L-Summing Method

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Abstract

We introduce a nice elementary method for summing, that we call it *L-Summing Method*. Applying this method on the elementary multiplication table we reprove a well-known identity. Also, if we let $\zeta_n(s)=\sum_{k=1}^n\frac{1}{k^s}$, applying L-Summing Method on another kind of multiplication table we yield

$$\sum_{k=1}^{n}\frac{\zeta_k(s)}{k^s}=\frac{\zeta_n^2(s)+\zeta_n(2s)}{2}, \qquad (s\in\mathbb{C}).$$

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Consider the following $n\times n$ Multiplication Table:

1	2	3	$\cdots$	n
2	4	6	$\cdots$	$2n$
3	6	9	$\cdots$	$3n$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
n	$2n$	$3n$	$\cdots$	n^2

If we let S the sum of all numbers in it, then by summing line by line, we have

$$S = \left(\frac{n(n+1)}{2}\right)^2.$$

In other hand we can find S by using other method; let L_k be the sum of boxed numbers in the following table (we call L_k, L-Summing Element)

1	2	$\cdots$	$\boxed{k}$	$\cdots$	n
2	4	$\cdots$	$\boxed{2k}$	$\cdots$	$2n$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
$\boxed{k}$	$\boxed{2k}$	$\cdots$	$\boxed{k^2}$	$\cdots$	kn
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
n	$2n$	$\cdots$	kn	$\cdots$	n^2

So, we have

$$L_k = k + 2k + \cdots + k^2 + \cdots + 2k + k = 2k(1 + 2 + \cdots + k) - k^2 = k^3.$$

Thus we yield $S = \sum_{k=1}^{n} L_k = \sum_{k=1}^{n} k^3$, and therefore

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

We call this method *L-Summing Method*, which briefly is

$$\sum (L - Summing\ Elements) = \sum .$$

Now, we apply the this method on the following table:

$\frac{1}{1^s}$	$\frac{1}{2^s}$	$\frac{1}{3^s}$	$\cdots$	$\frac{1}{n^s}$
$\frac{1}{2^s}$	$\frac{1}{4^s}$	$\frac{1}{6^s}$	$\cdots$	$\frac{1}{(2n)^s}$
$\frac{1}{3^s}$	$\frac{1}{6^s}$	$\frac{1}{9^s}$	$\cdots$	$\frac{1}{(3n)^s}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\frac{1}{n^s}$	$\frac{1}{(2n)^s}$	$\frac{1}{(3n)^s}$	$\cdots$	$\frac{1}{(n^2)^s}$

in which s is an arbitrary complex number. For $s \in \mathbb{C}$, let $\zeta_n(s) = \sum_{k=1}^{n} \frac{1}{k^s}$. L-Summing elements in above table are

$$L_k = \frac{2\zeta_k(s)}{k^s} - \frac{1}{k^{2s}}$$

and sum of all numbers, is equal to $\zeta_n^2(s)$. Thus, we have the following identity for all $s \in \mathbb{C}$

$$\sum_{k=1}^{n} \frac{\zeta_k(s)}{k^s} = \frac{\zeta_n^2(s) + \zeta_n(2s)}{2}. \tag{1}$$

If $\Re(s) > 1$, then $\lim_{n\to\infty} \zeta_n(s) = \zeta(s)$, and we yield the following identity for $\Re(s) > 1$

$$\sum_{k=1}^{\infty} \frac{\zeta_k(s)}{k^s} = \frac{\zeta^2(s) + \zeta(2s)}{2}.$$

It is known that $\sum_{k=1}^{\infty} \frac{H_k}{k^2} = 2\zeta(3)$ (see [1]). Now, if $s = 1$, then we have $\zeta_n(1) = H_n = \sum_{k=1}^{n} \frac{1}{k}$, and so, according to (1) and considering $H_n \sim \ln n$, when $n \to \infty$, we yield that

$$\sum_{k=1}^{n} \frac{H_k}{k} = \frac{H_n^2 + \zeta_n(2)}{2} \sim \frac{\ln^2 n}{2} \qquad (n \to \infty).$$

References

[1] Tom. M. Apostol and Ankur Basu, A new method for investigating Euler sums, *Ramanujan J.*, **4**(2000) no. 4, 397-419.