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Note On an Open Problem for Algebraic Inequality

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Abstract. In this paper, the open problem published in ([1]: Feng Qi, An algebraic inequality, J. Inequal. Pure Appl.Math., 2(2001), no.1, Art.13.) is solved by using analytic arguments. At the same time, the precise scope of r in the open problem is given. The lower bound of the theorem in [1] is refined.

1. Introduction

In[1],F. Qi, posed the following:

Open problem. Let $b > a > 0$ and $\delta > 0$ be real numbers. Then for any positive $r \in \mathbb{R}$, We have

$$(\frac{b+\delta-a}{b-a} \cdot \frac{b^{r+1}-a^{r+1}}{(b+\delta)^{r+1}-a^{r+1}})^{1/r} < \frac{[b^b/a^a]^{1/(b-a)}}{[(b+\delta)^{b+\delta}/a^a]^{1/(b+\delta-a)}} \tag{1}$$

The upper bound in (1) is best possible.

In [1], F.Qi proved :

Theorem. Let $b > a > 0$ and $\delta > 0$ be real numbers. Then for any positive $r \in \mathbb{R}$, We have

$$(\frac{b+\delta-a}{b-a} \cdot \frac{b^{r+1}-a^{r+1}}{(b+\delta)^{r+1}-a^{r+1}})^{1/r} > \frac{b}{b+\delta} \tag{2}$$

The lower bound is best possible.

There is much literature available on the study of Algebraic inequalities, see for example [1,2,3,4,5,6,7,8,9,10]. The purpose of this paper is to verify the above inequality (1) and refine the lower bound of inequality (2).

We believe that the open problem inequality (1) should be decomposed with variable r evaluation. The open problem should be stated as following.

Theorem 1. Let $b > a > 0$ and $\delta > 0$ be real numbers. Then for any positive $r \in \mathbb{R}$

(i) If $r > 1$, then

$$(\frac{b+\delta-a}{b-a} \cdot \frac{b^{r+1}-a^{r+1}}{(b+\delta)^{r+1}-a^{r+1}})^{1/r} < \frac{[b^b/a^a]^{1/(b-a)}}{[(b+\delta)^{b+\delta}/a^a]^{1/(b+\delta-a)}} \tag{3}$$

The upper bound in (3) is best possible.

(ii)If $0 < r < \frac{4}{5}$, then

$$(\frac{b+\delta-a}{b-a} \cdot \frac{b^{r+1}-a^{r+1}}{(b+\delta)^{r+1}-a^{r+1}})^{1/r} > \frac{[b^b/a^a]^{1/(b-a)}}{[(b+\delta)^{b+\delta}/a^a]^{1/(b+\delta-a)}} > \frac{b}{b+\delta} \tag{4}$$

2. Proof of theorem1

Proof. (i) The inequality (3) is equivalent to

$$[\frac{(b+\delta)^{b+\delta}}{a^a}]^{r/(b+\delta-a)} / \frac{(b+\delta)^{r+1}-a^{r+1}}{b+\delta-a} < [\frac{b^b}{a^a}]^{r/(b-a)} / \frac{b^{r+1}-a^{r+1}}{b-a} \tag{5}$$

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Therefore, it is sufficient to prove that the function

$$f(s) = [\frac{s^s}{a^a}]^{r/(s-a)} / \frac{s^{r+1} - a^{r+1}}{s-a} \tag{6}$$

is decreasing for $s > a$.

By direct computation, we have

$$f'(s) = [\frac{s^s}{a^a}]^{r/(s-a)} \cdot g(s)/(s-a)^6 \cdot (\frac{s^{r+1} - a^{r+1}}{s-a})^2 \tag{7}$$

Where

$$\begin{aligned} g(s) = (1-r)s^{r+2} + (2ar + alna - alns)s^{r+1} - a^2(r+1)s^r - 2a^{r+1}s \\ +a^{r+2}lns - a^{r+2}lna + 2a^{r+2} \end{aligned} \tag{8}$$

and $g(a) = 0$

Straightforward calculation produces

$$g'(s) = \tau(s)/s \tag{9}$$

Where

$$\begin{aligned} \tau(s) = (1-r)(r+2)s^{r+2} + [a(r+1)lna + 2ar(r+1) - a(r+1)lns - a]s^{r+1} \\ -a^2r(r+1)s^r - 2a^{r+1}s + a^{r+2} \end{aligned} \tag{10}$$

and $\tau(a) = 0$

Direct computation gives us

$$\begin{aligned} \tau'(s) = (1-r)(r+2)^2s^{r+1} + [a(r+1)^2lna + 2a(r+1)(r^2+r-1) - a(r+1)^2lns]s^r \\ -a^2r^2(r+1)s^{r-1} - 2a^{r+1} \end{aligned} \tag{11}$$

$$\tau''(s) = s^{r-1} \cdot h(s) \tag{12}$$

Where

$$\begin{aligned} h(s) = (1-r^2)(r+2)^2s - ar(r+1)^2lns - a^2r^2(r^2-1)s^{-1} + 2ar^4 \\ +4ar^3 - ar^2 - 4ar - a + ar(r+1)^2lna \end{aligned} \tag{13}$$

and $\tau'(a) = 0$,$h(a) = 3a(1-r^2)$

By direct computation, we obtain

$$h'(s) = p(s)/s^2 \tag{14}$$

Where

$$p(s) = (1-r^2)(r+2)^2s^2 - ar(r+1)^2s + a^2r^2(r^2-1) \tag{15}$$

$$p(a) = a^2(r+1)^2(4-5r)$$

$$p'(s) = 2(1-r^2)(r+2)^2s - ar(r+1)^2$$

$$P'(a) = a(r+1)(-2r^3 - 7r^2 - r + 8)$$

$$p''(s) = 2(1-r^2)(r+2)^2 \tag{16}$$

Then, when $r > 1$ and $s > a$, $p''(s) < 0$, $p'(s) \searrow$, $p'(s) < p'(a) < 0$; $p(s) \searrow$, $p(s) < p(a) < 0$, and thus $h'(s) < 0$, $h(s) \searrow$, $h(s) < h(a) < 0$; therefore $\tau''(s) < 0$, $\tau'(s) \searrow$, $\tau'(s) < \tau'(a) = 0$, and then $\tau(s) \searrow$, $\tau(s) < \tau(a) = 0$; $g'(s) < 0$, $g(s) \searrow$, $g(s) < g(a) = 0$; hence $f'(s) < 0$. The inequality (3) follows.

(ii). The left inequality in (4) is equivalent to

$$[\frac{(b+\delta)^{b+\delta}}{a^a}]^{r/(b+\delta-a)} / \frac{(b+\delta)^{r+1} - a^{r+1}}{b+\delta-a} > [\frac{b^b}{a^a}]^{r/(b-a)} / \frac{b^{r+1} - a^{r+1}}{b-a} \tag{17}$$

Therefore, it is sufficient to prove that the function

$$f(s) = [\frac{s^s}{a^a}]^{r/(s-a)} / \frac{s^{r+1} - a^{r+1}}{s-a} \tag{18}$$

is increasing for $s > a$.

Similar to arguments above in (1), when $0 < r < \frac{4}{5}$,and $s > a$, we have $p''(s) > 0$, $p'(s) \nearrow$, $p'(s) > p'(a) > 0$; $p(s) \nearrow$, $p(s) > p(a) > 0$, and thus $h'(s) > 0$, $h(s) \nearrow$, $h(s) > h(a) > 0$; therefore $\tau''(s) > 0$, $\tau'(s) \nearrow$, $\tau'(s) > \tau'(a) = 0$, and then $\tau(s) \nearrow$, $\tau(s) > \tau(a) = 0$; $g'(s) > 0$, $g(s) \nearrow$, $g(s) > g(a) = 0$; hence $f'(s) > 0$. The left inequality in (4) holds.

The right inequality in (4) is equivalent to

$$[\frac{b^b}{a^a}]^{1/(b-a)}/b > [\frac{(b+\delta)^{b+\delta}}{a^a}]^{1/(b+\delta-a)}/b+\delta \tag{19}$$

Therefore, it is sufficient to prove that the function

$$L(s) = [\frac{s^s}{a^a}]^{1/(s-a)}/s \tag{20}$$

is decreasing for $s > a$.

By direct computation, we obtain

$$L'(s) = [\frac{s^s}{a^a}]^{1/(s-a)} \cdot M(S)/s^2(s-a)^2 \tag{21}$$

Where

$$M(s) = as - aslns + aslna - a^2 \tag{22}$$

and $M'(s) = alna - alns$.

Therefore $M''(S) = -\frac{a}{s} < 0$, $M'(s) \searrow$, $M'(s) < M'(a) = 0$; $M(s) \searrow$, $M(s) < M(a) = 0$, and then $L'(s) < 0$. The right inequality in (4) follows.

Remark 1. Let $r \in \mathbb{R}$ and $0 < r < \frac{4}{5}$, inequality (4) refines the lower bound of inequality(2).

Remark 2. The inequalities in this paper are related to the study of monotonicity of the ratios and differences of mean values.

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