SEIFFERT MEANS IN A TRIANGLE

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Abstract. Simple geometric proofs of some old and new inequalities between the Seiffert mean and classical means.

Seiffert introduced his first mean in [3] as

$$\mathbf{P}(x,y)=\begin{cases}\frac{x-y}{2\arcsin\frac{x-y}{x+y}} & x\neq y,\\ x & x=y.\end{cases} \tag{1}$$

and proved in [4, 3] that for $x \neq y$

$$\mathbf{G} \leq \mathbf{L} \leq \mathbf{P} \leq \mathbf{I} \leq \mathbf{A} \tag{2}$$

where

$$\mathbf{G}(x,y) = \sqrt{xy}, \tag{3}$$

$$\mathbf{L}(x,y) = \frac{x-y}{\log x - \log y}, \tag{4}$$

$$\mathbf{I}(x,y) = \frac{1}{e}\left(\frac{x^x}{y^y}\right)^{\frac{1}{x-y}}, \tag{5}$$

$$\mathbf{A}(x,y) = \frac{x+y}{2} \tag{6}$$

are the geometric, logarithmic, identric and arithmetic means. Later in [6] he used series representation to show that

$$\mathbf{P} < \mathbf{A} < \frac{\pi}{2}\mathbf{P}. \tag{7}$$

and

$$\frac{3}{\mathbf{P}} < \frac{2}{\mathbf{A}} + \frac{1}{\mathbf{G}}. \tag{8}$$

Sándor in [2] obtained further refinement. Using Pfaff's algorithm he proved that

$$\frac{\mathbf{A}+\mathbf{G}}{2} < \mathbf{P} < \sqrt{\mathbf{A}\frac{\mathbf{A}+\mathbf{G}}{2}} \tag{9}$$

The second Seiffert mean [5] is defined by

$$\mathbf{T}(x,y)=\begin{cases}\frac{x-y}{2\arctan\frac{x-y}{x+y}} & x\neq y,\\ x & x=y.\end{cases} \tag{10}$$

The goal of this paper is to give simple geometric proofs of (2), (7), (8) and sharpen the inequality (9). We also use obtain similar inequalities for $\mathbf{T}$.

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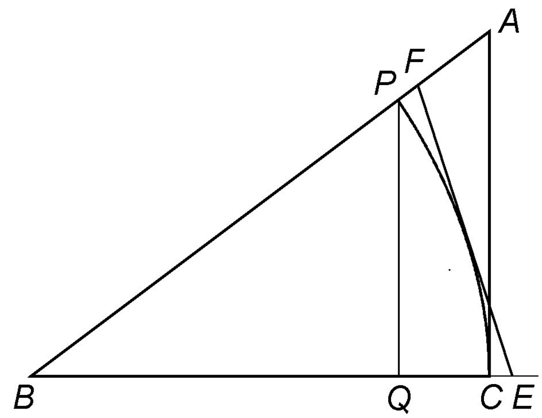

FIGURE 1

Consider a right triangle ΔABC with sides

$$|AB| = \frac{x+y}{2} = \mathbf{A}, \quad |AC| = \frac{|x-y|}{2}, \quad |BC| = \sqrt{xy} = \mathbf{G}$$

Let P be the intersection point of AB and the circle of radius $|BC|$ centered at B. Then

$$\angle B = \arcsin \frac{x-y}{x+y}$$

and

$$\mathbf{P} = \frac{|AC|}{\angle B} = \frac{|AC||BC|}{|\widehat{PC}|} \tag{11}$$

The following equations will be useful:

$$\begin{aligned} \sin \frac{B}{2} &= \sqrt{\frac{1-\cos B}{2}} = \sqrt{\frac{|AB|-|BC|}{2|AB|}} \\ &= \frac{|AC|}{2\sqrt{|AB|\frac{|AB|+|BC|}{2}}} \end{aligned} \tag{12}$$

$$\begin{aligned} \tan \frac{B}{2} &= \sqrt{\frac{1-\cos B}{1+\cos B}} = \sqrt{\frac{|AB|-|BC|}{|AB|+|BC|}} \\ &= \frac{|AC|}{|AB|+|BC|} \end{aligned} \tag{13}$$

Now we are ready to prove the first theorem:

Theorem 1. *For $x \neq y$*

$$\mathbf{G} < \mathbf{P} \tag{14}$$

and there is no constant c satisfying $\mathbf{P} < c\mathbf{G}$ for all x, y.

Proof. As $|\stackrel{\frown}{PC}| < |AC|$ and $|BC| = \mathbf{G}$ (14) follows form (11). On the other hand the ratio

$$\frac{|AC|}{|\stackrel{\frown}{PC}|} > \frac{2}{\pi}\frac{|AC|}{|BC|} = \frac{x-y}{\pi\sqrt{xy}} = \frac{1}{\pi}\left(\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}}\right)$$

can be made as large as we wish, so the ratio $\mathbf{P}/\mathbf{G}$ cannot be bounded from above. □

Let PQ be the height of the triangle ΔPBC. Then the following inequalities hold:

$$1 < \frac{|\stackrel{\frown}{PC}|}{|PQ|} < \frac{\pi}{2} \tag{15}$$

which implies

Theorem 2.

$$\frac{2}{\pi}\mathbf{A} < \mathbf{P} < \mathbf{A}$$

Proof. From (15) and (11) we get

$$\frac{2}{\pi}\frac{|AC||BC|}{|PQ|} < \mathbf{P} < \frac{|AC||BC|}{|PQ|}$$

and

$$\frac{|AC||BC|}{|PQ|} = \frac{|AC||BP|}{|PQ|} = |AB| = \mathbf{A}.$$

□

Another obvious inequality

$$1 < \frac{|\stackrel{\frown}{PC}|}{|PC|} < \frac{\pi}{2\sqrt{2}} \tag{16}$$

gives

Theorem 3.

$$\frac{2\sqrt{2}}{\pi}\sqrt{\mathbf{A}\frac{\mathbf{A}+\mathbf{G}}{2}} < \mathbf{P} < \sqrt{\mathbf{A}\frac{\mathbf{A}+\mathbf{G}}{2}}$$

Proof. From (16) and (11) we get

$$\frac{2\sqrt{2}}{\pi}\frac{|AC||BC|}{|PC|} < \mathbf{P} < \frac{|AC||BC|}{|PC|}$$

and

$$\frac{|AC||BC|}{|PC|} = \frac{|AC|}{2\sin\frac{B}{2}} = \sqrt{|AB|\frac{|AB|+|BC|}{2}} = \sqrt{\mathbf{A}\frac{\mathbf{A}+\mathbf{G}}{2}} \tag{17}$$

by (12). □

In the middle of $\stackrel{\frown}{PC}$ draw a tangent line that meets BA at F and BC at E. It is obvious that

$$\frac{\pi}{4} < \frac{|\stackrel{\frown}{PC}|}{|EF|} < 1 \tag{18}$$

and this implies

Theorem 4. *(see* [1]*, Cor. 1.11)*

$$\frac{\mathbf{A}+\mathbf{G}}{2}<\mathbf{P}<\frac{4}{\pi}\frac{\mathbf{A}+\mathbf{G}}{2}$$

Proof. From (18) and (11) we get

$$\frac{|AC||BC|}{|EF|}<\mathbf{P}<\frac{4}{\pi}\frac{|AC||BC|}{|EF|}$$

and

$$\frac{|AC||BC|}{|EF|}=\frac{|AC|}{2\tan\frac{B}{2}}=\frac{|AB|+|BC|}{2}=\frac{\mathbf{A}+\mathbf{G}}{2} \tag{19}$$

by 13. □

In order to show other inequalities for the Seiffert means we need the following lemma:

Lemma 1. *Let* $\phi_t(x)=(1-t)\sin x+t\tan x-x$, $0\le t\le 1$. *Then*

(a) $\phi_t(x)>0$ *for* $x\in(0,\frac{\pi}{2})$ *if and only if* $t\ge\frac{1}{3}$.
(b) $\phi_t(x)<0$ *for* $x\in(0,\frac{\pi}{4})$ *if and only if* $t\le\frac{\pi-2\sqrt{2}}{4-2\sqrt{2}}$.
(c) $\phi_t(x)<0$ *for* $x\in(0,\frac{\pi}{8})$ *if and only if* $t\le\frac{\pi-4\sqrt{2-\sqrt{2}}}{4\left[2(\sqrt{2}-1)-\sqrt{2-\sqrt{2}}\right]}$.

Proof. $\phi_t'(x)=(1-t)\cos x+t\cos^{-2}x-1$, so $\phi_t'(0)=0$.

$$\phi_t''(x)=2t\sin x\left(cos^{-3}x-\frac{1-t}{2t}\right). \tag{20}$$

From (20) we see that if $t\ge\frac{1}{3}$ then $\phi_t''>0$ so ϕ_t is convex, so from $\phi_t(0)=0$ and $\phi_t'(0)=0$ we deduce that $\phi>0$. On the other hand if $t<\frac{1}{3}$ then ϕ is concave for small x hence is negative.
To prove (b) note that if $t<\frac{1}{3}$ then ϕ_t is concave and negative for $x<x_0$ and then becomes convex, so ϕ_t has exactly one zero in $(0,\frac{\pi}{2})$. So $\phi_t<0$ in $(0,\frac{\pi}{4})$ if and only if $\phi_t(\frac{\pi}{4})<0$, which holds for $t\le\frac{\pi-2\sqrt{2}}{4-2\sqrt{2}}$.
Proof of (c) is exatly the same with $\pi/4$ replaced by $\pi/8$. □

Consider now the points $M_t=(1-t)Q+tC$ and $N_t=(1-t)P+tA$. We have

$$|M_tN_t|=(1-t)|QP|+t|CA|=|BC|((1-t)\sin B+t\tan B). \tag{21}$$

Theorem 5.

$$\frac{3}{\mathbf{P}}<\frac{2}{\mathbf{A}}+\frac{1}{\mathbf{G}}$$

Proof. From Lemma 1(a) we see that $|M_tN_t|>|\overset{\frown}{PC}|$ holds for every triangle if and only if $t\ge\frac{1}{3}$. (11) and (21) give

$$\frac{1}{\mathbf{P}}=\frac{|\overset{\frown}{PC}|}{|AC||BC|}<\frac{|M_tN_t|}{|AC||BC|}=\frac{(1-t)|QP|+t|CA|}{|AC||BC|}=\frac{1-t}{\mathbf{A}}+\frac{t}{\mathbf{G}}.$$

The right hand side of this expression increases with t, so the inequality in theorem is the strongest one. □

Similarly let $R_t = (1-t)C + tE$ and $S_t = (1-t)P + tF$. Then

$$|R_tS_t| = (1-t)|CP| + t|EF| = 2|BC|((1-t)\sin\frac{B}{2} + t\tan\frac{B}{2}). \tag{22}$$

The formula is similar to (21) but $B/2$ varies from 0 to $\pi/4$ and we can improve the inequalities (9)

Theorem 6.

$$\frac{1-r_1}{\sqrt{\mathbf{A}\frac{\mathbf{A}+\mathbf{G}}{2}}} + \frac{r_1}{\frac{\mathbf{A}+\mathbf{G}}{2}} < \frac{1}{\mathbf{P}} < \frac{2/3}{\sqrt{\mathbf{A}\frac{\mathbf{A}+\mathbf{G}}{2}}} + \frac{1/3}{\frac{\mathbf{A}+\mathbf{G}}{2}} \tag{23}$$

where $r_1 = \frac{\pi-2\sqrt{2}}{4-2\sqrt{2}} \approx .2673035$.

Proof. As in the proof of the previous theorem we see from (22) and lemma 1(b), that for $t > \frac{1}{3}$ $|R_tS_t| > |\overset{\frown}{PC}|$ and for $t < r_1$ $|R_tS_t| < |\overset{\frown}{PC}|$.
Using (11), (22),(17) and (19) we obtain the desired estimations. □

Similar inequalities for the second Seiffert mean **T** can be obtained in the same way by considering a triangle with sides

$$|AB| = \sqrt{\frac{x^2+y^2}{2}} = \mathbf{A_2} \quad |AC| = \frac{|x-y|}{2}, \quad |BC| = \frac{x+y}{2} = \mathbf{A}.$$

$\mathbf{A_2}$ is called the root-square-mean.
In this case

$$\mathbf{T} = \frac{|AC|}{\angle B} = \frac{|AC||BC|}{|\overset{\frown}{PC}|} \tag{24}$$

and we obtain similar results with **G** and **A** replaced with **A** and $\mathbf{A_2}$. Important difference between the two cases is that now $|AC| < |BC|$, so $0 < \angle B < \pi/4$ hence the constants in inequalities are different:

$$\begin{aligned}
\frac{\pi}{4} &< \frac{|\overset{\frown}{PC}|}{|AC|} < 1\\
1 &< \frac{|\overset{\frown}{PC}|}{|PQ|} < \frac{\pi\sqrt{2}}{4}\\
1 &< \frac{|\overset{\frown}{PC}|}{|PC|} < \frac{\pi}{4\sqrt{2-\sqrt{2}}}\\
\frac{\pi}{8(\sqrt{2}-1)} &< \frac{|\overset{\frown}{PC}|}{|EF|} < 1
\end{aligned}$$

which leads to

Theorem 7.

$$\begin{aligned}
\mathbf{A} &< \mathbf{T} < \frac{4}{\pi}\mathbf{A}\\
\frac{2\sqrt{2}}{\pi}\mathbf{A_2} &< \mathbf{T} < \mathbf{A_2}\\
\frac{4\sqrt{2-\sqrt{2}}}{\pi}\sqrt{\mathbf{A_2}\frac{\mathbf{A_2}+\mathbf{A}}{2}} &< \mathbf{T} < \sqrt{\mathbf{A_2}\frac{\mathbf{A_2}+\mathbf{A}}{2}}\\
\frac{\mathbf{A_2}+\mathbf{A}}{2} &< \mathbf{T} < \frac{8(\sqrt{2}-1)}{\pi}\frac{\mathbf{A_2}+\mathbf{A}}{2}\\
\frac{1-r_1}{\mathbf{A_2}}+\frac{r_1}{\mathbf{A}} &< \frac{1}{\mathbf{T}} < \frac{2/3}{\mathbf{A_2}}+\frac{1/3}{\mathbf{A}}\\
\frac{1-r_2}{\sqrt{\mathbf{A_2}\frac{\mathbf{A_2}+\mathbf{A}}{2}}}+\frac{r_2}{\frac{\mathbf{A_2}+\mathbf{A}}{2}} &< \frac{1}{\mathbf{T}} < \frac{2/3}{\sqrt{\mathbf{A_2}\frac{\mathbf{A_2}+\mathbf{A}}{2}}}+\frac{1/3}{\frac{\mathbf{A_2}+\mathbf{A}}{2}}
\end{aligned}$$

where $r_1 = \frac{\pi-2\sqrt{2}}{4-2\sqrt{2}} \approx .2673035$ *and* $r_2 = \frac{\pi-4\sqrt{2-\sqrt{2}}}{4\left[2(\sqrt{2}-1)-\sqrt{2-\sqrt{2}}\right]} \approx 0.3176533$

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