DENSITY OF n^{th}-POWER FREES

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Abstract

In this note we are going to analyze the density of n^{th}*-power free* integers.

1. Introduction

Let $\mathbb{P}$ the set of all primes and suppose M is a positive integer, with the following prime factoring:

$$M = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \quad (p_1, p_2, \cdots, p_k \in \mathbb{P}).$$

We call M, n^{th}*-power free* if for $1 \leq i \leq k$, $\alpha_i < n$. Let $f_n(x) =$ The number of n^{th}-power frees $\leq x$. By density we mean

$$\lim_{x \to \infty} \frac{f_n(x)}{x}.$$

It is well-know that [1],

$$f_2(x) = \frac{6x}{\pi^2} + O(\sqrt{x}).$$

So, the density of square frees is $\frac{6}{\pi^2}$ or approximately 61 percent!. Now, what about *cubic frees*? And generally the n^{th}-power frees?

2. Density Analysis

In this section we will show that the density of n^{th}-power free integers is $\frac{1}{\zeta(n)}$. Our main result is based on the following lemma.

Lemma 1 *Let $s > 1$ be a real number. We have*

$$\sum_{m=1}^{\infty} \frac{\mu(m)}{m^s} = \frac{1}{\zeta(s)}.$$

PROOF:

$$\sum_{m=1}^{\infty}\frac{\mu(m)}{m^s}=\sum_{k=1}^{\infty}\frac{(-1)^k}{p_1^s p_2^s\cdots p_k^s}=\prod_{p\in\mathbb{P}}(1-\frac{1}{p^s})=\prod_{p\in\mathbb{P}}\frac{1}{\sum_{k=1}^{\infty}\frac{1}{p^{sk}}}=\frac{1}{\sum_{m=1}^{\infty}\frac{1}{m^s}}=\frac{1}{\zeta(s)}.$$

Theorem 1 *For any integer $n\geq 2$ and any real $x\geq 1$, we have*

$$|\frac{x}{\zeta(n)}-f_n(x)|<\frac{n}{n-1}\sqrt[n]{x}-1. \tag{1}$$

PROOF: By a usual counting, we obtain

$$f_n(x)=x-\sum_{p\in\mathbb{P}}\lfloor\frac{x}{p^n}\rfloor+\sum_{p,q\in\mathbb{P},p\neq q}\lfloor\frac{x}{(pq)^n}\rfloor-\cdots=\sum_{k\leq\sqrt[n]{x}}\mu(k)\lfloor\frac{x}{k^n}\rfloor.$$

So, we have

$$|\frac{x}{\zeta(n)}-f_n(x)|=|\sum_{1<k\leq\sqrt[n]{x}}\mu(k)(\frac{x}{k^n}-\lfloor\frac{x}{k^n}\rfloor)+\sum_{k>\sqrt[n]{x}}\mu(k)\frac{x}{k^n}|$$

$$<(\sqrt[n]{x}-1)+x\sum_{k>\sqrt[n]{x}}\frac{1}{k^n}<\sqrt[n]{x}-1+x\int_{\sqrt[n]{x}}^{\infty}\frac{ds}{s^n}=\frac{n}{n-1}\sqrt[n]{x}-1.$$

This completes the proof.
A weak but nice form of the above theorem is

Corollary 1 *For any integer $n\geq 2$ and any real*

$$f_n(x)=\frac{x}{\zeta(n)}+O(\sqrt[n]{x}).$$

Corollary 2 *For any integer $n\geq 2$, the density of n^{th}-power free integers is*

$$\frac{1}{\zeta(n)}.$$

According the definition of $f_n(x)$ we obtain $0\leq\frac{f_n(x)}{x}<1$. We desire to find better lower bounds:

Lemma 2 *Let $n\geq 2$ is an integer. For any real $0\leq\alpha<\frac{1}{\zeta(n)}$ and any real $x>(\frac{n\zeta(n)}{(n-1)(1-\alpha\zeta(n))})^{\frac{n}{n-1}}$, we have*

$$\alpha<\frac{f_n(x)}{x}.$$

PROOF: From (1), we have

$$\frac{1}{\zeta(n)}-\frac{n}{(n-1)x^{1-\frac{1}{n}}}<\frac{f_n(x)}{x}.$$

Let $LB(n,x)$ denote the left hand side of the above inequality. If $0\leq\alpha<\frac{1}{\zeta(n)}$ and $x>(\frac{n\zeta(n)}{(n-1)(1-\alpha\zeta(n))})^{\frac{n}{n-1}}$, then $\alpha<LB(n,x)$. This completes the proof.
The obtained results are useful in study of distribution of n^{th}-power free integers. In the next section we do this.

3. Computational Results

The sequence $\frac{f_n(x)}{x}$ for any fixed n is convergent, and it may affairs minimum for some $x \in \mathbb{N}$. The lemma 2 led us to the following algorithm to find the minimum of $\frac{f_n(x)}{x}$ on $\mathbb{N}$.

Step(1). Find $x_0 \in \mathbb{N}$ such that

$$\frac{f_n(x_0)}{x_0} < \frac{1}{\zeta(n)}.$$

Step(2). For $\alpha = \frac{f_n(x_0)}{x_0}$, take

$$N = \left\lfloor \left(\frac{n\zeta(n)}{(n-1)(1-\alpha\zeta(n))}\right)^{\frac{n}{n-1}} \right\rfloor.$$

Step(3). Find

$$\min_{1\leq x\leq N} \{\frac{f_n(x)}{x}\}.$$

We note that there is no guarantee for the existence of x_0 in step(1), but there are some evidences for the following question.

Question Is there exists an x_0 in the interval $[5^n, 6^n]$ with $\frac{f_n(x_0)}{x_0} < \frac{1}{\zeta(n)}$?

Our computer program gave an affirmative answer to above question for $n = 2, \ldots, 10$. It is based on the following recursive relation:

$$f_n(x) = f_n(x-1) + \begin{cases} 1 & x \text{ is } n^{th}\text{-power free} \\ 0 & x \text{ other wise} \end{cases}$$

Since $f_n(2^n) = 2^n - 1$, we start from $x = 2^n$. Then divide our interval into sub intervals $[2^n, 3^n], [3^n, 5^n], [5^n, 7^n], \cdots$. The following table includes the value of x_0, exact value of the minimum of $\frac{f_n(x)}{x}$ and the value of x at which the minimum occur.

n	x_0	N	x	$f_n(x)$	$\min_{x\in\mathbb{N}} \frac{f_n(x)}{x}$
2	28	6503647	176	106	0.602273
3	136	55980	378	314	0.830688
4	656	171931	2512	2320	0.923567
5	3168	269627	3168	3055	0.964331
6	16064	1346593	31360	30825	0.982940
7	78732	10552627	236288	234331	0.991718
8	393728	25381201	1174528	1169758	0.995939
9	1968640	146390429	7814151	7798488	0.997996
10	9802752	816756521	48833536	48785015	0.999006

Now, we use our numerical results to get the following corollaries.

Corollary 3 *Let $n > 1$ is an integer. The number of cases that we can write n as sum of two square frees is greater than $\frac{n}{10}$.*

PROOF: More than 60 percent of integers between 1 and n are square free. The number of pairs $\{i, j\}$ such that $i + j = n$ is not greater than $\frac{n}{2}$, so, there are more than $\frac{n}{10}$ of this pairs with square free members. This complete the proof.

Corollary 4 *The probabilty that two successive positive integer numbers both be square free is more than 20%.*

PROOF: Obvious.

References

[H-W] G. H. HARDY AND E. M. WRIGHT, An Introduction to THE THEORY OF NUMBERS, fifth edition, Oxford University Press, London, 1979.