A NEW PROOF OF AN INEQUALITY INVOLVING THE GENERALIZED ELEMENTARY SYMMETRIC MEAN TO THE POWER MEAN

ZHEN-GANG XIAO, RONG TANG, AND ZHI-HUA ZHANG

Abstract. In this short note, a conjecture ([4]: J. K. Merikoski, *Extending means of two variables to several variables*, J. Ineq. Pure & Appl. Math., 5(2) (2004), Article 65) of an inequality involving the generalized elementary symmetric mean to the power mean is proved, and its generalization is given.

1. Introduction

Let $a = (a_1, a_2, \cdots, a_n)$ and r be a nonnegative integer, where a_i for $1 \le i \le n$ are nonnegative real numbers. Then

$$E_n^{[r]} = E_n^{[r]}(a) = \sum_{\substack{i_1+i_2+\cdots+i_n=r,\\ i_1,i_2,\cdots,i_n\ge 0 \text{ are integers}}} \prod_{k=1}^{n} a_k^{i_k} \tag{1.1}$$

with $E_n^{[0]} = E_n^{[0]}(a) = 1$ for $n \ge 1$ and $E_n^{[r]} = 0$ for $r < 0$ or $n \le 0$ is called the rth generalized elementary symmetric function of a.

The rth generalized elementary symmetric mean of a is defined by ([1, 2])

$$\sum_n^{[r]} = \sum_n^{[r]}(a) = \frac{E_n^{[r]}(a)}{\binom{n+r-1}{r}}. \tag{1.2}$$

If r be a real number, then the r-order power mean as follows [3]

$$M_r = M_r(a) = \begin{cases} \left(\dfrac{1}{n}\displaystyle\sum_{i=1}^{n} a_i^r\right)^{\frac{1}{r}}, & r \ne 0; \\ \left(\displaystyle\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}}, & r = 0. \end{cases} \tag{1.3}$$

In [5] and [4], S. Mustonen and J. K. Merikoski both posed the following Conjecture 1.1 that the inequality relating the generalized elementary symmetric mean to the power mean is true:

Conjecture 1.1. *If r be a nonnegative integer, and a_i for $1 \le i \le n$ are nonnegative real numbers, then*

$$\left[\sum_n^{[r]}(a)\right]^{\frac{1}{r}} \leqslant M_r(a). \tag{1.4}$$

In 1988, by using B-splines, E. Neuman obtained a solation of Conjecture 1.1 in [6].

In this paper, we shall prove inequality (1.4) again, and give its generalization.

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2. Proof of Conjecture 1.1

To prove Conjecture 1.1, the following lemma are necessary.

Lemma 2.1. (I. Schur [3, p. 182]) *If $r \in \mathbb{N}$, then*

$$\sum_{n}^{[r]}(a) = (n-1)! \int \cdots \int \left(\sum_{i=1}^{n} a_i x_i \right)^r dx_1 \cdots dx_{n-1}, \tag{2.1}$$

where $x_n = 1 - (x_1 + x_2 + \cdots + x_{n-1})$ and the integral is taken over $x_k \geq 0$ for $k = 1, 2, \cdots n-1$.

Let $r = 1$, and alter $a_i \to a_i^r$, $i = 1.2.\cdots, n$, Lemma 2.1 leads to

Corollary 2.1. *If $r \in \mathbb{N}$, then*

$$\Big[M_r(a) \Big]^r = (n-1)! \int \cdots \int \sum_{i=1}^{n} a_i^r x_i dx_1 \cdots dx_{n-1}, \tag{2.2}$$

where $x_n = 1 - (x_1 + x_2 + \cdots + x_{n-1})$ and the integral is taken over $x_k \geq 0$ for $k = 1, 2, \cdots n-1$.

Proof of Conjecture 1.1. From the well-known weighted power mean inequality, $x_n = 1-(x_1+x_2+\cdots + x_{n-1})$, and $r > 1$, we have

$$\left(\sum_{i=1}^{n} a_i x_i \right)^r \leqslant \sum_{i=1}^{n} a_i^r x_i. \tag{2.3}$$

Combination of Lemma 2.1, Corollary 2.1 and (2.3) easily find Conjecture 1.1. The proof is completed. ∎

3. Generalization of Conjecture 1.1

In this section, we assume

$$V(a;r) = \begin{vmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-2} & a_1^{n-1+r} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-2} & a_2^{n-1+r} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-2} & a_n^{n-1+r} \end{vmatrix}. \tag{3.1}$$

If $r = 0$, then $V(a;r) = V(a)$ is the Vandermonde determinant. Let $Vi(a)$ denote $V(a)$ subdeterminant obtained by omitting its last row and ith column, we have

$$V(a) = V(a;0) = \sum_{i=1}^{n} (-1)^{n+i} a_i^{n-1} V_i(a) = \prod_{1 \leq i < j \leq n} (a_j - a_i). \tag{3.2}$$

Definition 3.1. *Let r be a real number, and all the $a_i{}'s$ are unequal. Then*

$$S_r(a) = \begin{cases} \left[\dfrac{(n-1)!}{\prod_{k=1}^{n}(k+r)} \cdot \dfrac{V(a;r)}{V(a)} \right]^{1/r}, & r \neq 0, -1, -2, \cdots, -(n-1), \\ \exp\left[\dfrac{\sum_{i=1}^{n} (-1)^{n+i} a_i^{n-1} V_i(a) \ln a_i}{V(a)} - \displaystyle\sum_{k=1}^{n-1} \frac{1}{k} \right], & r = 0, \\ \left[\dfrac{(n-1)! \sum_{i=1}^{n} (-1)^{n+i} a_i^{n-1+r} V_i(a) \ln a_i}{(-1)^{r+1}(-r-1)!(n+r)! \cdot V(a)} \right]^{1/r}, & r = -1, \cdots, -(n-1), \end{cases} \tag{3.3}$$

is called the rth generalized Stolaesky's mean of a.

In 2000, we obtained a formulas relating $S_r(a)$ in [7]

Theorem 3.1. *Let $S_r(a)$ be the rth generalized Stolaesky's mean of a, and all the $a_i{}'s$ are unequal, then we have*

$$S_r(a) = \begin{cases} \left[(n-1)! \int \cdots \int \left(\sum_{i=1}^n a_i x_i \right)^r dx_1 \cdots dx_{n-1} \right]^{1/r}, & r \neq 0, \\ \exp\left[(n-1)! \int \cdots \int \ln(\sum_{i=1}^n a_i x_i) dx_1 \cdots dx_{n-1} \right], & r = 0. \end{cases} \tag{3.4}$$

By using same method of Section 2, we can easily lead to the following generalization of Conjecture 1.1.

Theorem 3.2. *Let r be a real number. If $r > 1$, then we have*

$$S_r(a) \leqslant M_r(a), \tag{3.5}$$

and inverse inequality of (3.5) *holds if $r < 1$, with equality in* (3.5) *holding if and only if $a_1 = a_2 = \cdots = a_n$.*

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(Zh.-G. Xiao) Hunan Institute of Science and Technology, Yueyang, Hunan 414006, China
E-mail address: xiaozg@163.com

(R. Tang) Xiangnan University, Chenzhou, Hunan 423000, China

(Zh.-H. Zhang) Zixing Educational Research Section, Chenzhou, Hunan 423400, China
E-mail address: zxzh1234@163.com