GEOMETRICALLY CONVEX FUNCTIONS AND SOLUTION OF A QUESTION

XIAO-MING ZHANG AND YU-DONG WU

Abstract. From the properties of geometrically convex functions, this paper presents the solution of question 11031 in American Mathematical Monthly.

1. Definition of Geometrically Convex Functions

Throughout the paper we assume $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$, and R^n be the n-dimensional Euclidean Space, $R_+^n = \{(x_1, x_2, \cdots, x_n), x_i > 0, i = 1, 2, \cdots, n\}$, and $\alpha x = (\alpha x_1, \alpha x_2, \cdots, \alpha x_n), e^x = (e^{x_1}, e^{x_2}, \cdots, e^{x_n}), x^\alpha = (x_1^\alpha, x_2^\alpha, \cdots, x_n^\alpha), \ln x = (\ln x_1, \ln x_2, \cdots, \ln x_n), x \cdot y = (x_1 y_1, x_2 y_2, \cdots, x_n y_n)$, where $\alpha \in R$, and $x = (x_1, x_2, \cdots, x_n) \in R^n$, $y = (y_1, y_2, \cdots, y_n) \in R^n$.And let $G(x) = \sqrt[n]{x_1 x_2 \cdots x_n}$ with $x \in R_+^n$.

Paper ([1, 2])presents the definition of geometrically convex functions on R_+.

Definition 1.1. ([1, 2, 3]) *Let $f : I \subseteq (0, +\infty) \to (0, +\infty)$ be a continuous function, then f is called a geometrically convex function on I, if existing $n \geqslant 2$, such that one of the following three inequalities holds for any $x_1, x_2, \cdots, x_n \in I$ and $\lambda_1, \lambda_2, \cdots, \lambda_n > 0$ with $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$.*

$$f(\sqrt{x_1 x_2}) \leqslant \sqrt{f(x_1) f(x_2)}, \tag{1.1}$$

$$f\left(\sqrt[n]{\prod\nolimits_{i=1}^n x_i}\right) \leqslant \sqrt[n]{\prod\nolimits_{i=1}^n f(x_i)}, \tag{1.2}$$

$$f\left(\prod\nolimits_{i=1}^n x_i^{\lambda_i}\right) \leqslant \prod\nolimits_{i=1}^n f^{\lambda_i}(x_i), \tag{1.3}$$

and f is called a geometrically concave function on I if one of three inequalities (1.1)-(1.3) *is inverse.*

References ([3]) presents the definition of geometrically convex sets on R_+^n.

Definition 1.2. ([3]) *$H \subseteq R_+^n$ is called a geometrically convex set if $x^\alpha y^\beta \in H$ for any $x, y \in H$.*

Paper ([4])presents the definition of geometrically convex functions on R_+^n.References ([3]) extends the definition of geometrically convex functions on geometrically convex sets.

Definition 1.3. ([3, 4]) *Let $H \subseteq R_+^n$ is a geometrically convex set, $f : H \to (0.+\infty)$ is a continuous function, f is called a geometrically convex function if $f(x^\alpha y^\beta) \leqslant f^\alpha(x) f^\beta(y)$ for any $x, y \in H$. f is called a geometrically concave function if the above inequality is inverse.*

Definition 1.4. ([3, 4, 5])*Let $x = (x_1, x_2, \cdots, x_n) \in R_+^n, y = (y_1, y_2, \cdots, y_n) \in R_+^n, (x_{[1]}, x_{[2]}, \cdots, x_{[n]})$ and $(y_{[1]}, y_{[2]}, \cdots, y_{[n]})$ are the decreasing queue of $(x_1, x_2, \cdots, x_n)$ and $(y_1, y_2, \cdots, y_n)$ respectively. We say $(x_1, x_2, \cdots, x_n)$ logarithm majorizes $(y_1, y_2, \cdots, y_n)$, denotes $\ln x \succ \ln y$ if*

$$\begin{cases} \prod_{i=1}^k x_{[i]} \geq \prod_{i=1}^k y_{[i]}, k = 1, 2, \cdots, n-1, \\ x_1 x_2 \cdots x_n = y_1 y_2 \cdots y_n. \end{cases} \tag{1.4}$$

Date: November 30, 2004.
2000 *Mathematics Subject Classification.* Primary 26D15.
Key words and phrases. Inequalities, Geometrically convex functions,S-geometrically convex functions.
This paper was typeset using $\mathcal{A}\mathcal{M}\mathcal{S}$-LaTeX.

Lemma 1.1. ([3]) *Let* $x \in R_+^n$,*then* x *logarithm majorizes* $\bar{G} = (G(x), G(x), \cdots, G(x))$.

Definition 1.5. *Suppose* $E \subseteq R_+^n$, $f : E \to [0, +\infty)$. *Then* f *is called S-geometrically convex function, if the following inequality*

$$f(x) \geq f(y). \tag{1.5}$$

holds for any $x, y \in E \subseteq R_+^n$, *when* $\ln x \succ \ln y$. *And* f *is called S-geometrically concave function, if the inequality* (1.5) *is reversed.*

Lemma 1.2. ([3]) *Suppose* $E \subseteq R_+^n$ *is symmetric geometrically convex set,* $f : E \to [0, +\infty)$ *is symmetric continuously differentiable function. Then* f *is S-geometrically concave function, if the following inequality*

$$(\ln x_1 - \ln x_2)\left(x_1 \frac{\partial f}{\partial x_1} - x_2 \frac{\partial f}{\partial x_2}\right) \geq 0 \tag{1.6}$$

holds for any $x = (x_1, x_2, \cdots, x_n) \in E \subseteq R_+^n$. *And* f *is S-geometrically concave function, if the inequality* (1.6) *is reversed.*

2. Question And Lemmas

Question 11031 of the American Mathematical Monthly: Define the monster mean $M(a,b)$ of two positive real number to be $\ln N(a,b)$, where $N(a,b)$ is the fraction

$$\frac{1 + \ln\left(\sqrt{1 + f(a,b)} + \sqrt{f(a,b)}\right)}{1 - \ln\left(\sqrt{1 + f(a,b)} + \sqrt{f(a,b)}\right)},$$

and

$$f(a,b) = \frac{1}{4}\left(e^{2(e^a-1)\cdot(e^a+1)^{-1}} - 1\right)\left(e^{2\left(e^b-1\right)\cdot\left(e^b+1\right)^{-1}} - 1\right) \cdot e^{-\left((e^a-1)\cdot(e^a+1)^{-1} + \left(e^b-1\right)\cdot\left(e^b+1\right)^{-1}\right)}.$$

Prove or disprove: the monster mean $M(a,b)$ is always less than or equal to the geometric mean $\sqrt{ab}$.

To solve the question in the next section, the following lemmas are necessary.

Lemma 2.1. *Let* $0 < t < 1$, *then*

$$e^{2t} > 1 + 2t + 2t^2 + \frac{4}{3}t^3. \tag{2.1}$$

$$\ln\frac{1+t}{1-t} > 2t. \tag{2.2}$$

$$\left(2t^2 - 1\right)e^{4t} > -1 - 4t - 6t^2 - \frac{8}{3}t^3 + \frac{16}{3}t^4 + \frac{64}{5}t^5 + \frac{704}{45}t^6. \tag{2.3}$$

Proof. The proof of inequalities (2.1) and (2.2) is easy. Suppose $f(t) = \left(2t^2 - 1\right)e^{4t} + 1 + 4t + 6t^2 + \frac{8}{3}t^3 - \frac{16}{3}t^4 - \frac{64}{5}t^5 - \frac{704}{45}t^6$, $0 < t < 1$, then

$$f'(t) = \left(8t^2 + 4t - 4\right)e^{4t} + 4 + 12t + 8t^2 - \frac{64}{3}t^3 - 64t^4 - \frac{1408}{15}t^5,$$

$$f''(t) = \left(32t^2 + 32t - 12\right)e^{4t} + 12 + 16t - 64t^2 - 256t^3 - \frac{1408}{3}t^4,$$

$$\left(\frac{1}{4}f''(t)\right)' = \left(32t^2 + 48t - 4\right)e^{4t} + 4 - 32t - 192t^2 - \frac{1408}{3}t^3,$$

$$\left(\frac{1}{16}f'''(t)\right)' = \left(32t^2 + 64t + 8\right)e^{4t} - 8 - 96t^{-}352t^2,$$

$$\left(\frac{1}{128}f^{(4)}(t)\right)' = \left(16t^2+40t+12\right)e^{4t} - 12 - 88t,$$

$$\left(\frac{1}{512}f^{(5)}(t)\right)' = \left(16t^2+48t+22\right)e^{4t} - 22.$$

With $f^{(6)}(t) > 0$ and $\lim_{t\to 0+} f^{(i)}(t) = 0, i = 1, 2, \cdots, 5$, therefore

$$f(t) = \left(2t^2-1\right)e^{4t} + 1 + 4t + 6t^2 + \frac{8}{3}t^3 - \frac{16}{3}t^4 - \frac{64}{5}t^5 - \frac{704}{45}t^6 > 0,$$

then the inequality (2.3) holds. ∎

Lemma 2.2. *Let* $h(t) = \left(1+\frac{2}{e^{2t}-1}\right)\cdot\left(1-t^2\right)\cdot\left(\ln(1+t) - \ln(1-t)\right)$*,then* h *is a decreasing function on* $(0,1)$*.*

Proof.

$$h'(t) = -2\cdot\frac{1-e^{4t} + \left(2e^{2t} - t + te^{4t} - 2t^2e^{2t}\right)\left(\ln(1+t) - \ln(1-t)\right)}{\left(e^{2t}-1\right)^2},$$

$$-\frac{1}{2}\left(e^{2t}-1\right)^2 h'(t) = 1 - e^{4t} + \left(2t^{2t} - t + te^{4t} - 2t^2e^{2t}\right)\left(\ln(1+t) - \ln(1-t)\right).$$

Form Lemma 2.1,

$$-\frac{1}{2}\left(e^{2t}-1\right)^2 h'(t) > 1 - e^{4t} + \left(2\left(1-t^2\right)\left(1+2t+2t^2+\frac{4}{3}t^3\right) - t + te^{4t}\right)\cdot 2t$$

$$= \left(2t^2-1\right)e^{4t} + 1 + 4t + 6t^2 + 4t^3 - \frac{8}{3}t^4 - 8t^5 - \frac{16}{3}t^6$$

$$> \frac{4}{3}t^3 + \frac{8}{3}t^4 + \frac{24}{5}t^5 + \frac{464}{45}t^6 > 0.$$

So $h'(t) < 0$, h is a decreasing function on open interval $(0,1)$. ∎

3. Solution of the question

Proof. Let $T(t) = \frac{e^t-1}{e^t+1}, t \in R$, and $(a,b) \in R_+^2$, then

$$f(a,b) = \frac{1}{4}\left(e^{T(a)} - e^{-T(a)}\right)\left(e^{T(b)} - e^{-T(b)}\right),$$

$$f_1'(a,b) = \frac{1}{4}\left(e^{T(a)} + e^{-T(a)}\right)\left(e^{T(b)} - e^{-T(b)}\right)\frac{2e^a}{\left(e^a+1\right)^2},$$

$$f_2'(a,b) = \frac{1}{4}\left(e^{T(a)} - e^{-T(a)}\right)\left(e^{T(b)} + e^{-T(b)}\right)\frac{2e^b}{\left(e^b+1\right)^2}.$$

And

$$(\ln a - \ln b)\left(af_1' - bf_2'\right)$$

$$= \tfrac{1}{2}(\ln a - \ln b)\left(\left(e^{T(a)} + e^{-T(a)}\right)\left(e^{T(b)} - e^{-T(b)}\right)\tfrac{ae^a}{\left(e^a+1\right)^2} - \left(e^{T(a)} - e^{-T(a)}\right)\left(e^{T(b)} + e^{-T(b)}\right)\tfrac{be^b}{\left(e^b+1\right)^2}\right).$$

Let $a > b > 0$, $T(a) = u = \frac{e^a-1}{e^a+1}$, $T(b) = v = \frac{e^b-1}{e^b+1}$, then $u > v$, $0 < u, v < 1$, $a = \ln(1+u) - \ln(1-u)$, $b = \ln(1+v) - \ln(1-v)$, so

$$(\ln a - \ln b)\left(af_1' - bf_2'\right) \le 0$$

$$\Leftrightarrow \frac{e^{T(a)} + e^{-T(a)}}{e^{T(a)} - e^{-T(a)}}\cdot\frac{ae^a}{\left(e^a+1\right)^2} - \frac{e^{T(b)} + e^{-T(b)}}{e^{T(b)} - e^{-T(b)}}\cdot\frac{be^b}{\left(e^b+1\right)^2} \le 0$$

$$\Leftrightarrow \frac{e^{2T(a)}+1}{e^{2T(a)}-1}\cdot\frac{ae^a}{(e^a+1)^2}-\frac{e^{2T(b)}+1}{e^{2T(b)}-1}\cdot\frac{be^b}{(e^b+1)^2}\le 0,$$

$$\Leftrightarrow \frac{e^{2u}+1}{e^{2u}-1}\cdot\frac{\frac{1+u}{1-u}\cdot\ln\frac{1+u}{1-u}}{\left(\frac{1+u}{1-u}+1\right)^2}-\frac{e^{2v}+1}{e^{2v}-1}\cdot\frac{\frac{1+v}{1-v}\cdot\ln\frac{1+v}{1-v}}{\left(\frac{1+v}{1-v}+1\right)^2}\le 0,$$

$$\Leftrightarrow \left(1+\frac{2}{e^{2u}-1}\right)\cdot\left(1-u^2\right)\cdot\ln\frac{1+u}{1-u}-\left(1+\frac{2}{e^{2v}-1}\right)\cdot\left(1-v^2\right)\cdot\ln\frac{1+v}{1-v}\le 0.$$

By Lemma 2.2 and $u > v$, the previous inequality holds. So f is a S-geometrically concave function. Let $g(a,b) = \sqrt{1+f(a,b)}+\sqrt{f(a,b)}$, $(a,b)\in R_+^2$, from Definition 1.5, we know g is a S-geometrically concave function. Then from Definition 1.5 and Lemma 1.1, we immediately get

$$\sqrt{1+f(a,b)}+\sqrt{f(a,b)}\le\sqrt{1+f\left(\sqrt{ab},\sqrt{ab}\right)}+\sqrt{f\left(\sqrt{ab},\sqrt{ab}\right)}=\exp\left(\frac{e^{\sqrt{ab}}-1}{e^{\sqrt{ab}}+1}\right)$$

$$\Leftrightarrow N(a,b)\le e^{\sqrt{ab}}\Leftrightarrow M(a,b)\le\sqrt{ab}.$$

Thus, the solution of the question is completed. ∎

References

[1] J. Matkowski, *L^p-Like Paranorms*, Selected Topics in Functional Equations and Iteration Theory, Proceedings of the Austrian-Polish seminar, Graz Math. Ber. 316 (1992),103-138.

[2] S.-J. Li *About Definition and Qualities of Generalized Convex Function*. Secondary School Mathematics , 1999 (5).(Chinese)

[3] X.-M. Zhang, *Geometrically Convex Functions*. Hefei: An'hui University Press, 2004.6.(Chinese)

[4] D.-H. Yang, *About Inequality of Geometrically Convex Function* , Hebei University Learned Journal, (Natural Science Edition), 22 (2002), no.4:325-328. (Chinese)

[5] C. P. Niculescu, *Convexity According to the Geometric Mean*, Mathematical Inequalities & Applications, 3(2000), no.2:155-167.

(X.-M. Zhang) Zhejiang Haining TV University, Haining, Zhejiang, 314400, P. R. China.
E-mail address: zjzxm79@sohu.com

(Y.-D. Wu) Xinchang Middle School ,Xinchang, Zhejiang , 312500,P.R.China.
E-mail address: zjxcwyd@tom.com