A FRACTAL VERSION OF THE SCHULTZ'S THEOREM

M. A. Navascués[a] and M. V. Sebastián[b]

[a]Departamento de Matemática Aplicada - Centro Politécnico Superior de Ingenieros
Universidad de Zaragoza. C/ María de Luna, 3.
50018 Zaragoza, Spain.
Tel.: 34-976-761983, Fax: 34-976-761886
e-mail: manavas@unizar.es (corresponding author)
[b]Departamento de Matemáticas - Facultad de Ciencias
Universidad de Zaragoza. Pza. San Francisco, s/n.
50009 Zaragoza, Spain.
Tel.: 34-976-761319, Fax: 34-976-761338
e-mail: msebasti@unizar.es

Odd Degree Polynomial Fractal Functions

Abstract

The approximation of experimental data can be envisaged in the light of fractal interpolation functions defined by iterated function systems. In the particular case of polynomial fractal interpolation functions, the method can be considered as a generalization of the splines of the same kind. That extension is verified under preservation of the smoothness of the function. A bound of the interpolation error by odd degree polynomial fractal interpolation functions is obtained here. The upper limit is also given for high-order derivatives, up to the $(2m-2)$th derivative if the polynomials are of degree $2m-1$. The result can be considered a fractal version of Schultz's theorem for odd degree polynomial splines.

Keywords: Fractal interpolation functions, iterated function systems, odd degree polynomial splines.

Mathematics Subject Classification: 37M10, 58C05

1. Introduction

The approximation and quantification of experimental data can be envisaged in the light of fractal interpolation functions defined by iterated function systems ([2]). In the article of M.F. Barnsley ([3]), the moments of an experimental signal are computed by explicit formulae involving the coefficients of the iterated function systems defining the function. A moment of any order can be used as an index of the signal, to perform comparisons and quantified measures.

Another important fact is that the graphs of these functions can possess a non-integer fractal dimension, and this parameter can be used as a measure of the complexity of a signal ([8]). We have also proved that the method of fractal interpolation is so general that contains any other approximation technique as a particular case ([9], [10]).

As explained in the paper of M.F. Barnsley & A.N. Harrington ([4]), the polynomial fractal interpolation functions can be integrated indefinitely and smooth functions generalizing splines can be obtained. The main difference with the classic procedures resides in the definition by a functional relation assuming a self-similarity on small scales. In this way, the interpolants are defined as fixed points of maps between spaces of functions. The properties of these correspondences allow to deduce some inequalities that express the sensitivity of the functions and their derivatives to those changes in the parameters defining them ([9], [10]).

In the particular case of polynomial fractal interpolation functions, the method can be considered as generalization of splines of the same kind. Some bounds of the interpolation error by odd degree polynomial fractal functions are obtained here. If the polynomials have degree $2m-1$, the bounds range from the function up to the $(2m-2)$th derivative. The degree of regularity required for the function being approximated is lightly superior to the chosen interpolant.

2. Differentiable Fractal Interpolation Functions

Let $t_0 < t_1 < ... < t_N$ be real numbers, and $I = [t_0, t_N] \subset \mathbb{R}$ the closed interval that contains them. Let a set of data points $\{(t_n, x_n) \in I \times \mathbb{R} : \ n = 0, 1, 2, ..., N\}$ be given. Set $I_n = [t_{n-1}, t_n]$ and let $L_n : I \to I_n, \ n \in \{1, 2, ..., N\}$ be contractive homeomorphisms such that:

$$L_n(t_0) = t_{n-1}, \ L_n(t_N) = t_n \tag{1}$$

$$|L_n(c_1) - L_n(c_2)| \leq l \ |c_1 - c_2| \quad \forall \ c_1, c_2 \ \in I \tag{2}$$

for some $0 \leq l < 1$.

Let $-1 < \alpha_n < 1$; $n = 1, 2, ..., N$, $F = I \times [c, d]$ for some $-\infty < c < d < +\infty$ and N continuous mappings, $F_n : F \to \mathbb{R}$ be given satisfying:

$$F_n(t_0, x_0) = x_{n-1}, \quad F_n(t_N, x_N) = x_n, \quad n = 1, 2, ..., N \tag{3}$$

$$|F_n(t, x) - F_n(t, y)| \leq \alpha_n |x - y|, \quad t \in I, \quad x, y \in \mathbb{R} \tag{4}$$

Now define functions $w_n(t, x) = (L_n(t), F_n(t, x)), \ \forall \ n = 1, 2, ..., N$.

Theorem (Barnsley [2]): The iterated function system (IFS)[6] $\{F, \ w_n : n = 1, 2, ..., N\}$ defined above admits a unique attractor G. G is the graph of a continuous function $f : I \to \mathbb{R}$ which obeys $f(t_n) = x_n$ for $n = 0, 1, 2, ..., N$.

The previous function is called a fractal interpolation function (FIF) corresponding to $\{(L_n(t), F_n(t, x))\}_{n=1}^{N}$. $f : I \to \mathbb{R}$ is the unique function satisfying the functional equation

$$f(L_n(t)) = F_n(t, f(t)), \quad n = 1, 2, ..., N, \quad t \in I$$

or,

$$f(t) = F_n(L_n^{-1}(t), f \circ L_n^{-1}(t)), \quad n = 1, 2, ..., N, \quad t \in I_n = [t_{n-1}, t_n] \tag{5}$$

Let $\mathcal{F}$ be the set of continuous functions $f : [t_0, t_N] \to [c, d]$ such that $f(t_0) = x_0$; $f(t_N) = x_N$. Define a metric on $\mathcal{F}$ by

$$\|f - g\|_\infty = \max \{|f(t) - g(t)| : \ t \in [t_0, t_N]\} \ \ \forall \ f, g \in \mathcal{F}$$

Then $(\mathcal{F}, \| \cdot \|_\infty)$ is a complete metric space.

Define a mapping $T : \mathcal{F} \to \mathcal{F}$ by:

$$(Tf)(t) = F_n(L_n^{-1}(t), f \circ L_n^{-1}(t)) \quad \forall \ t \in [t_{n-1}, t_n], \quad n = 1, 2, ..., N$$

Using (1)-(4), it can be proved that $(Tf)(t)$ is continuous on the interval $[t_{n-1}, t_n]$ for $n = 1, 2, ..., N$ and at each of the points $t_1, t_2, ..., t_{N-1}$. T is a contraction mapping on the metric space $(\mathcal{F}, \| \cdot \|_\infty)$,

$$\|Tf - Tg\|_\infty \leq |\boldsymbol{\alpha}|_\infty \|f - g\|_\infty \tag{6}$$

where $|\boldsymbol{\alpha}|_\infty = \max \{|\alpha_n|; \ n = 1, 2, ..., N\}$. Since $|\boldsymbol{\alpha}|_\infty < 1$, T possesses a unique fixed point on $\mathcal{F}$, that is to say, there is $f \in \mathcal{F}$ such that $(Tf)(t) = f(t) \ \ \forall \ t \in [t_0, t_N]$. This function is the FIF corresponding to w_n.

The most widely studied fractal interpolation functions so far are defined by the IFS

$$L_n(t) = a_n t + b_n \tag{7}$$

$$F_n(t, x) = \alpha_n x + q_n(t) \tag{8}$$

where $q_n(t)$ is an affine map [2,7]. α_n is called a vertical scaling factor of the transformation w_n and $a_n = (t_n - t_{n-1})/(t_N - t_0)$. If the data are evenly sampled:

$$a_n = \frac{1}{N} \tag{9}$$

We deal here with the case where q_n is a polynomial of odd-degree, that can be considered a generalization of polynomial spline functions.

The following theorem assures the existence of differentiable FIF.

Theorem (Barnsley and Harrington [4]): Let $t_0 < t_1 < t_2 < ... < t_N$ and $L_n(t)$, $n = 1, 2, ..., N$, the affine function $L_n(t) = a_n t + b_n$ satisfying (1)-(2). Let $a_n = L'_n(t) = \frac{t_n - t_{n-1}}{t_N - t_0}$ and $F_n(t, x) = \alpha_n x + q_n(t)$, $n = 1, 2, ..., N$ verifying (3)-(4). Suppose for some integer $p \geq 0$, $|\alpha_n| < a_n^p$ and $q_n \in C^p[t_0, t_N]$; $n = 1, 2, ..., N$. Let

$$F_{nk}(t, x) = \frac{\alpha_n x + q_n^{(k)}(t)}{a_n^k} \qquad k = 1, 2, ..., p \tag{10}$$

$$x_{0,k} = \frac{q_1^{(k)}(t_0)}{a_1^k - \alpha_1} \qquad x_{N,k} = \frac{q_N^{(k)}(t_N)}{a_N^k - \alpha_N} \qquad k = 1, 2, ..., p$$

If

$$F_{n-1,k}(t_N, x_{N,k}) = F_{nk}(t_0, x_{0,k}) \tag{11}$$

with $n = 2, 3, ..., N$ and $k = 1, 2, ..., p$, then $\{(L_n(t), F_n(t, x))\}_{n=1}^N$ determines a FIF $f \in C^p[t_0, t_N]$ and $f^{(k)}$ is the FIF determined by $\{(L_n(t), F_{nk}(t, x))\}_{n=1}^N$, for $k = 1, 2, ..., p$.

In the present paper, q_n are polynomials of degree $2m - 1$. Following the previous theorem, consider $p = 2m - 2$, $f \in C^{2m-2}$. The vertical scaling factor must satisfy $|\alpha_n| < a_n^{2m-2}$, $n = 1, 2, ..., N$.

If $\alpha_n = 0 \quad \forall\, n = 1, 2, ..., N$, $f(t) = f_0(t) = q_n \circ L_n^{-1}(t) \quad \forall\, t \in I_n$ (5), f_0 is a piecewise odd degree polynomial and $f_0 \in C^{2m-2}$, therefore is a polynomial spline ([1]). In this sense, we refer to this kind of functions as spline fractal interpolation functions (SFIF).

We consider here the case of constant scale factors $\alpha_n = \alpha$. If the condition (11) is imposed, the coefficients of q_n depend on α (see the reference [4] for the construction of q_n). In this sense we denote $q_n^\alpha(t) = q_n(\alpha, t)$. The polynomial spline is:

$$f_0 = q_n^0 \circ L_n^{-1} \quad \forall t \in I_n \tag{12}$$

3. Error bounds for the interpolation by odd degree polynomial fractal functions

In the first place, the error committed in the substitution of the function $x(t)$ by the SFIF $f_\alpha(t)$ with factor α will be bounded. A theorem concerning polynomial spline functions due to M.H. Schultz [11] is used.

By $\mathcal{H}^m(a,b)$ we mean the class of all functions $f(x)$ defined on $[a,b]$ which possess an absolutely continuous $(m-1)$th derivative on $[a,b]$ and whose mth derivative is in $L^2(a,b)$.

From here on $m > 1$, $m \in \mathbb{N}$, $N \geq 1$; $h = t_n - t_{n-1}$. The following constants will be used in the next theorem ([11]).

If $m-1 \leq z \leq 2m-2$ and $0 \leq j \leq m$:

$$K_{m,m,z,j} = \begin{cases} 1 & if \;\; m-1 \leq z \leq 2m-2, \;\; j = m \\ (\frac{1}{\pi})^{m-j} & if \;\; m-1 = z, \;\; 0 \leq j \leq m-1 \\ \frac{(z+2-m)!}{\pi^{m-j}} & if \;\; m-1 \leq z \leq 2m-2, \;\; 0 \leq j \leq 2m-2-z \\ \frac{(z+2-m)!}{j! \; \pi^{m-j}} & if \;\; m-1 \leq z \leq 2m-2, \; 2m-2-z \leq j \leq m-1 \end{cases}$$

If $m-1 \leq z \leq 2m-2$ and $0 \leq j \leq m$:

$$K_{m,2m,z,j} = K_{m,m,z,j} \; K_{m,m,z,0}$$

If $m < p < 2m$, $4m - 2p - 1 \le z \le 2m - 2$ and $0 \le j \le m$:

$$K_{m,p,z,j} = K_{p,p,2m-1,j} + K_{m,2m,z,j}\ 2^{\frac{1}{2}(2m-p)} \left(\frac{p!}{(2p-2m)!}\right)^2 \left(\frac{\|\Delta\|}{\underline{\Delta}}\right)^{2m-p}$$

with $\|\Delta\| = \underset{0 \le i \le N-1}{max}(t_{i+1} - t_i)$, $\quad \underline{\Delta} = \underset{0 \le i \le N-1}{min}(t_{i+1} - t_i)$.

If $m < p \le 2m$, $4m - 2p - 1 \le z \le 2m - 2$ and $m < j \le p$:

$$K_{m,p,z,j} = K_{p,p,p,j} + (K_{m,p,z,m} + K_{p,p,p,m})\ 2^{\frac{j-m}{2}} \left(\frac{(2p+m)!}{(2p-j)!}\right)^2 \left(\frac{\|\Delta\|}{\underline{\Delta}}\right)^{j-m}$$

If $m - 1 \le z \le 2m - 2$ and $0 \le j \le m - 1$:

$$K^{\infty}_{m,m,z,j} = \begin{cases} K_{m,m,z,j+1} & if \quad m-1 = z, \quad 0 \le j \le m-1 \\ K_{m,m,z,j+1} & if \quad m-1 < z \le 2m-2, \quad 0 \le j \le 2m-2-z \\ (j-2m+3+z)^{1/2}\ K_{m,m,z,j+1} & if \quad m-1 < z \le 2m-2, \\ & \qquad 2m-2-z < j \le m-1 \end{cases}$$

If $m - 1 \le z \le 2m - 2$ and $0 \le j \le m - 1$:

$$K^{\infty}_{m,2m,z,j+1} = \begin{cases} K_{m,2m,z,j+1} & if \quad m-1 = z, \quad 0 < j \le m-1 \\ K_{m,2m,z,j+1} & if \quad m-1 < z \le 2m-2, \quad 0 \le j \le 2m-2-z \\ (j-2m+3+z)^{1/2}\ K_{m,2m,z,j+1} & if \quad m-1 < z \le 2m-2 \\ & if \quad 2m-2-z < j \le m-1 \end{cases}$$

If $m < p < 2m$, $4m - 2p - 1 \le z \le 2m - 2$, $0 \le j \le m - 1$:

$$K^{\infty}_{m,p,z,j} = K^{\infty}_{p,p,2m-1,j} +$$

$$+K^{\infty}_{m,2m,z,j}\ 2^{\frac{2m-p}{2}}\left(\frac{p!}{(2p-2m)!}\right)^2\left(\frac{\|\Delta\|}{\underline{\Delta}}\right)^{2m-p} \qquad (13)$$

If $m < p \leq 2m,\ \ 4m-2p-1 \leq z \leq 2m-2,\ \ m \leq j \leq p-1$:

$$K^{\infty}_{m,p,z,j} = K^{\infty}_{p,p,p,j}+$$
$$+(K^{\infty}_{m,p,z,m-1}+K^{\infty}_{p,p,p,j})2^{j-m+1}\left(\frac{(2p-m)!}{(2p-j-1)!}\right)^2\left(\frac{\|\Delta\|}{\underline{\Delta}}\right)^{j-m+1}$$

We present here a simplified version of the Schultz's theorem.

Theorem ([11]). Let $x(t)$ be in $\mathcal{H}^{2m-1}(a,b)$ and let $\Delta : a = t_0 < t_1 < ... < t_N = b$ be a mesh of the interval. Let $S(x,t)$ be a spline of degree $(2m-1)$ to $x(t)$ on Δ satisfying $S^{(k)}(x,t_0) = x^{(k)}(t_0)$, $S^{(k)}(x,t_N) = x^{(k)}(t_N)$ for $0 \leq k \leq m-1$. Then we have

$$\|x^{(k)}(t) - S^{(k)}(x,t)\|_{\infty} \leq K^{\infty}_{m,2m-1,2m-2,k}\ \|x^{(2m-1)}\|_2\ \|\Delta\|^{2m-\frac{3}{2}-k} \qquad (14)$$

for $0 \leq k \leq 2m-2$, with

$$\|x^{(2m-1)}\|_2 = \left(\int_a^b (x^{(2m-1)}(t))^2 dt\right)^{1/2}$$

Collage Theorem ([5], [12]). Let (Y,d_Y) be a complete metric space and let T be a contraction map with contractivity factor $c \in [0,1)$. Then for any $y \in Y$,

$$d_Y(y,\overline{y}) \leq \frac{1}{(1-c)}d_Y(y,Ty) \qquad (15)$$

where $\overline{y}$ is the fixed point of T.

Theorem 1. *Interpolation error bound.* Let $x(t)$ be a function verifying $x(t) \in \mathcal{H}^{2m-1}(t_0,t_N)$ and Δ a partition of the interval such that $h = t_n - t_{n-1}$ is constant. Let $q_n(\alpha, t)$ be differentiable and such that $\exists\ D_0 \geq 0$ with $|\frac{\partial q_n}{\partial \alpha}(\xi, t)| \leq D_0\ \forall\ (\xi, t) \in J \times I$, $\forall\ n = 1, 2, ..., N$. Let $|\alpha| < \frac{1}{N^{2m-2}}$. Then

$$\|x - f_\alpha\|_\infty \leq \frac{N^{2m-2}}{N^{2m-2} - 1}\left[C_0 h^{2m-\frac{3}{2}} + \frac{(L_0 + D_0)}{T^{2m-2}} h^{2m-2}\right]$$

where $L_0 = \|x\|_\infty$, $T = t_N - t_0$ and $C_0 = K^\infty_{m,2m-1,2m-2,0}\ \|x^{(2m-1)}\|_2$.

Proof

In order to apply the Collage Theorem, consider Y=$\mathcal{F}$ where $\mathcal{F}$ is the set of continuous functions $f : [t_0, t_N] \to [c, d]$ such that $f(t_0) = x_0$; $f(t_N) = x_N$, $d_Y = \|.\|_\infty$ and T_α defined on $\mathcal{F}$ such that

$$(T_\alpha f)(t) = F_n(L_n^{-1}(t), f \circ L_n^{-1}(t)) = \alpha f \circ L_n^{-1}(t) + q_n^\alpha \circ L_n^{-1}(t)$$

$$\forall\ t \in [t_{n-1}, t_n], \quad n = 1, 2, ..., N$$

As previously explained, T_α is a contraction with factor $|\alpha|$. Let f_α be the FIF associated. According to the Collage Theorem, if $x(t)$ is the original function providing the data points then:

$$\|x - f_\alpha\|_\infty \leq \frac{1}{(1 - |\alpha|)}\|x - T_\alpha x\|_\infty \tag{16}$$

For $t \in I_n$:

$$|x(t) - T_\alpha x(t)| = |x(t) - \alpha x \circ L_n^{-1}(t) - q_n^\alpha \circ L_n^{-1}(t)|$$

Let f_0 be the spline of degree $2m-1$, $f_0(t) = q_n^0 \circ L_n^{-1}(t)$ for $t \in I_n = [t_{n-1}, t_n]$ (12). The former equality continues as:

$$|x(t) - f_0(t) - \alpha x \circ L_n^{-1}(t) - q_n^\alpha \circ L_n^{-1}(t) + q_n^0 \circ L_n^{-1}(t)| \leq \|x - f_0\|_\infty + |\alpha| \|x\|_\infty +$$

$$+ |q_n^\alpha \circ L_n^{-1}(t) - q_n^0 \circ L_n^{-1}(t)| \tag{17}$$

The last term can be bounded using the mean-value theorem and the given hypothesis:

$$|q_n^\alpha \circ L_n^{-1}(t) - q_n^0 \circ L_n^{-1}(t)| \leq |\alpha| |\frac{\partial}{\partial \alpha}(q_n^\alpha \circ L_n^{-1})(t)| \leq |\alpha| D_0 \tag{18}$$

Furthermore, by the theorem of Schultz with $k = 0$ and $h = \|\Delta\|$:

$$\|x - f_0\|_\infty \leq C_0 h^{2m - \frac{3}{2}} \tag{19}$$

From (16), (17), (18) and (19):

$$\|x - f_\alpha\|_\infty \leq \frac{1}{1 - |\alpha|} \left[C_0 h^{2m - \frac{3}{2}} + |\alpha|(L_0 + D_0) \right]$$

By the hypotheses of the theorem of differentiability of fractal interpolation functions [4]: $|\alpha| < a_n^{2m-2} = \frac{1}{N^{2m-2}} = \frac{h^{2m-2}}{T^{2m-2}}$ and, therefore, $\frac{1}{1-|\alpha|} \leq \frac{N^{2m-2}}{N^{2m-2}-1}$, and the former inequality is transformed in:

$$\|x - f_\alpha\|_\infty \leq \frac{N^{2m-2}}{N^{2m-2} - 1} \left[C_0 h^{2m - \frac{3}{2}} + \frac{(L_0 + D_0)}{T^{2m-2}} h^{2m-2} \right] \qquad (18) \ \diamond$$

According to the theorem of Barnsley & Harrington, the derivatives $f^{(k)}$ of f are FIF corresponding to the IFS $\{(L_n(t), F_{nk}(t, x))\}_{n=1}^N$ with

$$F_{nk}(t, x) = N^k \alpha x + N^k q_n^{(k)}(t)$$

Consequently, the results above can be generalized to the first derivatives of f. In order to preserve the order of convergence, we impose an additional condition to the vertical scale factor.

Theorem 2. *Derivatives interpolation error bounds.* Let $x(t)$ be a function verifying $x(t) \in \mathcal{H}^{2m-1}(t_0, t_N)$ and Δ a partition of the interval such that $h = t_n - t_{n-1}$ is constant. Let $\frac{\partial^k q_n}{\partial t^k}(\alpha, t)$ be differentiable and $\exists\ D_k \geq 0$ such that $|\frac{\partial^{k+1} q_n}{\partial \alpha \partial t^k}(\xi, t)| \leq D_k\ \forall\ (\xi, t) \in J \times I$ and $\forall\ n = 1, 2, ..., N$. Let $s > 0$ and α be such that $|\alpha| \leq \frac{1}{N^{2m-2+s}}$. Then:

$$\|x^{(k)} - f_\alpha^{(k)}\|_\infty \leq \frac{N^{2m-2+s-k}}{N^{2m-2+s-k} - 1}\left[C_k h^{2m-\frac{3}{2}-k} + \frac{(L_k + D_k)}{T^{2m-2+s-k}} h^{2m-2+s-k}\right]$$

for $k = 0, 1, ..., 2m-2$, being $L_k = \|x^{(k)}\|_\infty$, $h = t_n - t_{n-1}$, $T = t_N - t_0$ and $C_k = K^\infty_{m,2m-1,2m-2,k}\ \|x^{(2m-1)}\|_2$.

Proof

The proof is analogous to the theorem 1.

5. Conclusions

The bounds of error in the approximation by polinomial splines are generalized to differentiable polynomial Barnsley-Harrington functions. The error obtained is comparable to other precision procedures, such as the interpolation by piecewise polynomials. The property of good fit of the derivatives is also verified here. The possible loss of precision is counteracted by the generality of the method, as the fractal interpolants contain odd degree polynomial spline functions as a particular case. That extension is verified under preservation of the smoothness of the function.

6. Acknowledgements

This paper is part of a research project financed by the Consejo Superior de

Investigación y Desarrollo de la Diputación General de Aragón (P072/2001) and by the University of Zaragoza (Spain) (226-55).

7. References

[1] Ahlberg JH, Nilson EN and Walsh JL. The Theory of Splines and Their Applications. New York: Academic Press, 1967.

[2] Barnsley MF. Fractals Everywhere. Academic Press Inc., 1988.

[3] Barnsley MF. Fractal functions and interpolation. Constr. Approx. 1986; 2, 4: 303-329.

[4] Barnsley MF and Harrington AN. The calculus of fractal interpolation functions. J. Approx. Theory, 1989; 57: 14-34.

[5] Barnsley MF, Ervin V, Hardin D, Lancaster J. Solution of an inverse problem for fractals and other sets. Proc. Nat. Acad. Sci. USA, 1985; 83: 1975-1977.

[6] Edgar GA. Measure, Topology and Fractal Geometry. New York: Springer-Verlag, 1990.

[7] Hardin DP, Kessler B and Massopust PR. Multiresolution analyses based on fractal functions. J. Approx. Theory, 1992; 71: 104-120.

[8] Navascués MA and Sebastián MV. Fitting curves by fractal interpolation. In: Thinking in Patterns: Fractals and Related Phenomena in Nature, 143-154. M.M. Novak (ed.), World Scientific, 2004.

[9] Navascués MA and Sebastián MV. Generalization of Hermite functions by fractal interpolation. J. Approx. Theory, 2004; 131(1): 19-29.

[10] Navascués MA and Sebastián MV. Some results of convergence of cubic spline fractal interpolation functions. Fractals 2003; 11, 1: 1-8.

[11] Schultz MH. Error bounds for polynomial spline interpolation. Math.Comp. 1970; 24 (111): 507-515.

[12] Smart DR. Fixed Point Theorems. Cambridge Univ. Press, London, 1974.