^{-s}-1)+(s-1)x(x^{-s-2}-1)\right]. \tag{5.4}$$

respectively.

Using the fact that

$$x^k\begin{cases}\geqslant 1, & x\geqslant 1,\ k>0 \text{ or } x\leqslant 1,\ k<0\\ \leqslant 1, & x\leqslant 1,\ k>0 \text{ or } x\geqslant 1,\ k<0\end{cases}. \tag{5.5}$$

we can write the expression (5.4) as follow:

$$g'_{(\psi_s,\phi_1)}(x)\begin{cases}\leqslant 0, & (x\geqslant 1,\ -2\leqslant s\leqslant 0) \text{ or } (x\leqslant 1,\ s\geqslant 1)\\ \geqslant 0, & (x\leqslant 1,\ -2\leqslant s\leqslant 0) \text{ or } (x\geqslant 1,\ s\geqslant 1)\end{cases}. \tag{5.6}$$

From (5.6), we conclude that the function $g_{(\psi_s,\phi_1)}(x)$ is monotonically decreasing (resp. increasing) in $(1,\infty)$ and increasing (resp. decreasing) in $(0,1)$ for all $-2\leqslant s\leqslant 0$ (resp. $s\geqslant 1$). This gives

$$M=\sup_{x\in(0,\infty)} g_{(\psi_s,\phi_1)}(x)=g_{(\psi_s,\phi_1)}(1)=\frac{1}{8},\ -2\leqslant s\leqslant 0, \tag{5.7}$$

and

$$m=\inf_{x\in(0,\infty)} g_{(\psi_s,\phi_1)}(x)=g_{(\psi_s,\phi_1)}(1)=\frac{1}{8},\ s\geqslant 1. \tag{5.8}$$

By the application of the inequality (3.43) given in Theorem 3.6 with the expressions (5.7) and (5.8), we conclude the following inequality among the measures $\mathcal{W}_s(P||Q)$ and $J(P||Q)$:

$$\mathcal{W}_s(P||Q)\begin{cases}\leqslant \frac{1}{8}J(P||Q), & -2\leqslant s\leqslant 0\\ \\ \geqslant \frac{1}{8}J(P||Q), & s\geqslant 1\end{cases}. \tag{5.9}$$

In particular the expression (5.9) lead us to to the inequality

$$I(P||Q)\leqslant \frac{1}{8}J(P||Q)\leqslant T(P||Q). \tag{5.10}$$

5.2. Inequalities Among $\mathcal{W}_s(P||Q)$ and $h(P||Q)$. Take $t=1/2$ in (5.1) and (5.2), we get

$$g_{(\psi_s,\phi_{1/2})}(x)=\frac{\psi_s''(x)}{\phi_{1/2}''(x)}=\frac{x^{-1-s}+1}{16x^{-3/2}}\left(\frac{x+1}{2}\right)^{s-2},\ x\in(0,\infty) \tag{5.11}$$

and

$$\begin{aligned} g'_{(\psi_s,\phi_{1/2})}(x) &= -\frac{\sqrt{x}}{32(x+1)^2}\left(\frac{x+1}{2}\right)^{s-2}\times\\ &\qquad\times\left[x^{-s-1}\left(3x+(2s-1)\right)-(2s-1)x-3\right]\\ &= -\frac{\sqrt{x}}{32(x+1)^2}\left(\frac{x+1}{2}\right)^{s-2}\left[3(x^{-s}-1)+(2s-1)x(x^{-s-2}-1)\right]. \end{aligned} \tag{5.12}$$

respectively.

Again in view of (5.5) we can write

$$g'_{(\psi_s,\phi_{1/2})}(x)\begin{cases}\leqslant 0, & (x\geqslant 1,\ -2\leqslant s\leqslant 0) \text{ or } (x\leqslant 1,\ s\geqslant \frac{1}{2})\\ \\ \geqslant 0, & (x\leqslant 1,\ -2\leqslant s\leqslant 0) \text{ or } (x\geqslant 1,\ s\geqslant \frac{1}{2})\end{cases}. \tag{5.13}$$

From (5.13), we conclude that the function $g_{(\psi_s,\phi_{1/2})}(x)$ is monotonically decreasing (resp. increasing) in $(1,\infty)$ and increasing (resp. decreasing) in $(0,1)$ for all $-2 \leqslant s \leqslant 0$ (resp. $s \geqslant \frac{1}{2}$). This gives

$$M = \sup_{x\in(0,\infty)} g_{(\psi_s,\phi_{1/2})}(x) = g_{(\psi_s,\phi_{1/2})}(1) = \frac{1}{8},\ -2 \leqslant s \leqslant 0, \tag{5.14}$$

and

$$m = \inf_{x\in(0,\infty)} g_{(\psi_s,\phi_{1/2})}(x) = g_{(\psi_s,\phi_{1/2})}(1) = \frac{1}{8},\ s \geqslant \frac{1}{2}. \tag{5.15}$$

By the application of the inequality (3.43) given in Theorem 3.6 with the expressions (5.14) and (5.15), we conclude the following inequality among the measures $\mathcal{W}_s(P||Q)$ and $h(P||Q)$:

$$\mathcal{W}_s(P||Q) \begin{cases} \leqslant h(P||Q), & -2 \leqslant s \leqslant 0 \\ \\ \geqslant h(P||Q), & s \geqslant \frac{1}{2} \end{cases}. \tag{5.16}$$

In particular this gives

$$I(P||Q) \leqslant h(P||Q) \leqslant T(P||Q). \tag{5.17}$$

The expressions (4.9), (4.36), (5.10) and (5.17) together give an inequality among the six *symmetric divergence measures* as follows:

$$\frac{1}{4}\Delta(P||Q) \leqslant I(P||Q) \leqslant h(P||Q) \leqslant \frac{1}{8}J(P||Q) \leqslant T(P||Q) \leqslant \frac{1}{16}\Psi(P||Q). \tag{5.18}$$

Remark 5.1. (*i*) *In view of (5.18) and (1.10), we have the following inequality:*

$$\frac{1}{4}\Delta(P||Q) \leqslant I(P||Q) \leqslant h(P||Q) \leqslant \frac{1}{8}J(P||Q) \leqslant T(P||Q) \leqslant \frac{1}{4}J(P||Q). \tag{5.19}$$

(*ii*) *It is well known* [18] *that*

$$\frac{1}{4}\Delta(P||Q) \leqslant h(P||Q) \leqslant \frac{1}{2}\Delta(P||Q). \tag{5.20}$$

In view of (5.18) and (5.20), we have the following inequality:

$$\frac{1}{4}\Delta(P||Q) \leqslant I(P||Q) \leqslant h(P||Q) \leqslant \frac{1}{2}\Delta(P||Q). \tag{5.21}$$

5.3. Inequalities Among $\mathcal{W}_s(P||Q)$, $\mathcal{V}_t(P||Q)$ and $\Psi(P||Q)$. For $t = 2$, we have

$$\mathcal{V}_2(P||Q) = \frac{1}{2}\Psi(P||Q)$$

and

$$\mathcal{W}_2(P||Q) = \frac{1}{16}\Psi(P||Q).$$

Since the measure $\mathcal{W}_s(P||Q)$ is monotonically increasing in s for all $s \geqslant -1$. This gives

$$\mathcal{W}_s(P||Q) \begin{cases} \leqslant \frac{1}{16}\Psi(P||Q), & -1 \leqslant s \leqslant 2 \\ \\ \geqslant \frac{1}{16}\Psi(P||Q), & s \geqslant 2 \end{cases}. \tag{5.22}$$

Also, in view of monotonicity of $\mathcal{V}_t(P||Q)$ with respect to t, we have

$$\mathcal{V}_t(P||Q) \begin{cases} \leqslant \frac{1}{2}\Psi(P||Q), & \frac{1}{2} \leqslant t \leqslant 2 \\ \\ \geqslant \frac{1}{2}\Psi(P||Q), & s \leqslant \frac{1}{2} \end{cases}. \tag{5.23}$$

5.4. Inequalities Among $\mathcal{V}_t(P||Q)$ and $\Delta(P||Q)$. Take $s = -1$ in (5.1) and (5.2), we get

$$g_{(\psi_{-1},\phi_t)}(x) = \frac{\psi''_{-1}(x)}{\phi''_t(x)} = \frac{1}{2(x^{-t-1} + x^{t-2})(x+1)}, \ x \in (0,\infty) \tag{5.24}$$

and

$$\begin{aligned} g'_{(\psi_{-1},\phi_t)}(x) &= \frac{2}{x(x+1)^4(x^{-t-1}+x^{t-2})^2} \times \\ &\quad \times \left[(t-2)(x^{-t} - x^{t-2}) + (t+1)(x^{-t-1} - x^{t-1})\right] \\ &= \frac{2}{x(x+1)^4(x^{-t-1}+x^{t-2})^2} \times \\ &\quad \left[(t-2)x^{t-2}(x^{2(1-t)} - 1) + (t+1)x^{t-1}(x^{-2t} - 1)\right]. \end{aligned} \tag{5.25}$$

respectively.

Again in view of (5.5) we can write

$$g'_{(\psi_{-1},\phi_t)}(x) \begin{cases} \leqslant 0, & x \geqslant 1, \ t \geqslant 0 \text{ or } t \leqslant -1 \\ \geqslant 0, & x \leqslant 1, \ t \geqslant 0 \text{ or } t \leqslant -1 \end{cases}. \tag{5.26}$$

From (5.26) we conclude that the function $g_{(\psi_{-1},\phi_t)}(x)$ is monotonically decreasing (resp. increasing) in $(1,\infty)$ and increasing (resp. decreasing) in $(0,1)$ for all $t \geqslant 0$ or $t \leqslant -1$. This gives

$$M = \sup_{x\in(0,\infty)} g_{(\psi_{-1},\phi_t)}(x) = g_{(\psi_{-1},\phi_t)}(1) = \frac{1}{8}, \ t \geqslant 0 \text{ or } t \leqslant -1. \tag{5.27}$$

By the application of the inequality (3.43) given in Theorem 3.6 with the expression (5.27), we conclude the following inequality among the measures $\mathcal{V}_t(P||Q)$ and $\Delta(P||Q)$:

$$\Delta(P||Q) \leqslant \frac{1}{2}\mathcal{V}_t(P||Q), \ t \geqslant 0 \text{ or } t \leqslant -1. \tag{5.28}$$

5.5. Inequalities Among $\mathcal{V}_t(P||Q)$ and $I(P||Q)$. Take $s=0$ in (5.1) and (5.2), we get

$$g_{(\psi_0,\phi_t)}(x)=\frac{\psi_0''(x)}{\phi_t''(x)}=\frac{1}{4x(x+1)(x^{-t-1}+x^{t-2})},\ x\in(0,\infty) \tag{5.29}$$

and

$$\begin{aligned} g'_{(\psi_0,\phi_t)}(x)&=\frac{1}{2x^2(x+1)^2(x^{-t-1}+x^{t-2})^2}\times\\ &\quad\times\left[(t-1)(x^{-t}-x^{t-2})+t(x^{-t-1}-x^{t-1})\right]\\ &=\frac{2}{2x^2(x+1)^2(x^{-t-1}+x^{t-2})^2}\times\\ &\quad\left[(t-1)x^{t-2}(x^{2(1-t)}-1)+tx^{t-1}(x^{-2t}-1)\right]. \end{aligned} \tag{5.30}$$

respectively.

Again in view of (5.5) we can write

$$g'_{(\psi_0,\phi_t)}(x)\begin{cases}\leqslant 0, & x\geqslant 1,\\ \geqslant 0, & x\leqslant 1,\end{cases}. \tag{5.31}$$

for all $t\in(-\infty,\infty)$.

From (5.31) we conclude that the function $g_{(\psi_0,\phi_t)}(x)$ is monotonically decreasing in $(1,\infty)$ and increasing in $(0,1)$ for all $-\infty<t<\infty$. This gives

$$M=\sup_{x\in(0,\infty)}g_{(\psi_0,\phi_t)}(x)=g_{(\psi_0,\phi_t)}(1)=\frac{1}{8},\ -\infty<t<\infty. \tag{5.32}$$

By the application of the inequality (3.43) with the expression (5.32), we conclude the following inequality among the measures $\mathcal{V}_s(P||Q)$ and $I(P||Q)$:

$$I(P||Q)\leqslant\frac{1}{8}\mathcal{V}_t(P||Q),\ -\infty<t<\infty. \tag{5.33}$$

5.6. Inequalities Among $\mathcal{V}_t(P||Q)$ and $T(P||Q)$. Take $s=1$ in (5.1) and (5.2), we get

$$g_{(\psi_1,\phi_t)}(x)=\frac{\psi_1''(x)}{\phi_t''(x)}=\frac{x^2+1}{4x^2(x+1)(x^{-t-1}+x^{t-2})},\ x\in(0,\infty) \tag{5.34}$$

and

$$\begin{aligned} g'_{(\psi_1,\phi_t)}(x)&=\frac{1}{4x^4(x+1)^2(x^{-t-1}+x^{t-2})^2}\times\\ &\quad\times\big[t(x^{2-t}-x^{t-2})+(t+1)(x^{-t+1}-x^{t-1})\\ &\qquad+(t-2)(x^{-t}-x^{t})+(t-1)(x^{-t-1}-x^{t+1})\big]\\ &=\frac{1}{4x^4(x+1)^2(x^{-t-1}+x^{t-2})^2}\times\\ &\quad\times\big[tx^{t-2}(x^{2(2-t)}-1)+(t+1)x^{t-1}(x^{2(1-t)}-1)\\ &\qquad+(t-2)x^{t}(x^{-2t}-1)+(t-1)x^{t+1}(x^{-2(t+1)}-1)\big]. \end{aligned} \tag{5.35}$$

respectively.

Again in view of (5.5) we can write

$$g'_{(\psi_1,\phi_t)}(x)\begin{cases}\geqslant 0, & (x\geqslant 1,\ 0\leqslant t\leqslant 1),\ (x\leqslant 1,\ t\geqslant 2 \text{ or } t\leqslant -1)\\ \leqslant 0, & (x\leqslant 1,\ 0\leqslant t\leqslant 1),\ (x\geqslant 1,\ t\geqslant 2 \text{ or } t\leqslant -1)\end{cases}. \tag{5.36}$$

From (5.36) we conclude that the function $g_{(\psi_1,\phi_t)}(x)$ is monotonically decreasing (resp. increasing) in $(1,\infty)$ and increasing (resp. decreasing) in $(0,1)$ for all $0\leqslant t\leqslant 1$(resp. $t\geqslant 2$ or $t\leqslant -1$). This gives

$$m=\sup_{x\in(0,\infty)} g_{(\psi_1,\phi_t)}(x)=g_{(\psi_1,\phi_t)}(1)=\frac{1}{8},\ 0\leqslant t\leqslant 1, \tag{5.37}$$

and

$$M=\inf_{x\in(0,\infty)} g_{(\psi_1,\phi_t)}(x)=g_{(\psi_1,\phi_t)}(1)=\frac{1}{8},\ t\geqslant 2 \text{ or } t\leqslant -1. \tag{5.38}$$

By the application of the inequality (3.43) with the expressions (5.37) and (5.38), we conclude the following inequality among the measures $\mathcal{V}_t(P||Q)$ and $T(P||Q)$:

$$T(P||Q)\begin{cases}\leqslant \frac{1}{8}\mathcal{V}_t(P||Q), & t\geqslant 2 \text{ or } t\geqslant -1\\ \\ \geqslant \frac{1}{8}\mathcal{V}_t(P||Q), & 0\leqslant t\leqslant 1\end{cases}. \tag{5.39}$$

5.7. Inequalities Among $\mathcal{W}_s(P||Q)$ and $\mathcal{V}_s(P||Q)$. When $t=s$, we shall obtain the results in two different ways, one by using the Theorem 3.6, and another applying Jensen's inequality.

(i) Take $t=s$ in (5.1) and (5.2), we get

$$g_{(\psi_s,\phi_s)}(x)=\frac{\psi''_s(x)}{\phi''_s(x)}=\frac{x^{-1-s}+1}{8(x^{-s-1}+x^{s-2})}\left(\frac{x+1}{2}\right)^{s-2},\ x\in(0,\infty) \tag{5.40}$$

and

$$\begin{aligned}g'_{(\psi_s,\phi_t)}(x)=&\frac{1}{8x(x+1)(x^{-s-1}+x^{s-2})^2}\left(\frac{x+1}{2}\right)^{s-2}\times\\ &\times\left[(s+1)x^{-2}(x^{1-s}-1)+(s-2)x^{s-2}(x^{1-3s}-1)\right.\\ &\left.+(2s-1)x^{-3}(x^{3-s}-1)\right]\end{aligned} \tag{5.41}$$

respectively.

Again in view of (5.5) we can write

$$g'_{(\psi_s,\phi_s)}(x)\begin{cases}\geqslant 0, & (x\geqslant 1,\ \frac{1}{2}\leqslant s\leqslant 1),\ (x\leqslant 1,\ s\geqslant 2 \text{ or } s\leqslant -1)\\ \leqslant 0, & (x\leqslant 1,\ \frac{1}{2}\leqslant s\leqslant 1),\ (x\geqslant 1,\ s\geqslant 2 \text{ or } s\leqslant -1)\end{cases}. \tag{5.42}$$

From (5.42) we conclude that the function $g_{(\psi_s,\phi_s)}(x)$ is monotonically increasing (resp. decreasing) in $(1,\infty)$ and decreasing (resp. increasing) in $(0,1)$ for all $\frac{1}{2} \leqslant s \leqslant 1$(resp. $s \geqslant 2$ or $s \leqslant -1$). This gives

$$m = \inf_{x\in(0,\infty)} g_{(\psi_s,\phi_s)}(x) = g_{(\psi_s,\phi_s)}(1) = \frac{1}{8},\ \frac{1}{2} \leqslant s \leqslant 1, \tag{5.43}$$

and

$$M = \sup_{x\in(0,\infty)} g_{(\psi_s,\phi_s)}(x) = g_{(\psi_s,\phi_s)}(1) = \frac{1}{8},\ s \geqslant 2 \text{ or } s \leqslant -1. \tag{5.44}$$

By the application of the inequality (3.43) with the expressions (5.43) and (5.44), we conclude the following inequality among the measures $\mathcal{W}_s(P||Q)$ and $\mathcal{V}_s(P||Q)$:

$$\mathcal{W}_s(P||Q) \begin{cases} \leqslant \frac{1}{8}\mathcal{V}_s(P||Q), & s \geqslant 2 \text{ or } s \geqslant -1 \\ \\ \geqslant \frac{1}{8}\mathcal{V}_s(P||Q), & \frac{1}{2} \leqslant s \leqslant 1 \end{cases}. \tag{5.45}$$

Some particular cases of (5.45) can be seen in (518) or (5.19).

(ii) By applying Jensen's inequality we can easily check that

$$\frac{p_i^s + q_i^s}{2} \begin{cases} \leqslant \left(\frac{p_i+q_i}{2}\right)^s, & 0 < s < 1 \\ \\ \geqslant \left(\frac{p_i+q_i}{2}\right)^s, & s > 1 \text{ or } s < 0 \end{cases}, \tag{5.46}$$

for all $i = 1, 2, ..., n$, where $P, Q \in \Gamma_n$.

Multiplying (5.46) by p_i^{1-s}, summing over all $i = 1, 2, ..., n$ and simplifying, we get

$$\frac{1}{2}\left(1 + \sum_{i=1}^n p_i^{1-s} q_i^s\right) \begin{cases} \leqslant \sum_{i=1}^n p_i^{1-s} \left(\frac{p_i+q_i}{2}\right)^s, & 0 < s < 1 \\ \\ \geqslant \sum_{i=1}^n p_i^{1-s} \left(\frac{p_i+q_i}{2}\right)^s, & s > 1 \text{ or s} < 0 \end{cases}. \tag{5.47}$$

Similarly, we can write

$$\frac{1}{2}\left(1 + \sum_{i=1}^n q_i^{1-s} p_i^s\right) \begin{cases} \leqslant \sum_{i=1}^n q_i^{1-s} \left(\frac{p_i+q_i}{2}\right)^s, & 0 < s < 1 \\ \\ \geqslant \sum_{i=1}^n q_i^{1-s} \left(\frac{p_i+q_i}{2}\right)^s, & s > 1 \text{ or } s < 0 \end{cases}. \tag{5.48}$$

Adding (5.47) and (5.48) and making some adjustments, we get

$$\frac{1}{4}\left(\sum_{i=1}^{n} p_i^{1-s}q_i^s + q_i^{1-s}p_i^s - 2\right) \tag{5.49}$$
$$\begin{cases} \leqslant \sum_{i=1}^{n}\left(\frac{p_i^{1-s}+q_i^{1-s}}{2}\right)\left(\frac{p_i+q_i}{2}\right)^s - 1, & 0 < s < 1 \\ \geqslant \sum_{i=1}^{n}\left(\frac{p_i^{1-s}+q_i^{1-s}}{2}\right)\left(\frac{p_i+q_i}{2}\right)^s - 1, & s > 1 \text{ or } s < 0 \end{cases}.$$

Since, $s(s-1) < 0$ for all $0 < s < 1$ and $s(s-1) > 0$ for all $s > 1$ or $s < 0$. This together with (5.50) proves that

$$\mathcal{V}_s(P||Q) \geqslant 4\mathcal{W}_s(P||Q),\ -\infty < s < \infty, \tag{5.50}$$

where for $s = 0$ and $s = 1$, the result is obtained by the continuity of the measures with respect to the parameter s.

In view of (5.45) and (5.50), we can write

$$\frac{1}{8}\mathcal{V}_s(P||Q) \leqslant \mathcal{W}_s(P||Q) \leqslant \frac{1}{4}\mathcal{V}_s(P||Q),\ \frac{1}{2} \leqslant s \leqslant 1. \tag{5.51}$$

Remark 5.2. (i) *For $s = 1$ in (5.51), we get a part of the inequality (5.19). For $s = \frac{1}{2}$ in (5.51), we get an interesting bound on a new measure given by (2.5) in terms of Hellinger's discrimination:*

$$\frac{1}{4}h(P||Q) \leqslant d(P||Q) \leqslant \frac{1}{2}h(P||Q). \tag{5.52}$$

(ii) *In particular for $s = \frac{1}{2}$ and $t = 1$ in (5.1), we can easily show that*

$$4\,d(P||Q) \leqslant \frac{1}{8}J(P||Q). \tag{5.53}$$

The inequalities (5.18) together with (5.52) and (5.53) give the following inequalities among some particular cases of the measures (2.2) and (2.4):

$$\begin{aligned} \frac{1}{4}\Delta(P||Q) &\leqslant I(P||Q) \leqslant h(P||Q) \leqslant 4\,d(P||Q) \\ &\leqslant \frac{1}{8}J(P||Q) \leqslant T(P||Q) \leqslant \frac{1}{16}\Psi(P||Q). \end{aligned} \tag{5.54}$$

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