Equations and Inequalities Involving $v_p(n!)$

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Abstract

In this paper we study $v_p(n!)$, the greatest power of prime p in factorization of $n!$. We find some lower and upper bounds for $v_p(n!)$, and we show that $v_p(n!) = \frac{n}{p-1} + O(\ln n)$. By using above mentioned bounds, we study the equation $v_p(n!) = v$ for a fixed positive integer v. Also, we study the triangle inequality about $v_p(n!)$, and show that the inequality $p^{v_p(n!)} > q^{v_q(n!)}$ holds for primes $p < q$ and sufficiently large values of n.

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1 Introduction

As we know, for every $n \in \mathbb{N}$, $n! = 1 \times 2 \times 3 \times \cdots \times n$. Let $v_p(n!)$ be the highest power of prime p in factorization of $n!$ to prime numbers. It is well-known that (see [3] or [5])

$$v_p(n!) = \sum_{k=1}^{\infty} \left[\frac{n}{p^k}\right] = \sum_{k=1}^{\left[\frac{\ln n}{\ln p}\right]} \left[\frac{n}{p^k}\right], \tag{1}$$

in which $[x]$ is the largest integer less than or equal to x. An elementary problem about $n!$ is finding the number of zeros at the end of it, in which clearly its answer is $v_5(n!)$. The inverse of this problem is very nice; for example finding values of n in which $n!$ terminates in 37 zeros [3], and generally finding values of n such that $v_p(n!) = v$. We show that if $v_p(n!) = v$ has a solution then it has exactly p solutions. For doing these, we need some properties of $[x]$, such as

$$[x] + [y] \leq [x + y] \qquad (x, y \in \mathbb{R}), \tag{2}$$

and

$$\left[\frac{x}{n}\right] = \left[\frac{[x]}{n}\right] \qquad (x \in \mathbb{R}, n \in \mathbb{N}). \tag{3}$$

2 Estimating $v_p(n!)$

Theorem 1 *For every $n \in \mathbb{N}$ and prime p, such that $p \leq n$, we have:*

$$\frac{n-p}{p-1} - \frac{\ln n}{\ln p} < v_p(n!) \leq \frac{n-1}{p-1}. \tag{4}$$

Proof: According to the relation (1), we have $v_p(n!) = \sum_{k=1}^{m} [\frac{n}{p^k}]$ in which $m = [\frac{\ln n}{\ln p}]$, and since $x - 1 < [x] \leq x$, we obtain

$$n \sum_{k=1}^{m} \frac{1}{p^k} - m < v_p(n!) \leq n \sum_{k=1}^{m} \frac{1}{p^k},$$

considering $\sum_{k=1}^{m} \frac{1}{p^k} = \frac{1-\frac{1}{p^m}}{p-1}$, we yield that

$$\frac{n}{p-1}(1 - \frac{1}{p^m}) - m < v_p(n!) \leq \frac{n}{p-1}(1 - \frac{1}{p^m}),$$

and combining this inequality with $\frac{\ln n}{\ln p} - 1 < m \leq \frac{\ln n}{\ln p}$ completes the proof. □

Corollary 1 *For every $n \in \mathbb{N}$ and prime p, such that $p \leq n$, we have:*

$$v_p(n!) = \frac{n}{p-1} + O(\ln n).$$

Proof: By using (4), we have

$$0 < \frac{\frac{n}{p-1} - v_p(n!)}{\ln n} < \frac{1}{\ln p},$$

and this yields the result. □

Note that the above corollary asserts that $n!$ ends approximately in $\frac{n}{4}$ zeros [1].

Corollary 2 *For every $n \in \mathbb{N}$ and prime p, such that $p \leq n$, and for all $a \in (0, +\infty)$ we have:*

$$\frac{n-p}{p-1} - \frac{1}{\ln p}\left(\frac{n}{a} + \ln a - 1\right) < v_p(n!). \tag{5}$$

Proof: Consider the function $f(x) = \ln x$. Since, $f''(x) = -\frac{1}{x^2}$, $\ln x$ is a concave function and so, for every $a \in (0, +\infty)$ we have

$$\ln x \leq \ln a + \frac{1}{a}(x - a),$$

combining this with the left hand side of (4) completes the proof. □

3 Study of the Equation $v_p(n!) = v$

Suppose $v \in \mathbb{N}$ is given. We are interested to find the values of n such that in factorization of $n!$, the highest power of p, is equal to v. First, we find some lower and upper bounds for these n's.

Lemma 1 *Suppose $v \in \mathbb{N}$ and p is a prime and $v_p(n!) = v$, then we have*

$$1 + (p-1)v \leq n < \frac{v + \frac{p}{p-1} + \frac{\ln(1+(p-1)v)}{\ln p} - \frac{1}{\ln p}}{\frac{1}{p-1} - \frac{1}{(1+(p-1)v)\ln p}}. \tag{6}$$

Proof: For Proving the left hand side of (6), use right hand side of (4) with assumption $v_p(n!) = v$, and for proving the right hand side of (6), use (5) with $a = 1+(p-1)v$. □

The lemma 1 suggest an interval for the solution of $v_p(n!) = v$. In the next lemma we show that it is sufficient one check only multiples of p in above interval.

Lemma 2 *Suppose $m \in \mathbb{N}$ and p is a prime, then we have*

$$v_p((pm+p)!) - v_p((pm)!) \geq 1. \tag{7}$$

Proof: By using (1) and (2) we have

$$v_p((pm+p)!) = \sum_{k=1}^{\infty} \left[\frac{pm+p}{p^k}\right] \geq \sum_{k=1}^{\infty} \left[\frac{pm}{p^k}\right] + \sum_{k=1}^{\infty} \left[\frac{p}{p^k}\right] = 1 + v_p((pm)!),$$

and this completes the proof. □

In the next lemma, we show that if $v_p(n!) = v$ has a solution, then it has exactly p solutions. In fact, the next lemma asserts that if $v_p((mp)!) = v$ holds, then for all $0 \leq r \leq p-1$, $v_p((mp+r)!) = v$ also holds.

Lemma 3 *Suppose $m \in \mathbb{N}$ and p is a prime, then we have*

$$v_p((m+1)!) \geq v_p(m!), \tag{8}$$

and

$$v_p((pm+p-1)!) = v_p((pm)!). \tag{9}$$

Proof: For proving (8), use (1) and (2) as follows

$$v_p((m+1)!) = \sum_{k=1}^{\infty} \left[\frac{m+1}{p^k}\right] \geq \sum_{k=1}^{\infty} \left[\frac{m}{p^k}\right] + \sum_{k=1}^{\infty} \left[\frac{1}{p^k}\right] = \sum_{k=1}^{\infty} \left[\frac{m}{p^k}\right] = v_p(m!).$$

For proving (9), it is enough to show that for all $k \in \mathbb{N}$, $[\frac{pm+p-1}{p^k}] = [\frac{pm}{p^k}]$ and we do this by induction on k; for $k=1$, clearly $[\frac{pm+p-1}{p}] = [\frac{pm}{p}]$. Now, by using (3) we have

$$\left[\frac{pm+p-1}{p^{k+1}}\right] = \left[\frac{\frac{pm+p-1}{p^k}}{p}\right] = \left[\frac{\left[\frac{pm+p-1}{p^k}\right]}{p}\right] = \left[\frac{\left[\frac{pm}{p^k}\right]}{p}\right] = \left[\frac{\frac{pm}{p^k}}{p}\right] = \left[\frac{pm}{p^{k+1}}\right].$$

This completes the proof. □

So, we have proved that

Theorem 2 *Suppose $v \in \mathbb{N}$ and p is a prime. For solving the equation $v_p(n!) = v$, it is sufficient to check the values $n = mp$, in which $m \in \mathbb{N}$ and*

$$\left[\frac{1+(p-1)v}{p}\right] \leq m \leq \left[\frac{v + \frac{p}{p-1} + \frac{\ln(1+(p-1)v)}{\ln p} - \frac{1}{\ln p}}{\frac{p}{p-1} - \frac{p}{(1+(p-1)v)\ln p}}\right]. \tag{10}$$

Also, if $n = mp$ is a solution of $v_p(n!) = v$, then it has exactly p solutions $n = mp + r$, in which $0 \leq r \leq p-1$.

Note and Problem 1 *As we see, there is no guarantee for existing a solution for $v_p(n!) = v$. In fact we need to show that $\{v_p(n!) | n \in \mathbb{N}\} = \mathbb{N}$; however, computational observations suggest that $n = p||\frac{1+(p-1)v}{p}||$ usually is a solution, such that $||x||$ is the nearest integer to x, but we can't prove it.*

Note and Problem 2 *Other problems can lead us to other equations involving $v_p(n!)$; for example, suppose $n, v \in \mathbb{N}$ given, find the value of prime p such that $v_p(n!) = v$. Or, suppose p and q are primes and $f : \mathbb{N}^2 \to \mathbb{N}$ is a prime value function, for which n's we have $v_p(n!) + v_q(n!) = v_{f(p,q)}(n!)$? And many other problems!*

4 Triangle Inequality Concerning $v_p(n!)$

In this section we are going to compare $v_p((m+n)!)$ and $v_p(m!)+v_p(n!)$.

Theorem 3 *For every $m,n\in\mathbb{N}$ and prime p, such that $p\le\min\{m,n\}$, we have*

$$v_p((m+n)!)\ge v_p(m!)+v_p(n!), \tag{11}$$

and

$$v_p((m+n)!)-v_p(m!)-v_p(n!)=O(\ln(mn)). \tag{12}$$

Proof: By using (1) and (2), we have

$$v_p((m+n)!)=\sum_{k=1}^{\infty}\left[\frac{m+n}{p^k}\right]\ge\sum_{k=1}^{\infty}\left[\frac{m}{p^k}\right]+\sum_{k=1}^{\infty}\left[\frac{n}{p^k}\right]=v_p(m!)+v_p(n!).$$

Also, by using (4) and (11) we obtain

$$0\le v_p((m+n)!)-v_p(m!)-v_p(n!)<\frac{2p-1}{p-1}+\frac{\ln(mn)}{\ln p}\le 3+\frac{\ln(mn)}{\ln 2},$$

this completes the proof. □

More generally, if $n_1,n_2,\cdots,n_t\in\mathbb{N}$ and p is a prime, in which $p\le\min\{n_1,n_2,\cdots,n_t\}$, by using an extension of (2), we obtain

$$v_p((\sum_{k=1}^{t}n_k)!)\ge\sum_{k=1}^{t}v_p(n_k!),$$

and by using this inequality and (4), we yield that

$$0\le v_p((\sum_{k=1}^{t}n_k)!)-\sum_{k=1}^{t}v_p(n_k!)<\frac{kp-1}{p-1}+\frac{\ln(n_1n_2\cdots n_t)}{\ln p}\le 2k-1+\frac{\ln(n_1n_2\cdots n_t)}{\ln p},$$

and consequently we have

$$v_p((\sum_{k=1}^{t}n_k)!)-\sum_{k=1}^{t}v_p(n_k!)=O(\ln(n_1n_2\cdots n_t)).$$

Note and Problem 3 *Suppose $f:\mathbb{N}^t\to\mathbb{N}$ is a function and p is a prime. For which $n_1,n_2,\cdots,n_t\in\mathbb{N}$, we have*

$$v_p((f(n_1,n_2,\cdots,n_t)!)\ge f(v_p(n_1!),v_p(n_2!),\cdots,v_p(n_t!))?$$

Also, we can consider the above question in other view points.

5 The Inequality $p^{v_p(n!)} > q^{v_q(n!)}$

Suppose p and q are primes and $p < q$. Since $v_p(n!) \geq v_q(n!)$, comparing $p^{v_p(n!)}$ and $q^{v_q(n!)}$ become a nice problem. In [2], by using elementary properties about $[x]$, it is considered the inequality $p^{v_p(n!)} > q^{v_q(n!)}$ in some special cases, beside it is shown that $2^{v_2(n!)} > 3^{v_3(n!)}$ holds for all $n \geq 4$. In this section we study $p^{v_p(n!)} > q^{v_q(n!)}$ in more general case and also reprove $2^{v_2(n!)} > 3^{v_3(n!)}$.

Lemma 4 *Suppose p and q are primes and $p < q$, then*

$$p^{q-1} > q^{p-1}.$$

Proof: Consider the function

$$f(x) = x^{\frac{1}{x-1}} \qquad (x \geq 2).$$

A simple calculation yields that for $x \geq 2$ we have

$$f'(x) = -\frac{x^{\frac{x-2}{x-1}}(x \ln x - x + 1)}{(x-1)^2} < 0,$$

so, f is strictly decreasing and $f(p) > f(q)$. This completes the proof. □

Theorem 4 *Suppose p and q are primes and $p < q$, then for sufficiently large n's we have*

$$p^{v_p(n!)} > q^{v_q(n!)}. \tag{13}$$

Proof: Since $p < q$, the lemma 4 yields that $\frac{p^{q-1}}{q^{p-1}} > 1$ and so, there exits $N \in \mathbb{N}$ such that for $n > N$ we have

$$\left(\frac{p^{q-1}}{q^{p-1}}\right)^n \geq \frac{p^{p(q-1)}}{q^{p-1}} n^{(p-1)(q-1)}.$$

Thus,

$$\frac{p^{n(q-1)}}{n^{(p-1)(q-1)} p^{p(q-1)}} \geq \frac{q^{n(p-1)}}{q^{p-1}},$$

and therefor,

$$\frac{p^{\frac{n}{p-1}}}{np^{\frac{p}{p-1}}} \geq \frac{q^{\frac{n}{q-1}}}{q^{\frac{1}{q-1}}}.$$

So, we obtain

$$p^{\frac{n-p}{p-1} - \frac{\ln n}{\ln p}} \geq q^{\frac{n-1}{q-1}},$$

and considering this inequality with (4), completes the proof. □

Corollary 3 *For $n = 2$ and $n \geq 4$ we have*

$$2^{v_2(n!)} > 3^{v_3(n!)}. \tag{14}$$

Proof: It is easy to see that for $n \geq 30$ we have

$$(\frac{4}{3})^n \geq \frac{16}{3}n^2,$$

and by theorem 4, we yield (14) for $n \geq 30$. For $n = 2$ and $4 < n < 30$ check it by a computer. □

A Computational Note. In the theorem 4, the relation (13) holds for $n > N$ (see its proof). We can check (13) for $n \leq N$ at most by checking the following number of cases:

$$R(N) := \#\left\{(p,q,n)|\ p,q \in \mathbb{P}, n = 3,4,\cdots,N, \text{ and } p < q \leq N\right\},$$

in which $\mathbb{P}$ is the set of all primes. If, $\pi(x) =$ The number of primes$\leq x$, then we have

$$R(N) = \sum_{n=3}^{N} \#\left\{(p,q)|\ p,q \in \mathbb{P}, \text{ and } p < q \leq n\right\} = \frac{1}{2}\sum_{n=3}^{N} \pi(n)(\pi(n)-1).$$

But, clearly $\pi(n) < n$ and this yields that

$$R(N) < \frac{N^3}{6}.$$

Of course, we have other bounds for $\pi(n)$ sharper that n such as [4]

$$\pi(n) < \frac{n}{\ln n}\left(1 + \frac{1}{\ln n} + \frac{2.25}{\ln^2 n}\right) \qquad (n \geq 355991),$$

and by using this bound we can find sharper bounds for $R(N)$.

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