ABC Conjecture and Riemann Hypothesis

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Abstract

In this paper, we consider two great unproven problems in mathematics in the language of inequalities; ABC conjecture and Riemann hypothesis. It is shown that the Riemann hypothesis is true in some initial cases. Then we study radical function, which is contained in the heart of ABC conjecture; we find an upper bound for it by assuming Riemann hypothesis and finally by using this bound, we combine Riemann hypothesis and ABC conjecture.

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1 Riemann Hypothesis

The Riemann zeta-function is defined for $Re(s) > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

and extended by analytic continuation to the complex plan with one singularity at $s = 1$; in fact a simple pole with residues 1. The Riemann hypothesis [1], states that the non-real zeros of the Riemann zeta-function all lie on the line $Re(s) = \frac{1}{2}$. Now, let $\sigma(n)$ denote the sum of positive divisors of n, in 2002 Lagarias [3], showed that Riemann hypothesis holds if and only if

$$\sigma(n) \leq H_n + e^{H_n} \ln H_n, \tag{1}$$

for every $\mathbb{N}$, where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.
In [2], it is shown that the inequality (1) holds, when n is a power of a prime number and for some sufficiently large square free values of n; by square free integer we mean one that in its factoring to primes, the power of factors all are equal to 1. Here, we will recall all of them. Also, by using Maple software we have:

Note 1. The inequality (1) holds for $1 \leq n \leq 454013$.

Theorem 1 *Let $\mathbb{P}$ be the set of all primes. The inequality (1) holds for all $p \in \mathbb{P}$.*

Theorem 2 *The inequality (1) holds for all $n = p^a$, in which $p \in \mathbb{P}$ and $a \in \mathbb{N}$.*

Theorem 3 *The inequality (1) holds for all $n > e^{(e^{1+\frac{k}{2}})}$ and n square free with k distinct prime factors.*

Note 2. In the theorem 3, $n = p_1 p_2 \cdots p_k > k! > \Gamma(k)$ and so,

$$k < \Gamma^{-1}(n).$$

Corollary 1 *The inequality (1) holds for all $n = pq$, in which $p, q \in \mathbb{P}$ and $2 \leq p < q$.*

Similarly, by using Theorem 3, for $n \geq 195338$ and note 1 for $n \leq 195339$, we can yield the following result.

Corollary 2 *The inequality (1) holds for all $n = pqr$, in which $p, q, r \in \mathbb{P}$ and $2 \leq p < q < r$.*

2 Radical Function and ABC Conjecture

Suppose $n \in \mathbb{N}$ and $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, in which $p_1, p_2, \cdots, p_k \in \mathbb{P}$. The radical function defined as follows [4],

$$\mathrm{rad}(n) = p_1 p_2 \cdots p_k = \prod_{p|n} p.$$

This function has many nice properties; for example suppose $m, n \in \mathbb{N}$, then we have

$$\mathrm{rad}(mn) = \frac{\mathrm{rad}(m)\mathrm{rad}(n)}{\mathrm{rad}(gcd(m,n))},$$

in which this yields that "rad" is multiplicative. Now, since $\sum_{d|p^a} \mathrm{rad}(d) = 1 + ap$ holds for $p \in \mathbb{P}$ and $a \in \mathbb{N}$, we have

$$\sum_{d|n} \mathrm{rad}(d) = \prod_{i=1}^{k} (1 + a_i p_i), \qquad (n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}).$$

We introduce our necessary property in the following lemma.

Lemma 1 *Suppose $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, then*

$$\mathrm{rad}(n) = (-1)^k \sum_{d|n} \mu(d)\sigma(d), \tag{2}$$

in which μ is the well-known mobius function and defined by

$$\mu(m) = \begin{cases} 1 & m = 1, \\ (-1)^k & m = p_1 p_2 \cdots p_k, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: The functions μ and σ are multiplicative. So, for $p \in \mathbb{P}$ and $a \in \mathbb{N}$, we have

$$\sum_{d|p^a} \mu(d)\sigma(d) = -p,$$

and this completes the proof. $\square$

Above lemma help us to connect ABC conjecture with Riemann hypothesis. Now, lets to review ABC conjecture [4]:

ABC Conjecture. For every $\epsilon > 0$, there exists constant $c(\epsilon) \in \mathbb{R}$ such that for every $a, b \in \mathbb{N}$, we have

$$\frac{a+b}{gcd(a,b)} \leq c(\epsilon)\mathrm{rad}\left(\frac{ab(a+b)}{gcd(a,b)^3}\right)^{1+\epsilon}.$$

Now, let $L(n) = H_n + e^{H_n} \ln H_n$ and consider Riemann hypothesis; $\sigma(n) \leq L(n)$. Since both of the functions L and σ are positive, we have $-L(d) \leq \mu(d)\sigma(d) \leq L(n)$ and by lemma 1, we yield the following bound for radical function under assumption of Riemann hypothesis,

$$\mathrm{rad}(n) = \left| \sum_{d|n} \mu(d)\sigma(d) \right| \leq \sum_{d|n} L(d),$$

and now, we can combine ABC conjecture and Riemann hypothesis:

ABC Conjecture and Riemann hypothesis. For every $\epsilon > 0$, there exists constant $c(\epsilon) \in \mathbb{R}$ such that for every $a, b \in \mathbb{N}$, we have

$$\frac{a+b}{gcd(a,b)} \leq c(\epsilon)\text{rad}\left(\sum_{d|m} L(d)\right)^{1+\epsilon},$$

in which $m = \frac{ab(a+b)}{gcd(a,b)^3}$.

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