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# Sequence Inequalities for the Logarithmic Convex(Concave) Function

**Jian-She Sun**

**Abstract.** Let $f$ be a positive strictly increasing logarithmic convex (or logarithmic concave) function on $(0,1]$, then, for $k$ being a nonnegative integer and $n$ a natural number, the sequence $\frac{1}{n}\sum_{i=k+1}^{n+k} lnf(\frac{i}{n+k})$ is decreasing in $n$ and $k$ and has a lower bound $\int_0^1 f(t)dt$. From this, some new inequalities involving $\sqrt[n]{(n+k)!/k!}$ are deduced.

## 1. Introduction

In [1], H. Alzer, using mathematical induction and other techniques, proved that for $r > 0$ and $n \in N$,

$$\frac{n}{n+1} \le (\frac{1}{n}\sum_{i=1}^{n} i^r / \frac{1}{n+1}\sum_{i=1}^{n+1} i^r)^{1/r} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} \tag{1}$$

By Cauchy's mean-value theorem and mathematical induction, F. Qi in [7] presented that, if $n$ and $m$ are natural numbers, $k$ is a nonnegative integer, $r > 0$, then

$$\frac{n+k}{n+m+k} < (\frac{1}{n}\sum_{i=k+1}^{n+k} i^r / \frac{1}{n+m}\sum_{i=k+1}^{n+m+k} i^r)^{1/r} \tag{2}$$

The lower is best possible.

From Stirling's formula, for all nonnegative integers $k$ and natural numbers $n$ and $m$, F. Qi in [8] obtained

$$(\prod_{i=k+1}^{n+k} i)^{1/n} / (\prod_{i=k+1}^{n+m+k} i)^{1/(n+m)} \le \sqrt{\frac{n+m}{n+m+k}} \tag{3}$$

Let $f$ be a strictly increasing convex (or concave) function in $(0,1]$, J.-C.Kuang in [2] verified that

$$\frac{1}{n}\sum_{k=1}^{n} f(\frac{k}{n}) > \frac{1}{n+1}\sum_{k=1}^{n+1} f(\frac{k}{n+1}) > \int_0^1 f(x)dx \tag{4}$$

In [10], F.Qi, considering the convexity of a function proved the following: Let $f$ be a strictly increasing convex (or concave) function in $(0,1]$, then the sequence $\frac{1}{n}\sum_{i=k+1}^{n+k} f(\frac{i}{n+k})$ is decreasing in $n$ and $k$ and has a lower bound $\int_0^1 f(t)dt$. That is

$$\frac{1}{n}\sum_{i=k+1}^{n+k} f(\frac{i}{n+k}) > \frac{1}{n+1}\sum_{i=k+1}^{n+k+1} f(\frac{i}{n+k+1}) > \int_0^1 f(t)dt \tag{5}$$

Where $k$ is a nonnegative integer, $n$ a natural number.

There is much literature studying Alzer's and Minc-Sathre's inequality has many literature, for example, $[1]-[13]$.

In this article, motivated by $[2,7,10]$, i.e. the inequalities in (2), (3), (4) and (5), considering the logarithmic convexity of a function, we get

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**Theorem 1.** Let $f$ be a positive strictly increasing logarithmic convex (or logarithmic concave) function on $(0,1]$, then, for $k$ being a nonnegative integer and $n$ a natural number, the sequence $\frac{1}{n}\sum_{i=k+1}^{n+k} lnf(\frac{i}{n+k})$ is decreasing in $n$ and $k$ and has a lower bound $\int_0^1 f(t)dt$, that is

$$\frac{1}{n}\sum_{i=k+1}^{n+k} lnf(\frac{i}{n+k}) > \frac{1}{n+1}\sum_{i=k+1}^{n+k+1} lnf(\frac{i}{n+k+1}) > \int_0^1 f(t)dt \tag{6}$$

Where $k$ is a nonnegative integer, $n$ a natural number.

If let $f(x) = a^{x^r}$, $r > 0$, or let $k = 0$ in (6), then the inequalities in (1), (2) and (4) could be deduced. If we let $f(x) = e^{g(x)}$, $g(x)$ be a strictly increasing logarithmic convex (or logarithmic concave) function in $(0,1]$, the inequalities in (5) could be deduced. Therefore, inequality (6) generalizes Alzer's and Kuang's inequality in [1, 2] and inequality (2) above.

**Corollary 1.**([10]). For a nonnegative integer $k$ and a natural number $n > 1$, we have

$$\frac{n+k}{n+k+1} < [\frac{(2n+2k)!}{(n+2k)!}]^{1/n}/[\frac{(2n+2k+2)!}{(n+2k+1)!}]^{1/(n+1)}$$
$$< [\frac{(n+k)!}{k!}]^{1/n}/[\frac{(n+k+1)!}{k!}]^{1/(n+1)} \tag{7}$$

**Theorem 2.** For a natural number $n > 1$, then

$$[n^{(n+1)^2}/(n+1)^{n^2}]^{1/(2n+1)} < (\prod_{i=1}^{n} i^i)^{2/n(n+1)} < \frac{2n+1}{3} \tag{8}$$

## 2. Proofs of theorems

Proof of Theorem 1. Let us first assume that $f$ is a positive strictly increasing logarithmic convex function in $(0,1]$. Taking $x_1 = \frac{i-1}{n+k}$, $x_2 = \frac{i}{n+k}$, $\lambda = \frac{i-k-1}{n}$ and using the logarithmic convexity and monotonicity of $f$ yields

$$\frac{i-k-1}{n} lnf(\frac{i-1}{n+k}) + (1 - \frac{i-k-1}{n}) lnf(\frac{i}{n+k})$$
$$\geq lnf(\frac{i-k-1}{n}\cdot\frac{i-1}{n+k} + \frac{n-i+k+1}{n}\cdot\frac{i}{n+k})$$
$$= lnf(\frac{ni-i+k+1}{n(n+k)}) > lnf(\frac{i}{n+k+1}) \tag{9}$$

for $i = k+1, k+2, ..., n+k+1$. Summing up leads to

$$\sum_{i=k+1}^{n+k}[\frac{i-k-1}{n} lnf(\frac{i-1}{n+k}) + \frac{n-i+k+1}{n} lnf(\frac{i}{n+k})]$$
$$> \sum_{i=k+1}^{n+k} lnf(\frac{i}{n+k+1}) \tag{10}$$

$$\sum_{i=k+1}^{n+k}[(i-k-1)lnf(\frac{i-1}{n+k}) + (n-i+k+1)lnf(\frac{i}{n+k})]$$
$$> n\sum_{i=k+1}^{n+k} lnf(\frac{i}{n+k+1}) \tag{11}$$

$$(n+1)\sum_{i=k+1}^{n+k} lnf(\frac{i}{n+k}) - nlnf(1) > n\sum_{i=k+1}^{n+k} lnf(\frac{i}{n+k+1}) \tag{12}$$

$$(n+1)\sum_{i=k+1}^{n+k} lnf(\frac{i}{n+k}) > nlnf(1) + n\sum_{i=k+1}^{n+k} lnf(\frac{i}{n+k+1})$$

$$= n\sum_{i=k+1}^{n+k+1} lnf(\frac{i}{n+k+1}) \tag{13}$$

the left inequality in (6) is proved.

By a similar procedure, if $f$ is a strictly increasing logarithmic concave function in $(0,1]$, then for $i = k+1, k+2, ..., n+k+1$, we have

$$\frac{i-k}{n+1}lnf(\frac{i+1}{n+k+1}) + (1 - \frac{i-k}{n+1})lnf(\frac{i}{n+k+1})$$

$$\leq lnf(\frac{i-k}{n+1}\cdot\frac{i+1}{n+k+1} + \frac{n-i+k+1}{n+1}\cdot\frac{i}{n+k+1})$$

$$= lnf(\frac{ni+2i-k}{(n+1)(n+k+1)}) < lnf(\frac{i}{n+k}) \tag{14}$$

Summing up leads to

$$\sum_{i=k+1}^{n+k}[\frac{i-k}{n+1}lnf(\frac{i+1}{n+k+1}) + (1-\frac{i-k}{n+1})lnf(\frac{i}{n+k+1})]$$

$$= \frac{n}{n+1}\sum_{i=k+1}^{n+k} lnf(\frac{i}{n+k+1}) + \frac{n}{n+1}lnf(1)$$

$$< \sum_{i=k+1}^{n+k} lnf(\frac{i}{n+k}) \tag{15}$$

$$\frac{n}{n+1}\sum_{i=k+1}^{n+k+1} lnf(\frac{i}{n+k+1}) < \sum_{i=k+1}^{n+k} lnf(\frac{i}{n+k}) \tag{16}$$

The final line in (16) implies the left inequality in (6).

Finally, by definition of definite integral, the right inequality in (6) follows.

The proof is complete.

Proof of Corollary 1. Substituting $f$ with $(x+1)^r$, $r > 0$ or with $\frac{x}{x+1}$ in (6) and simplifying yields the first or the second inequality in (7), respectively.

Proof of Theorem 2. Substituting $f$ by $x^x$ and $k = 0$ in Theorem 1,we have

$$\frac{1}{n}\sum_{i=1}^{n}(\frac{i}{n})ln(\frac{i}{n}) > \frac{1}{n+1}\sum_{i=1}^{n+1}(\frac{i}{n+1})ln(\frac{i}{n+1}) \tag{17}$$

$$\frac{1}{n^2}\sum_{i=1}^{n}[i(lni - lnn)] > \frac{1}{(n+1)^2}\sum_{i=1}^{n+1}[i(lni - ln(n+1))] \tag{18}$$

$$[\frac{1}{n^2} - \frac{1}{(n+1)^2}]\sum_{i=1}^{n}(ilni) > [\frac{lnn}{n^2} - \frac{ln(n+1)}{(n+1)^2}]\sum_{i=1}^{n} i$$

$$= [\frac{lnn}{n^2} - \frac{ln(n+1)}{(n+1)^2}]\frac{n(n+1)}{2} \tag{19}$$

$$(2n+1)ln(\prod_{i=1}^{n} i^i) > \frac{n(n+1)}{2}ln[n^{(n+1)^2}/(n+1)^{n^2}] \tag{20}$$

In $[3, p.89]$,the following inequalities were given for $n > 1$, $n \in N$.

$$(\frac{n+1}{2})^{a_n} < \prod_{i=1}^{n} i^i < (\frac{2n+1}{3})^{a_n}, \qquad a_n = \frac{n(n+1)}{2} \tag{21}$$

Taking the logarithm yields

$$a_n ln(\frac{n+1}{2}) < ln(\prod_{i=1}^{n} i^i) < a_n ln(\frac{2n+1}{3}) \tag{22}$$

By substituting the inequalities in (22) into the left term of inequality (20), we obtain

$$(2n+1)\frac{n(n+1)}{2}ln(\frac{2n+1}{3}) > (2n+1)ln(\prod_{i=1}^{n} i^i)$$

$$> \frac{n(n+1)}{2}ln[n^{(n+1)^2}/(n+1)^{n^2}] \tag{23}$$

$$[n^{(n+1)^2}/(n+1)^{n^2}]^{1/(2n+1)} < (\prod_{i=1}^{n} i^i)^{2/n(n+1)} < \frac{2n+1}{3} \tag{24}$$

The proof is complete.

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(J.-Sh. Sun) Departments of Mathematics (Northern of school), Jiaozuo Teacher's College, Jiaozuo City, Henan 454150, China.

E-mail address: sunjianshe@126.com