A note on Grüss type integral inequality *

DEEPAK B. PACHPATTE

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Abstract

In the present note we establish a new integral inequality similar to Grüss integral inequality by using a variant of the mean value theorem.

1 Introduction

G Grüss [3] has proved the following useful and interesting inequality (See also [4]):

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \left(\frac{1}{b-a} \int_a^b f(x)dx \right) \left(\frac{1}{b-a} \int_a^b g(x)dx \right) \right|$$

$$\leq \frac{1}{4} (P-p)(Q-q) \tag{1.1}$$

provided that f and g are two integrable functions on [a,b] such that

$$p \leq f(x) \leq P, \quad q \leq g(x) \leq Q$$

for all $x \in [a,b]$,where p,P,q,Q are real constants.

During the past few years, many researchers have obtained various generalizations , variants and extensions of the inequality (1.1), see [1,2] and the references cited therein . The main purpose of the present note is to establish a new inequality of the type (1.1) by using the variant of the well known Lagrange's mean value theorem.

2 Statement of Results

In what follows , R and $'$ denotes the set of real numbers and the derivative of a function.

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In the proof of our result we make use of the following theorem, which is a variant of the well known Lagrange's mean value theorem (See[8]).

Theorem 1 . Suppose f is continous on $[a,b]$ and differentiable in (a,b). Then there exists a point $c \in (a,b)$ such that

$$f(y) = f(x) + (y-x)f'(x) + \frac{1}{2}(y-x)^2 f''(c).$$

The proof can be completed by applying Rolle's theorem for the function

$$\Phi(t) = f(y) - f(t) - (y-t)f'(t) - (y-t)^2 A,$$

where A is constant.

Our main result is given in the following theorem.

Theorem 2 .Let $f,g : [a,b] \to R$ be continous on $[a,b]$ $a < b$; $a,b \in R$ and differentiable on (a,b). Then

$$\Bigg| \frac{1}{b-a}\int_a^b f(y)g(y)dy - \left(\frac{1}{b-a}\int_a^b f(y)dy\right)\left(\frac{1}{b-a}\int_a^b g(y)dy\right)$$

$$-\frac{1}{2(b-a)^2}\left[\{f(b)-f(a)\}\left(\int_a^b yg(y)\,dy\right) + \{g(b)-g(a)\}\left(\int_a^b yf(y)\,dy\right)\right]$$

$$+\frac{1}{2(b-a)^2}\left[\left(\int_a^b g(y)dy\right)\left(\int_a^b yf'(y)dy\right) + \left(\int_a^b f(y)dy\right)\left(\int_a^b yg'(y)dy\right)\right]\Bigg|$$

$$\leq \frac{1}{4(b-a)^2}\int_a^b \left\{\int_a^b (y-x)^2 \{|g(y)|\,\|f''\|_\infty + |f(y)|\,\|g''\|_\infty\}\,dx\right\}dy\,,$$

where

$$\|f''\|_\infty = \underset{t\in(a,b)}{Sup}|f''(t)| < \infty, \|g''\|_\infty = \underset{t\in(a,b)}{Sup}|g''(t)| < \infty.$$

Proof: Let $x,y \in [a,b]$with $y \neq x$.From the hypotheses and applying Theorem1, there exists a point c between y and x such that

$$f(y) = f(x) + (y-x)f'(x) + \frac{1}{2}(y-x)^2 f''(c) \tag{2.2}$$

$$g(y) = g(x) + (y-x)g'(x) + \frac{1}{2}(y-x)^2 g''(c)\,. \tag{2.3}$$

Multiplying both sides of (2.2) and (2.3) by $g(y)$ and $f(y)$ respectively and adding the resulting identities we get

$$2f(y)g(y) = g(y)f(x) + f(y)g(x) + y\{g(y)f'(x) + f(y)g'(x)\}$$

$$-x\left\{g\left(y\right)f'\left(x\right)+f\left(y\right)g'\left(x\right)\right\}+\frac{1}{2}\left(y-x\right)^{2}\left\{f\left(y\right)g''\left(c\right)+g\left(y\right)f''\left(c\right)\right\}. \quad (2.4)$$

Integrating both sides of (2.4) with respect to x over $[a,b]$ we have

$$\begin{aligned}2\left(b-a\right)f\left(y\right)g\left(y\right)&=g\left(y\right)\int_a^b f(x)dx+f\left(y\right)\int_a^b g(x)dx\\&+yg(y)\int_a^b f'(x)dx+yf(y)\int_a^b g'(x)dx\\&-g(y)\int_a^b xf'(x)dx-f(y)\int_a^b xg'(x)dx\\&+\frac{1}{2}\int_a^b\left(y-x\right)^{2}\left[g(y)f''(c)+f(y)g''(c)\right]dx. \quad (2.5)\end{aligned}$$

Integrating both sides of (2.5) with respect to y over $[a,b]$ we have

$$\begin{aligned}&2\left(b-a\right)\int_a^b f(y)g(y)dy\\&=\left(\int_a^b g(y)dy\right)\left(\int_a^b f(x)dx\right)+\left(\int_a^b f(y)dy\right)\left(\int_a^b g(x)dx\right)\\&+\left(\int_a^b yg(y)dy\right)\left(\int_a^b f'(x)dx\right)+\left(\int_a^b yf(y)dy\right)\left(\int_a^b g'(x)dx\right)\\&-\left(\int_a^b g(y)dy\right)\left(\int_a^b xf'(x)dx\right)-\left(\int_a^b f(y)dy\right)\left(\int_a^b xg'(x)dx\right)\\&+\frac{1}{2}\int_a^b\left\{\int_a^b\left(y-x\right)^{2}\left\{g(y)f''(c)+f(y)g''(c)\right\}dx\right\}dy. \quad (2.6)\end{aligned}$$

Multiplying both sides of (2.6) by $\frac{1}{2(b-a)^2}$ and rewriting we get

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(y)g(y)dy-\left(\frac{1}{b-a}\int_a^b f(y)dy\right)\left(\frac{1}{b-a}\int_a^b g(y)dy\right)\\&-\frac{1}{2(b-a)^2}\left[\left(f(b)-f(a)\right)\left(\int_a^b yg(y)dy\right)+\left(g(b)-g(a)\right)\left(\int_a^b yf(y)dy\right)\right]\\&+\frac{1}{2(b-a)^2}\left[\left(\int_a^b g(y)dy\right)\left(\int_a^b yf'(y)dy\right)+\left(\int_a^b f(y)dy\right)\left(\int_a^b yg'(y)dy\right)\right]\\&=\frac{1}{4(b-a)^2}\int_a^b\left\{\int_a^b\left(y-x\right)^{2}\left\{g(y)f''(c)+f(y)g''(c)\right\}dx\right\}dy. \quad (2.7)\end{aligned}$$

From (2.7) and using properties of modulus we have

$$\left| \frac{1}{b-a} \int_a^b f(y)g(y)dy - \left(\frac{1}{b-a} \int_a^b f(y)dy \right) \left(\frac{1}{b-a} \int_a^b g(y)dy \right) \right.$$

$$-\frac{1}{2(b-a)^2} \left[\{f(b)-f(a)\} \left(\int_a^b yg(y)\,dy \right) + \{g(b)-g(a)\} \left(\int_a^b yf(y)\,dy \right) \right]$$

$$\left. +\frac{1}{2(b-a)^2} \left[\left(\int_a^b g(y)dy \right) \left(\int_a^b yf'(y)dy \right) + \left(\int_a^b f(y)dy \right) \left(\int_a^b yg'(y)dy \right) \right] \right|$$

$$\leq \frac{1}{4(b-a)^2} \int_a^b \left\{ \int_a^b (y-x)^2 \{|g(y)| \, \|f''\|_\infty + |f(y)| \, \|g''\|_\infty\} \, dx \right\} dy ,$$

which is the required inequality in (2.1) and the proof is complete.

References

1. S.S.DRAGOMIR, Some integral inequalities of Grüss type, Indian J.Pure and Appl.Math. 31(2000),379-415.

2. A.M.FINK, A treatise on Grüss inequality, Analytic and Geometric Inequalities and Applications, T.M.Rassias and H.M.Srivastava (eds.), Kluwer Academic Publishers, Dordrecht 1999,93-113.

3. G.GRÜSS, Über das maximum des absoluten Betrages von

$$\frac{1}{b-a} \int_a^b f(x)\, g(x)\, dx - \frac{1}{(b-a)^2} \int_a^b f(x)\, dx \int_a^b g(x)\, dx,$$

Math.Z. 39(1935),215-226.

4. D.S.MITRINOVIĆ,J.E.PEČARIĆ and A.M.FINK, Classical and new inequalities in Analysis, Kluwer Academic Publishers, Dordrecht, 1993.

5. B.G.PACHPATTE, On Grüss type inequalities for double integrals, J.Math.Anal.Appl. 267(2002),454-459.

6. B.G.PACHPATTE, On Grüss type integral inequalities, J.Inequal.Pure and Appl.Math. 3(1)(2002),Art.11.

7. B.G.PACHPATTE, New weighhted multivariate Grüss type inequalities, J.Inequal.Pure and Appl.Math. 4(5) (2003),Art.108.

8. W.RUDIN, Principles of Mathematical Analysis, McGraw-Hill Book Company, Inc. 1953.

DEEPAK B. PACHPATTE 57 Shri Niketan Colony, Near Abhinay Talkies, Aurangabad 431 001 (Maharashtra) India

E-mail: pachpatte@yahoo.co.in