Reverses of the Triangle Inequality in Banach Spaces

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Abstract. Recent reverses for the discrete generalised triangle inequality and its continuous version for vector-valued integrals in Banach spaces are surveyed. New results are also obtained. Particular instances of interest in Hilbert spaces and for complex numbers and functions are pointed out as well.

1. Introduction

The *generalised triangle inequality*, namely

$$\left\| \sum_{i=1}^{n} x_i \right\| \leq \sum_{i=1}^{n} \|x_i\|,$$

provided $(X, \|.\|)$ is a normed linear space over the real or complex filed $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ and $x_i, i \in \{1, ..., n\}$ are vectors in X plays a fundamental role in establishing various analytic and geometric properties of such spaces.

With no less importance, the *continuous* version of it, i.e.,

$$\left\| \int_a^b f(t)\, dt \right\| \leq \int_a^b \|f(t)\|\, dt, \tag{1.1}$$

where $f : [a, b] \subset \mathbb{R} \to X$ is a Bochner measurable function on the compact interval $[a, b]$ with values in the Banach space X and $\|f(.)\|$ is Lebesgue integrable on $[a, b]$, is crucial in the Analysis of vector-valued functions with countless applications in Functional Analysis, Operator Theory, Differential Equations, Semigroups Theory and related fields.

Surprisingly enough, the reverses of these, i.e., inequalities of the following type

$$\sum_{i=1}^{n} \|x_i\| \leq C \left\| \sum_{i=1}^{n} x_i \right\|, \int_a^b \|f(t)\|\, dt \leq C \left\| \int_a^b f(t)\, dt \right\|,$$

with $C \geq 1$, which we call *multiplicative reverses*, or

$$\sum_{i=1}^{n} \|x_i\| \leq \left\| \sum_{i=1}^{n} x_i \right\| + M, \int_a^b \|f(t)\|\, dt \leq \left\| \int_a^b f(t)\, dt \right\| + M,$$

with $M \geq 0$, which we call *additive reverses*, under suitable assumptions for the involved vectors or functions, are far less known in the literature.

2000 *Mathematics Subject Classification.* Primary 46B05, 46C05; Secondary 26D15, 26D10.

It is worth mentioning though, the following reverse of the generalised triangle inequality for complex numbers

$$\cos\theta\sum_{k=1}^{n}|z_k|\leq\left|\sum_{k=1}^{n}z_k\right|,$$

provided the complex numbers z_k, $k\in\{1,\dots,n\}$ satisfy the assumption

$$a-\theta\leq\arg(z_k)\leq a+\theta,\quad\text{for any}\quad k\in\{1,\dots,n\},$$

where $a\in\mathbb{R}$ and $\theta\in\left(0,\frac{\pi}{2}\right)$ was first discovered by M. Petrovich in 1917, [**22**] (see [**20**, p. 492]) and subsequently was rediscovered by other authors, including J. Karamata [**14**, p. 300 – 301], H.S. Wilf [**23**], and in an equivalent form by M. Marden [**18**]. Marden and Wilf have outlined in their work the important fact that reverses of the generalised triangle inequality may be successfully applied to the location problem for the roots of complex polynomials.

In 1966, J.B. Diaz and F.T. Metcalf [**2**] proved the following reverse of the triangle inequality in the more general case of inner product spaces:

Theorem 1 (Diaz-Metcalf, 1966). *Let a be a unit vector in the inner product space $(H;\langle\cdot,\cdot\rangle)$ over the real or complex number field $\mathbb{K}$. Suppose that the vectors $x_i\in H\backslash\{0\}$, $i\in\{1,\dots,n\}$ satisfy*

$$0\leq r\leq\frac{\operatorname{Re}\langle x_i,a\rangle}{\|x_i\|},\qquad i\in\{1,\dots,n\}.$$

Then

$$r\sum_{i=1}^{n}\|x_i\|\leq\left\|\sum_{i=1}^{n}x_i\right\|,$$

where equality holds if and only if

$$\sum_{i=1}^{n}x_i=r\left(\sum_{i=1}^{n}\|x_i\|\right)a.$$

A generalisation of this result for orthonormal families is incorporated in the following result [**2**].

Theorem 2 (Diaz-Metcalf, 1966). *Let $a_1,\dots,a_n$ be orthonormal vectors in H. Suppose the vectors $x_1,\dots,x_n\in H\backslash\{0\}$ satisfy*

$$0\leq r_k\leq\frac{\operatorname{Re}\langle x_i,a_k\rangle}{\|x_i\|},\qquad i\in\{1,\dots,n\},\ k\in\{1,\dots,m\}.$$

Then

$$\left(\sum_{k=1}^{m}r_k^2\right)^{\frac{1}{2}}\sum_{i=1}^{n}\|x_i\|\leq\left\|\sum_{i=1}^{n}x_i\right\|,$$

where equality holds if and only if

$$\sum_{i=1}^{n}x_i=\left(\sum_{i=1}^{n}\|x_i\|\right)\sum_{k=1}^{m}r_ka_k.$$

Similar results valid for semi-inner products may be found in [**15**], [**16**] and [**19**].

Now, for the scalar continuous case.

It appears, see [**20**, p. 492], that the first reverse inequality for (1.1) in the case of complex valued functions was obtained by J. Karamata in his book from 1949, [**14**]. It can be stated as

$$\cos\theta\int_a^b |f(x)|\,dx \le \left|\int_a^b f(x)\,dx\right|$$

provided

$$-\theta \le \arg f(x) \le \theta, \quad x \in [a,b]$$

for given $\theta \in \left(0, \frac{\pi}{2}\right)$.

This result has recently been extended by the author for the case of Bochner integrable functions with values in a Hilbert space H. If by $L([a,b];H)$, we denote the space of Bochner integrable functions with values in a Hilbert space H, i.e., we recall that $f \in L([a,b];H)$ if and only if $f:[a,b]\to H$ is Bochner measurable on $[a,b]$ and the Lebesgue integral $\int_a^b \|f(t)\|\,dt$ is finite, then

$$\int_a^b \|f(t)\|\,dt \le K\left\|\int_a^b f(t)\,dt\right\|, \tag{1.2}$$

provided that f satisfies the condition

$$\|f(t)\| \le K\,\mathrm{Re}\langle f(t), e\rangle \quad \text{for a.e. } t\in[a,b],$$

where $e \in H$, $\|e\| = 1$ and $K \ge 1$ are given. The case of equality holds in (1.2) if and only if

$$\int_a^b f(t)\,dt = \frac{1}{K}\left(\int_a^b \|f(t)\|\,dt\right)e.$$

The aim of the present paper is to survey some of the recent results concerning multiplicative and additive reverses for both the discrete and continuous version of the triangle inequalities in Banach spaces. New results and applications for the important case of Hilbert spaces and for complex numbers and complex functions have been provided as well.

2. Diaz-Metcalf Type Inequalities

In [**2**], Diaz and Metcalf established the following reverse of the generalised triangle inequality in real or complex normed linear spaces.

Theorem 3 (Diaz-Metcalf, 1966). *If $F: X \to \mathbb{K}$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$ is a linear functional of a unit norm defined on the normed linear space X endowed with the norm $\|\cdot\|$ and the vectors $x_1, \dots, x_n$ satisfy the condition*

$$0 \le r \le \mathrm{Re}\,F(x_i), \qquad i \in \{1,\dots,n\}; \tag{2.1}$$

then

$$r\sum_{i=1}^n \|x_i\| \le \left\|\sum_{i=1}^n x_i\right\|, \tag{2.2}$$

where equality holds if and only if both

$$F\left(\sum_{i=1}^{n} x_i\right) = r\sum_{i=1}^{n} \|x_i\| \tag{2.3}$$

and

$$F\left(\sum_{i=1}^{n} x_i\right) = \left\|\sum_{i=1}^{n} x_i\right\|. \tag{2.4}$$

If $X = H$, $(H; \langle\cdot,\cdot\rangle)$ is an inner product space and $F(x) = \langle x, e\rangle$, $\|e\| = 1$, then the condition (2.1) may be replaced with the simpler assumption

$$0 \leq r\|x_i\| \leq \operatorname{Re}\langle x_i, e\rangle, \qquad i = 1, \dots, n, \tag{2.5}$$

which implies the reverse of the generalised triangle inequality (2.2). In this case the equality holds in (2.2) if and only if [**2**]

$$\sum_{i=1}^{n} x_i = r\left(\sum_{i=1}^{n} \|x_i\|\right) e. \tag{2.6}$$

Theorem 4 (Diaz-Metcalf, 1966). *Let $F_1, \dots, F_m$ be linear functionals on X, each of unit norm. As in* [**2**]*, let consider the real number c defined by*

$$c = \sup_{x\neq 0}\left[\frac{\sum_{k=1}^{m} |F_k(x)|^2}{\|x\|^2}\right];$$

it then follows that $1 \leq c \leq m$. Suppose the vectors $x_1, \dots, x_n$ whenever $x_i \neq 0$, satisfy

$$0 \leq r_k\|x_i\| \leq \operatorname{Re} F_k(x_i), \qquad i = 1, \dots, n,\ k = 1, \dots, m. \tag{2.7}$$

Then one has the following reverse of the generalised triangle inequality [**2**]

$$\left(\frac{\sum_{k=1}^{m} r_k^2}{c}\right)^{\frac{1}{2}} \sum_{i=1}^{n} \|x_i\| \leq \left\|\sum_{i=1}^{n} x_i\right\|, \tag{2.8}$$

where equality holds if and only if both

$$F_k\left(\sum_{i=1}^{n} x_i\right) = r_k\sum_{i=1}^{n} \|x_i\|, \qquad k = 1, \dots, m \tag{2.9}$$

and

$$\sum_{k=1}^{m}\left[F_k\left(\sum_{i=1}^{n} x_i\right)\right]^2 = c\left\|\sum_{i=1}^{n} x_i\right\|^2. \tag{2.10}$$

If $X = H$, an inner product space, then, for $F_k(x) = \langle x, e_k\rangle$, where $\{e_k\}_{k=\overline{1,n}}$ is an orthonormal family in H, i.e., $\langle e_i, e_j\rangle = \delta_{ij}$, $i, j \in \{1, \dots, k\}$, δ_{ij} is Kronecker delta, the condition (2.7) may be replaced by

$$0 \leq r_k\|x_i\| \leq \operatorname{Re}\langle x_i, e_k\rangle, \qquad i = 1, \dots, n,\ k = 1, \dots, m; \tag{2.11}$$

implying the following reverse of the generalised triangle inequality

$$\left(\sum_{k=1}^{m} r_k^2\right)^{\frac{1}{2}} \sum_{i=1}^{n} \|x_i\| \leq \left\|\sum_{i=1}^{n} x_i\right\|, \tag{2.12}$$

where the equality holds if and only if

$$\sum_{i=1}^{n} x_i = \left(\sum_{i=1}^{n} \|x_i\|\right) \sum_{k=1}^{m} r_k e_k. \tag{2.13}$$

The aim of the following sections is to present recent reverses of the triangle inequality obtained by the author in [**5**] and [**6**]. New results are established for the general case of normed spaces. Their versions in inner product spaces are analyzed and applications for complex numbers are given as well.

For various classical inequalities related to the triangle inequality, see Chapter XVII of the book [**20**] and the references therein.

3. Inequalities of Diaz-Metcalf Type for m Functionals

3.1. The Case of Normed Spaces.

The following result may be stated [**5**].

THEOREM 5 (Dragomir, 2004). *Let $(X, \|\cdot\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$ and $F_k : X \to \mathbb{K}$, $k \in \{1, \dots, m\}$ continuous linear functionals on X. If $x_i \in X \setminus \{0\}$, $i \in \{1, \dots, n\}$ are such that there exists the constants $r_k \geq 0$, $k \in \{1, \dots, m\}$ with $\sum_{k=1}^{m} r_k > 0$ and*

$$\operatorname{Re} F_k(x_i) \geq r_k \|x_i\| \tag{3.1}$$

for each $i \in \{1, \dots, n\}$ and $k \in \{1, \dots, m\}$, then

$$\sum_{i=1}^{n} \|x_i\| \leq \frac{\left\|\sum_{k=1}^{m} F_k\right\|}{\sum_{k=1}^{m} r_k} \left\| \sum_{i=1}^{n} x_i \right\|. \tag{3.2}$$

The case of equality holds in (3.2) if both

$$\left(\sum_{k=1}^{m} F_k\right)\left(\sum_{i=1}^{n} x_i\right) = \left(\sum_{k=1}^{m} r_k\right) \sum_{i=1}^{n} \|x_i\| \tag{3.3}$$

and

$$\left(\sum_{k=1}^{m} F_k\right)\left(\sum_{i=1}^{n} x_i\right) = \left\|\sum_{k=1}^{m} F_k\right\| \left\|\sum_{i=1}^{n} x_i\right\|. \tag{3.4}$$

PROOF. Utilising the hypothesis (3.1) and the properties of the modulus, we have

$$\begin{aligned} I &:= \left|\left(\sum_{k=1}^{m} F_k\right)\left(\sum_{i=1}^{n} x_i\right)\right| \geq \left|\operatorname{Re}\left[\left(\sum_{k=1}^{m} F_k\right)\left(\sum_{i=1}^{n} x_i\right)\right]\right| \\ &\geq \sum_{k=1}^{m} \operatorname{Re} F_k\left(\sum_{i=1}^{n} x_i\right) = \sum_{k=1}^{m}\sum_{i=1}^{n} \operatorname{Re} F_k(x_i) \\ &\geq \left(\sum_{k=1}^{m} r_k\right) \sum_{i=1}^{n} \|x_i\|. \end{aligned} \tag{3.5}$$

On the other hand, by the continuity property of F_k, $k \in \{1, \dots, m\}$ we obviously have

$$I = \left|\left(\sum_{k=1}^{m} F_k\right)\left(\sum_{i=1}^{n} x_i\right)\right| \leq \left\|\sum_{k=1}^{m} F_k\right\| \left\|\sum_{i=1}^{n} x_i\right\|. \tag{3.6}$$

Making use of (3.5) and (3.6), we deduce the desired inequality (3.2).

Now, if (3.3) and (3.4) are valid, then, obviously, the case of equality holds true in the inequality (3.2).

Conversely, if the case of equality holds in (3.2), then it must hold in all the inequalities used to prove (3.2). Therefore we have

$$\operatorname{Re} F_k(x_i) = r_k \|x_i\| \tag{3.7}$$

for each $i \in \{1, \ldots, n\}$, $k \in \{1, \ldots, m\}$;

$$\sum_{k=1}^{m} \operatorname{Im} F_k\left(\sum_{i=1}^{n} x_i\right) = 0 \tag{3.8}$$

and

$$\sum_{k=1}^{m} \operatorname{Re} F_k\left(\sum_{i=1}^{n} x_i\right) = \left\|\sum_{k=1}^{m} F_k\right\| \left\|\sum_{i=1}^{n} x_i\right\|. \tag{3.9}$$

Note that, from (3.7), by summation over i and k, we get

$$\operatorname{Re}\left[\left(\sum_{k=1}^{m} F_k\right)\left(\sum_{i=1}^{n} x_i\right)\right] = \left(\sum_{k=1}^{m} r_k\right) \sum_{i=1}^{n} \|x_i\|. \tag{3.10}$$

Since (3.8) and (3.10) imply (3.3), while (3.9) and (3.10) imply (3.4) hence the theorem is proved. ∎

Remark 1. *If the norms $\|F_k\|$, $k \in \{1, \ldots, m\}$ are easier to find, then, from (3.2), one may get the (coarser) inequality that might be more useful in practice:*

$$\sum_{i=1}^{n} \|x_i\| \le \frac{\sum_{k=1}^{m} \|F_k\|}{\sum_{k=1}^{m} r_k} \left\|\sum_{i=1}^{n} x_i\right\|. \tag{3.11}$$

3.2. The Case of Inner Product Spaces. The case of inner product spaces, in which we may provide a simpler condition for equality, is of interest in applications [**5**].

Theorem 6 (Dragomir, 2004). *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, e_k, $x_i \in H\setminus\{0\}$, $k \in \{1, \ldots, m\}$, $i \in \{1, \ldots, n\}$. If $r_k \ge 0$, $k \in \{1, \ldots, m\}$ with $\sum_{k=1}^{m} r_k > 0$ satisfy*

$$\operatorname{Re}\langle x_i, e_k\rangle \ge r_k \|x_i\| \tag{3.12}$$

for each $i \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$, then

$$\sum_{i=1}^{n} \|x_i\| \le \frac{\left\|\sum_{k=1}^{m} e_k\right\|}{\sum_{k=1}^{m} r_k} \left\|\sum_{i=1}^{n} x_i\right\|. \tag{3.13}$$

The case of equality holds in (3.13) if and only if

$$\sum_{i=1}^{n} x_i = \frac{\sum_{k=1}^{m} r_k}{\left\|\sum_{k=1}^{m} e_k\right\|^2} \left(\sum_{i=1}^{n} \|x_i\|\right) \sum_{k=1}^{m} e_k. \tag{3.14}$$

PROOF. By the properties of inner product and by (3.12), we have

$$\begin{aligned} &\left|\left\langle \sum_{i=1}^{n} x_i, \sum_{k=1}^{m} e_k \right\rangle\right| \\ &\geq \left|\sum_{k=1}^{m} \operatorname{Re}\left\langle \sum_{i=1}^{n} x_i, e_k \right\rangle\right| \geq \sum_{k=1}^{m} \operatorname{Re}\left\langle \sum_{i=1}^{n} x_i, e_k \right\rangle \\ &= \sum_{k=1}^{m}\sum_{i=1}^{n} \operatorname{Re}\langle x_i, e_k\rangle \geq \left(\sum_{k=1}^{m} r_k\right) \sum_{i=1}^{n} \|x_i\| > 0. \end{aligned} \tag{3.15}$$

Observe also that, by (3.15), $\sum_{k=1}^{m} e_k \neq 0$.

On utlising Schwarz's inequality in the inner product space $(H; \langle\cdot,\cdot\rangle)$ for $\sum_{i=1}^{n} x_i$, $\sum_{k=1}^{m} e_k$, we have

$$\left\|\sum_{i=1}^{n} x_i\right\| \left\|\sum_{k=1}^{m} e_k\right\| \geq \left|\left\langle \sum_{i=1}^{n} x_i, \sum_{k=1}^{m} e_k \right\rangle\right|. \tag{3.16}$$

Making use of (3.15) and (3.16), we can conclude that (3.13) holds.

Now, if (3.14) holds true, then, by taking the norm, we have

$$\begin{aligned} \left\|\sum_{i=1}^{n} x_i\right\| &= \frac{\left(\sum_{k=1}^{m} r_k\right)\sum_{i=1}^{n} \|x_i\|}{\left\|\sum_{k=1}^{m} e_k\right\|^2} \left\|\sum_{k=1}^{m} e_k\right\| \\ &= \frac{\left(\sum_{k=1}^{m} r_k\right)}{\left\|\sum_{k=1}^{m} e_k\right\|} \sum_{i=1}^{n} \|x_i\|, \end{aligned}$$

i.e., the case of equality holds in (3.13).

Conversely, if the case of equality holds in (3.13), then it must hold in all the inequalities used to prove (3.13). Therefore, we have

$$\operatorname{Re}\langle x_i, e_k\rangle = r_k \|x_i\| \tag{3.17}$$

for each $i \in \{1, \dots, n\}$ and $k \in \{1, \dots, m\}$,

$$\left\|\sum_{i=1}^{n} x_i\right\| \left\|\sum_{k=1}^{m} e_k\right\| = \left|\left\langle \sum_{i=1}^{n} x_i, \sum_{k=1}^{m} e_k \right\rangle\right| \tag{3.18}$$

and

$$\operatorname{Im}\left\langle \sum_{i=1}^{n} x_i, \sum_{k=1}^{m} e_k \right\rangle = 0. \tag{3.19}$$

From (3.17), on summing over i and k, we get

$$\operatorname{Re}\left\langle \sum_{i=1}^{n} x_i, \sum_{k=1}^{m} e_k \right\rangle = \left(\sum_{k=1}^{m} r_k\right) \sum_{i=1}^{n} \|x_i\|. \tag{3.20}$$

By (3.19) and (3.20), we have

$$\left\langle \sum_{i=1}^{n} x_i, \sum_{k=1}^{m} e_k \right\rangle = \left(\sum_{k=1}^{m} r_k\right) \sum_{i=1}^{n} \|x_i\|. \tag{3.21}$$

On the other hand, by the use of the following identity in inner product spaces

$$\left\| u - \frac{\langle u, v\rangle v}{\|v\|^2} \right\|^2 = \frac{\|u\|^2 \|v\|^2 - |\langle u, v\rangle|^2}{\|v\|^2}, \quad v \neq 0, \tag{3.22}$$

the relation (3.18) holds if and only if

$$\sum_{i=1}^{n} x_i = \frac{\langle \sum_{i=1}^{n} x_i, \sum_{k=1}^{m} e_k \rangle}{\|\sum_{k=1}^{m} e_k\|^2} \sum_{k=1}^{m} e_k. \tag{3.23}$$

Finally, on utilising (3.21) and (3.23), we deduce that the condition (3.14) is necessary for the equality case in (3.13). ∎

Before we give a corollary of the above theorem, we need to state the following lemma that has been basically obtained in [**4**]. For the sake of completeness, we provide a short proof here as well.

Lemma 1 (Dragomir, 2004). *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $x, a \in H$, $r > 0$ such that:*

$$\|x - a\| \leq r < \|a\|. \tag{3.24}$$

Then we have the inequality

$$\|x\| \left(\|a\|^2 - r^2 \right)^{\frac{1}{2}} \leq \operatorname{Re} \langle x, a\rangle \tag{3.25}$$

or, equivalently

$$\|x\|^2 \|a\|^2 - [\operatorname{Re} \langle x, a\rangle]^2 \leq r^2 \|x\|^2. \tag{3.26}$$

The case of equality holds in (3.25) (or in (3.26)) if and only if

$$\|x - a\| = r \quad \text{and} \quad \|x\|^2 + r^2 = \|a\|^2. \tag{3.27}$$

Proof. From the first part of (3.24), we have

$$\|x\|^2 + \|a\|^2 - r^2 \leq 2 \operatorname{Re} \langle x, a\rangle. \tag{3.28}$$

By the second part of (3.24) we have $\left(\|a\|^2 - r^2 \right)^{\frac{1}{2}} > 0$, therefore, by (3.28), we may state that

$$0 < \frac{\|x\|^2}{\left(\|a\|^2 - r^2 \right)^{\frac{1}{2}}} + \left(\|a\|^2 - r^2 \right)^{\frac{1}{2}} \leq \frac{2 \operatorname{Re} \langle x, a\rangle}{\left(\|a\|^2 - r^2 \right)^{\frac{1}{2}}}. \tag{3.29}$$

Utilising the elementary inequality

$$\frac{1}{\alpha} q + \alpha p \geq 2\sqrt{pq}, \quad \alpha > 0,\ p > 0,\ q \geq 0;$$

with equality if and only if $\alpha = \sqrt{\frac{q}{p}}$, we may state (for $\alpha = \left(\|a\|^2 - r^2 \right)^{\frac{1}{2}}$, $p = 1$, $q = \|x\|^2$) that

$$2\|x\| \leq \frac{\|x\|^2}{\left(\|a\|^2 - r^2 \right)^{\frac{1}{2}}} + \left(\|a\|^2 - r^2 \right)^{\frac{1}{2}}. \tag{3.30}$$

The inequality (3.25) follows now by (3.29) and (3.30).

From the above argument, it is clear that the equality holds in (3.25) if and only if it holds in (3.29) and (3.30). However, the equality holds in (3.29) if and only if $\|x-a\|=r$ and in (3.30) if and only if $\left(\|a\|^2-r^2\right)^{\frac{1}{2}}=\|x\|$.

The proof is thus completed. ∎

We may now state the following corollary [**5**].

Corollary 1. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, e_k, $x_i\in H\backslash\{0\}$, $k\in\{1,\dots,m\}$, $i\in\{1,\dots,n\}$. If $\rho_k\geq 0$, $k\in\{1,\dots,m\}$ with*

$$\|x_i-e_k\|\leq\rho_k<\|e_k\| \tag{3.31}$$

for each $i\in\{1,\dots,n\}$ and $k\in\{1,\dots,m\}$, then

$$\sum_{i=1}^n\|x_i\|\leq\frac{\left\|\sum_{k=1}^m e_k\right\|}{\sum_{k=1}^m\left(\|e_k\|^2-\rho_k^2\right)^{\frac{1}{2}}}\left\|\sum_{i=1}^n x_i\right\|. \tag{3.32}$$

The case of equality holds in (3.32) if and only if

$$\sum_{i=1}^n x_i=\frac{\sum_{k=1}^m\left(\|e_k\|^2-\rho_k^2\right)^{\frac{1}{2}}}{\left\|\sum_{k=1}^m e_k\right\|^2}\left(\sum_{i=1}^n\|x_i\|\right)\sum_{k=1}^m e_k.$$

Proof. Utilising Lemma 1, we have from (3.31) that

$$\|x_i\|\left(\|e_k\|^2-\rho_k^2\right)^{\frac{1}{2}}\leq\operatorname{Re}\langle x_i,e_k\rangle$$

for each $k\in\{1,\dots,m\}$ and $i\in\{1,\dots,n\}$.

Applying Theorem 6 for

$$r_k:=\left(\|e_k\|^2-\rho_k^2\right)^{\frac{1}{2}},\quad k\in\{1,\dots,m\},$$

we deduce the desired result. ∎

Remark 2. *If $\{e_k\}_{k\in\{1,\dots,m\}}$ are orthogonal, then (3.32) becomes*

$$\sum_{i=1}^n\|x_i\|\leq\frac{\left(\sum_{k=1}^m\|e_k\|^2\right)^{\frac{1}{2}}}{\sum_{k=1}^m\left(\|e_k\|^2-\rho_k^2\right)^{\frac{1}{2}}}\left\|\sum_{i=1}^n x_i\right\| \tag{3.33}$$

with equality if and only if

$$\sum_{i=1}^n x_i=\frac{\sum_{k=1}^m\left(\|e_k\|^2-\rho_k^2\right)^{\frac{1}{2}}}{\sum_{k=1}^m\|e_k\|^2}\left(\sum_{i=1}^n\|x_i\|\right)\sum_{k=1}^m e_k.$$

Moreover, if $\{e_k\}_{k\in\{1,\dots,m\}}$ is assumed to be orthonormal and

$$\|x_i-e_k\|\leq\rho_k\ \text{ for } k\in\{1,\dots,m\},\ i\in\{1,\dots,n\}$$

where $\rho_k\in[0,1)$ for $k\in\{1,\dots,m\}$, then

$$\sum_{i=1}^n\|x_i\|\leq\frac{\sqrt{m}}{\sum_{k=1}^m\left(1-\rho_k^2\right)^{\frac{1}{2}}}\left\|\sum_{i=1}^n x_i\right\| \tag{3.34}$$

with equality if and only if

$$\sum_{i=1}^{n} x_i = \frac{\sum_{k=1}^{m}\left(1-\rho_k^2\right)^{\frac{1}{2}}}{m}\left(\sum_{i=1}^{n}\|x_i\|\right)\sum_{k=1}^{m} e_k.$$

The following lemma may be stated as well [**3**].

Lemma 2 (Dragomir, 2004). *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $x,y\in H$ and $M\geq m>0$. If*

$$\operatorname{Re}\langle My-x, x-my\rangle \geq 0 \tag{3.35}$$

or, equivalently,

$$\left\|x-\frac{m+M}{2}y\right\| \leq \frac{1}{2}(M-m)\|y\|, \tag{3.36}$$

then

$$\|x\|\,\|y\| \leq \frac{1}{2}\cdot\frac{M+m}{\sqrt{mM}}\operatorname{Re}\langle x,y\rangle. \tag{3.37}$$

The equality holds in (3.37) if and only if the case of equality holds in (3.35) and

$$\|x\| = \sqrt{mM}\,\|y\|. \tag{3.38}$$

Proof. Obviously,

$$\operatorname{Re}\langle My-x, x-my\rangle = (M+m)\operatorname{Re}\langle x,y\rangle - \|x\|^2 - mM\|y\|^2.$$

Then (3.35) is clearly equivalent to

$$\frac{\|x\|^2}{\sqrt{mM}} + \sqrt{mM}\,\|y\|^2 \leq \frac{M+m}{\sqrt{mM}}\operatorname{Re}\langle x,y\rangle. \tag{3.39}$$

Since, obviously,

$$2\|x\|\,\|y\| \leq \frac{\|x\|^2}{\sqrt{mM}} + \sqrt{mM}\,\|y\|^2, \tag{3.40}$$

with equality iff $\|x\| = \sqrt{mM}\,\|y\|$, hence (3.39) and (3.40) imply (3.37).

The case of equality is obvious and we omit the details. ∎

Finally, we may state the following corollary of Theorem 6, see [**5**].

Corollary 2. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, e_k, $x_i\in H\backslash\{0\}$, $k\in\{1,\dots,m\}$, $i\in\{1,\dots,n\}$. If $M_k>\mu_k>0$, $k\in\{1,\dots,m\}$ are such that either*

$$\operatorname{Re}\langle M_k e_k - x_i, x_i-\mu_k e_k\rangle \geq 0 \tag{3.41}$$

or, equivalently,

$$\left\|x_i-\frac{M_k+\mu_k}{2}e_k\right\| \leq \frac{1}{2}(M_k-\mu_k)\|e_k\|$$

for each $k\in\{1,\dots,m\}$ and $i\in\{1,\dots,n\}$, then

$$\sum_{i=1}^{n}\|x_i\| \leq \frac{\left\|\sum_{k=1}^{m} e_k\right\|}{\sum_{k=1}^{m}\frac{2\cdot\sqrt{\mu_k M_k}}{\mu_k+M_k}\|e_k\|}\left\|\sum_{i=1}^{n} x_i\right\|. \tag{3.42}$$

The case of equality holds in (3.42) if and only if

$$\sum_{i=1}^{n} x_i = \frac{\sum_{k=1}^{m} \frac{2\cdot\sqrt{\mu_k M_k}}{\mu_k + M_k} \|e_k\|}{\left\|\sum_{k=1}^{m} e_k\right\|^2} \sum_{i=1}^{n} \|x_i\| \sum_{k=1}^{m} e_k.$$

PROOF. Utilising Lemma 2, by (3.41) we deduce

$$\frac{2\cdot\sqrt{\mu_k M_k}}{\mu_k + M_k} \|x_i\| \|e_k\| \leq \operatorname{Re}\langle x_i, e_k\rangle$$

for each $k \in \{1, \ldots, m\}$ and $i \in \{1, \ldots, n\}$.

Applying Theorem 6 for

$$r_k := \frac{2\cdot\sqrt{\mu_k M_k}}{\mu_k + M_k} \|e_k\|, \quad k \in \{1, \ldots, m\},$$

we deduce the desired result. ∎

4. Diaz-Metcalf Inequality for Semi-Inner Products

In 1961, G. Lumer [**17**] introduced the following concept.

DEFINITION 1. *Let X be a linear space over the real or complex number field $\mathbb{K}$. The mapping $[\cdot,\cdot] : X \times X \to \mathbb{K}$ is called a semi-inner product on X, if the following properties are satisfied (see also* [**3**, p. 17]*):*

(i) $[x+y, z] = [x,z] + [y,z]$ *for all* $x, y, z \in X$;
(ii) $[\lambda x, y] = \lambda [x,y]$ *for all* $x, y \in X$ *and* $\lambda \in \mathbb{K}$;
(iii) $[x,x] \geq 0$ *for all* $x \in X$ *and* $[x,x] = 0$ *implies* $x = 0$;
(iv) $|[x,y]|^2 \leq [x,x][y,y]$ *for all* $x, y \in X$;
(v) $[x, \lambda y] = \bar{\lambda}[x,y]$ *for all* $x, y \in X$ *and* $\lambda \in \mathbb{K}$.

It is well known that the mapping $X \ni x \longmapsto [x,x]^{\frac{1}{2}} \in \mathbb{R}$ is a norm on X and for any $y \in X$, the functional $X \ni x \xmapsto{\varphi_y} [x,y] \in \mathbb{K}$ is a continuous linear functional on X endowed with the norm $\|\cdot\|$ generated by $[\cdot,\cdot]$. Moreover, one has $\|\varphi_y\| = \|y\|$ (see for instance [**3**, p. 17]).

Let $(X, \|\cdot\|)$ be a real or complex normed space. If $J : X \to_2 X^*$ is the *normalised duality mapping* defined on X, i.e., we recall that (see for instance [**3**, p. 1])

$$J(x) = \{\varphi \in X^* | \varphi(x) = \|\varphi\| \|x\|, \ \|\varphi\| = \|x\|\}, \quad x \in X,$$

then we may state the following representation result (see for instance [**3**, p. 18]):

Each semi-inner product $[\cdot,\cdot] : X \times X \to K$ that generates the norm $\|\cdot\|$ of the normed linear space $(X, \|\cdot\|)$ over the real or complex number field K, is of the form

$$[x,y] = \left\langle \tilde{J}(y), x \right\rangle \quad \text{for any} \quad x, y \in X,$$

where $\tilde{J}$ is a selection of the normalised duality mapping and $\langle \varphi, x\rangle := \varphi(x)$ for $\varphi \in X^*$ and $x \in X$.

Utilising the concept of semi-inner products, we can state the following particular case of the Diaz-Metcalf inequality.

Corollary 3. *Let* $(X, \|\cdot\|)$ *be a normed linear space,* $[\cdot,\cdot] : X \times X \to \mathbb{K}$ *a semi-inner product generating the norm* $\|\cdot\|$ *and* $e \in X$, $\|e\| = 1$. *If* $x_i \in X$, $i \in \{1, \dots, n\}$ *and* $r \geq 0$ *such that*

$$r \|x_i\| \leq \operatorname{Re}[x_i, e] \quad \text{for each} \quad i \in \{1, \dots, n\}, \tag{4.1}$$

then we have the inequality

$$r \sum_{i=1}^{n} \|x_i\| \leq \left\| \sum_{i=1}^{n} x_i \right\|. \tag{4.2}$$

The case of equality holds in (4.2) if and only if both

$$\left[\sum_{i=1}^{n} x_i, e \right] = r \sum_{i=1}^{n} \|x_i\| \tag{4.3}$$

and

$$\left[\sum_{i=1}^{n} x_i, e \right] = \left\| \sum_{i=1}^{n} x_i \right\|. \tag{4.4}$$

The proof is obvious from the Diaz-Metcalf theorem [**2**, Theorem 3] applied for the continuous linear functional $F_e(x) = [x, e]$, $x \in X$.

Before we provide a simpler necessary and sufficient condition of equality in (4.2), we need to recall the concept of strictly convex normed spaces and a classical characterisation of these spaces.

Definition 2. *A normed linear space* $(X, \|\cdot\|)$ *is said to be strictly convex if for every* x, y *from* X *with* $x \neq y$ *and* $\|x\| = \|y\| = 1$, *we have* $\|\lambda x + (1-\lambda) y\| < 1$ *for all* $\lambda \in (0,1)$.

The following characterisation of strictly convex spaces is useful in what follows (see [**1**], [**13**], or [**3**, p. 21]).

Theorem 7. *Let* $(X, \|\cdot\|)$ *be a normed linear space over* $\mathbb{K}$ *and* $[\cdot,\cdot]$ *a semi-inner product generating its norm. The following statements are equivalent:*

(*i*) $(X, \|\cdot\|)$ *is strictly convex;*
(*ii*) *For every* $x, y \in X$, $x, y \neq 0$ *with* $[x, y] = \|x\| \|y\|$, *there exists a* $\lambda > 0$ *such that* $x = \lambda y$.

The following result may be stated.

Corollary 4. *Let* $(X, \|\cdot\|)$ *be a strictly convex normed linear space,* $[\cdot,\cdot]$ *a semi-inner product generating the norm and* e, x_i $(i \in \{1, \dots, n\})$ *as in Corollary 3. Then the case of equality holds in (4.2) if and only if*

$$\sum_{i=1}^{n} x_i = r \left(\sum_{i=1}^{n} \|x_i\| \right) e. \tag{4.5}$$

Proof. If (4.5) holds true, then, obviously

$$\left\| \sum_{i=1}^{n} x_i \right\| = r \left(\sum_{i=1}^{n} \|x_i\| \right) \|e\| = r \sum_{i=1}^{n} \|x_i\|,$$

which is the equality case in (4.2).

Conversely, if the equality holds in (4.2), then by Corollary 3, we have that (4.3) and (4.4) hold true. Utilising Theorem 7, we conclude that there exists a $\mu > 0$ such that

$$\sum_{i=1}^{n} x_i = \mu e. \tag{4.6}$$

Inserting this in (4.3) we get

$$\mu \|e\|^2 = r \sum_{i=1}^{n} \|x_i\|$$

giving

$$\mu = r \sum_{i=1}^{n} \|x_i\| . \tag{4.7}$$

Finally, by (4.6) and (4.7) we deduce (4.5) and the corollary is proved. ∎

5. Other Multiplicative Reverses for m Functionals

Assume that F_k, $k \in \{1, \dots, m\}$ are bounded linear functionals defined on the normed linear space X.

For $p \in [1, \infty)$, define

$$c_p := \sup_{x \neq 0} \left[\frac{\sum_{k=1}^{m} |F_k(x)|^p}{\|x\|^p} \right]^{\frac{1}{p}} \tag{c_p}$$

and for $p = \infty$,

$$c_\infty := \sup_{x \neq 0} \left[\max_{1 \le k \le m} \left\{ \frac{|F_k(x)|}{\|x\|} \right\} \right]. \tag{c_∞}$$

Then, by the fact that $|F_k(x)| \le \|F_k\| \|x\|$ for any $x \in X$, where $\|F_k\|$ is the norm of the functional F_k, we have that

$$c_p \le \left(\sum_{k=1}^{m} \|F_k\|^p \right)^{\frac{1}{p}}, \quad p \ge 1$$

and

$$c_\infty \le \max_{1 \le k \le m} \|F_k\| .$$

We may now state and prove a new reverse inequality for the generalised triangle inequality in normed linear spaces.

Theorem 8. *Let x_i, r_k, F_k, $k \in \{1, \dots, m\}$, $i \in \{1, \dots, n\}$ be as in the hypothesis of Theorem 5. Then we have the inequalities*

$$(1 \le) \frac{\sum_{i=1}^{n} \|x_i\|}{\left\| \sum_{i=1}^{n} x_i \right\|} \le \frac{c_\infty}{\max\limits_{1 \le k \le m} \{r_k\}} \left(\le \frac{\max\limits_{1 \le k \le m} \|F_k\|}{\max\limits_{1 \le k \le m} \{r_k\}} \right). \tag{5.1}$$

The case of equality holds in (5.1) if and only if

$$\operatorname{Re}\left[F_k \left(\sum_{i=1}^{n} x_i \right) \right] = r_k \sum_{i=1}^{n} \|x_i\| \quad \textit{for each} \;\; k \in \{1, \dots, m\} \tag{5.2}$$

and

$$\max_{1\leq k\leq m}\operatorname{Re}\left[F_k\left(\sum_{i=1}^{n}x_i\right)\right]=c_\infty\left\|\sum_{i=1}^{n}x_i\right\|. \tag{5.3}$$

Proof. Since, by the definition of c_∞, we have

$$c_\infty\|x\|\geq\max_{1\leq k\leq m}|F_k(x)|,\quad\text{for any }\ x\in X,$$

then we can state, for $x=\sum_{i=1}^{n}x_i$, that

$$\begin{aligned}c_\infty\left\|\sum_{i=1}^{n}x_i\right\|&\geq\max_{1\leq k\leq m}\left|F_k\left(\sum_{i=1}^{n}x_i\right)\right|\geq\max_{1\leq k\leq m}\left[\left|\operatorname{Re}F_k\left(\sum_{i=1}^{n}x_i\right)\right|\right]\\&\geq\max_{1\leq k\leq m}\left[\operatorname{Re}\sum_{i=1}^{n}F_k(x_i)\right]=\max_{1\leq k\leq m}\left[\sum_{i=1}^{n}\operatorname{Re}F_k(x_i)\right].\end{aligned}\tag{5.4}$$

Utilising the hypothesis (3.1) we obviously have

$$\max_{1\leq k\leq m}\left[\sum_{i=1}^{n}\operatorname{Re}F_k(x_i)\right]\geq\max_{1\leq k\leq m}\{r_k\}\cdot\sum_{i=1}^{n}\|x_i\|.$$

Also, $\sum_{i=1}^{n}x_i\neq0$, because, by the initial assumptions, not all r_k and x_i with $k\in\{1,\dots,m\}$ and $i\in\{1,\dots,n\}$ are allowed to be zero. Hence the desired inequality (5.1) is obtained.

Now, if (5.2) is valid, then, taking the maximum over $k\in\{1,\dots,m\}$ in this equality we get

$$\max_{1\leq k\leq m}\operatorname{Re}\left[F_k\left(\sum_{i=1}^{n}x_i\right)\right]=\max_{1\leq k\leq m}\{r_k\}\left\|\sum_{i=1}^{n}x_i\right\|,$$

which, together with (5.3) provides the equality case in (5.1).

Now, if the equality holds in (5.1), it must hold in all the inequalities used to prove (5.1), therefore, we have

$$\operatorname{Re}F_k(x_i)=r_k\|x_i\|\quad\text{for each }\ i\in\{1,\dots,n\}\ \text{ and }\ k\in\{1,\dots,m\}\tag{5.5}$$

and, from (5.4),

$$c_\infty\left\|\sum_{i=1}^{n}x_i\right\|=\max_{1\leq k\leq m}\operatorname{Re}\left[F_k\left(\sum_{i=1}^{n}x_i\right)\right],$$

which is (5.3).

From (5.5), on summing over $i\in\{1,\dots,n\}$, we get (5.2), and the theorem is proved. ∎

The following result in normed spaces also holds.

Theorem 9. *Let x_i,r_k,F_k, $k\in\{1,\dots,m\}$, $i\in\{1,\dots,n\}$ be as in the hypothesis of Theorem 5. Then we have the inequality*

$$(1\leq)\,\frac{\sum_{i=1}^{n}\|x_i\|}{\left\|\sum_{i=1}^{n}x_i\right\|}\leq\frac{c_p}{\left(\sum_{k=1}^{m}r_k^p\right)^{\frac{1}{p}}}\left(\leq\frac{\sum_{k=1}^{m}\|F_k\|^p}{\sum_{k=1}^{m}r_k^p}\right)^{\frac{1}{p}},\tag{5.6}$$

where $p\geq1$.

The case of equality holds in (5.6) if and only if

$$\operatorname{Re}\left[F_k\left(\sum_{i=1}^n x_i\right)\right]=r_k\sum_{i=1}^n\|x_i\|\quad \textit{for each}\quad k\in\{1,\dots,m\} \tag{5.7}$$

and

$$\sum_{k=1}^m\left[\operatorname{Re}F_k\left(\sum_{i=1}^n x_i\right)\right]^p=c_p^p\left\|\sum_{i=1}^n x_i\right\|^p. \tag{5.8}$$

Proof. By the definition of c_p, $p\geq 1$, we have

$$c_p^p\|x\|^p\geq\sum_{k=1}^m|F_k(x)|^p\quad\text{for any}\quad x\in X,$$

implying that

$$\begin{aligned}c_p^p\left\|\sum_{i=1}^n x_i\right\|^p&\geq\sum_{k=1}^m\left|F_k\left(\sum_{i=1}^n x_i\right)\right|^p\geq\sum_{k=1}^m\left|\operatorname{Re}F_k\left(\sum_{i=1}^n x_i\right)\right|^p\\&\geq\sum_{k=1}^m\left[\operatorname{Re}F_k\left(\sum_{i=1}^n x_i\right)\right]^p=\sum_{k=1}^m\left[\sum_{i=1}^n\operatorname{Re}F_k(x_i)\right]^p.\end{aligned} \tag{5.9}$$

Utilising the hypothesis (3.1), we obviously have that

$$\sum_{k=1}^m\left[\sum_{i=1}^n\operatorname{Re}F_k(x_i)\right]^p\geq\sum_{k=1}^m\left[\sum_{i=1}^n r_k\|x_i\|\right]^p=\sum_{k=1}^m r_k^p\left(\sum_{i=1}^n\|x_i\|\right)^p. \tag{5.10}$$

Making use of (5.9) and (5.10), we deduce

$$c_p^p\left\|\sum_{i=1}^n x_i\right\|^p\geq\left(\sum_{k=1}^m r_k^p\right)\left(\sum_{i=1}^n\|x_i\|\right)^p,$$

which implies the desired inequality (5.6).

If (5.7) holds true, then, taking the power p and summing over $k\in\{1,\dots,m\}$, we deduce

$$\sum_{k=1}^m\left[\operatorname{Re}\left[F_k\left(\sum_{i=1}^n x_i\right)\right]\right]^p=\sum_{k=1}^m r_k^p\left(\sum_{i=1}^n\|x_i\|\right)^p,$$

which, together with (5.8) shows that the equality case holds true in (5.6).

Conversely, if the case of equality holds in (5.6), then it must hold in all inequalities needed to prove (5.6), therefore, we must have:

$$\operatorname{Re}F_k(x_i)=r_k\|x_i\|\quad\text{for each}\quad i\in\{1,\dots,n\}\quad\text{and}\quad k\in\{1,\dots,m\} \tag{5.11}$$

and, from (5.9),

$$c_p^p\left\|\sum_{i=1}^n x_i\right\|^p=\sum_{k=1}^m\left[\operatorname{Re}F_k\left(\sum_{i=1}^n x_i\right)\right]^p,$$

which is exactly (5.8).

From (5.11), on summing over i from 1 to n, we deduce (5.7), and the theorem is proved. ∎

6. An Additive Reverse for the Triangle Inequality

6.1. The Case of One Functional. In the following we provide an alternative of the Diaz-Metcalf reverse of the generalised triangle inequality [**6**].

THEOREM 10 (Dragomir, 2004). *Let $(X, \|\cdot\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$ and $F : X \to \mathbb{K}$ a linear functional with the property that $|F(x)| \le \|x\|$ for any $x \in X$. If $x_i \in X$, $k_i \ge 0$, $i \in \{1, \dots, n\}$ are such that*

$$(0 \le) \|x_i\| - \operatorname{Re} F(x_i) \le k_i \quad \text{for each} \quad i \in \{1, \dots, n\}, \tag{6.1}$$

then we have the inequality

$$(0 \le) \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| \le \sum_{i=1}^{n} k_i. \tag{6.2}$$

The equality holds in (6.2) if and only if both

$$F\left(\sum_{i=1}^{n} x_i\right) = \left\| \sum_{i=1}^{n} x_i \right\| \quad \text{and} \quad F\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} \|x_i\| - \sum_{i=1}^{n} k_i. \tag{6.3}$$

PROOF. If we sum in (6.1) over i from 1 to n, then we get

$$\sum_{i=1}^{n} \|x_i\| \le \operatorname{Re}\left[F\left(\sum_{i=1}^{n} x_i\right)\right] + \sum_{i=1}^{n} k_i. \tag{6.4}$$

Taking into account that $|F(x)| \le \|x\|$ for each $x \in X$, then we may state that

$$\begin{aligned} \operatorname{Re}\left[F\left(\sum_{i=1}^{n} x_i\right)\right] &\le \left|\operatorname{Re} F\left(\sum_{i=1}^{n} x_i\right)\right| \\ &\le \left|F\left(\sum_{i=1}^{n} x_i\right)\right| \le \left\| \sum_{i=1}^{n} x_i \right\|. \end{aligned} \tag{6.5}$$

Now, making use of (6.4) and (6.5), we deduce (6.2).

Obviously, if (6.3) is valid, then the case of equality in (6.2) holds true.

Conversely, if the equality holds in (6.2), then it must hold in all the inequalities used to prove (6.2), therefore we have

$$\sum_{i=1}^{n} \|x_i\| = \operatorname{Re}\left[F\left(\sum_{i=1}^{n} x_i\right)\right] + \sum_{i=1}^{n} k_i$$

and

$$\operatorname{Re}\left[F\left(\sum_{i=1}^{n} x_i\right)\right] = \left|F\left(\sum_{i=1}^{n} x_i\right)\right| = \left\| \sum_{i=1}^{n} x_i \right\|,$$

which imply (6.3). ∎

The following corollary may be stated [**6**].

COROLLARY 5. *Let $(X, \|\cdot\|)$ be a normed linear space, $[\cdot, \cdot] : X \times X \to \mathbb{K}$ a semi-inner product generating the norm $\|\cdot\|$ and $e \in X$, $\|e\| = 1$. If $x_i \in X$, $k_i \ge 0$, $i \in \{1, \dots, n\}$ are such that*

$$(0 \le) \|x_i\| - \operatorname{Re}[x_i, e] \le k_i \quad \text{for each} \quad i \in \{1, \dots, n\}, \tag{6.6}$$

then we have the inequality

$$(0 \leq) \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| \leq \sum_{i=1}^{n} k_i. \tag{6.7}$$

The equality holds in (6.7) if and only if both

$$\left[\sum_{i=1}^{n} x_i, e \right] = \left\| \sum_{i=1}^{n} x_i \right\| \quad \text{and} \quad \left[\sum_{i=1}^{n} x_i, e \right] = \sum_{i=1}^{n} \|x_i\| - \sum_{i=1}^{n} k_i. \tag{6.8}$$

Moreover, if $(X, \|\cdot\|)$ is strictly convex, then the case of equality holds in (6.7) if and only if

$$\sum_{i=1}^{n} \|x_i\| \geq \sum_{i=1}^{n} k_i \tag{6.9}$$

and

$$\sum_{i=1}^{n} x_i = \left(\sum_{i=1}^{n} \|x_i\| - \sum_{i=1}^{n} k_i \right) \cdot e. \tag{6.10}$$

Proof. The first part of the corollary is obvious by Theorem 10 applied for the continuous linear functional of unit norm F_e, $F_e(x) = [x, e]$, $x \in X$. The second part may be shown on utilising a similar argument to the one from the proof of Corollary 4. We omit the details. ∎

Remark 3. *If $X = H$, $(H; \langle \cdot, \cdot \rangle)$ is an inner product space, then from Corollary 5 we deduce the additive reverse inequality obtained in Theorem 7 of* [**12**]. *For further similar results in inner product spaces, see* [**4**] *and* [**12**].

6.2. The Case of m Functionals. The following result generalising Theorem 10 may be stated [**6**].

Theorem 11 (Dragomir, 2004). *Let $(X, \|\cdot\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$. If F_k, $k \in \{1, \ldots, m\}$ are bounded linear functionals defined on X and $x_i \in X$, $M_{ik} \geq 0$ for $i \in \{1, \ldots, n\}$, $k \in \{1, \ldots, m\}$ are such that*

$$\|x_i\| - \operatorname{Re} F_k(x_i) \leq M_{ik} \tag{6.11}$$

for each $i \in \{1, \ldots, n\}$, $k \in \{1, \ldots, m\}$, then we have the inequality

$$\sum_{i=1}^{n} \|x_i\| \leq \left\| \frac{1}{m} \sum_{k=1}^{m} F_k \right\| \left\| \sum_{i=1}^{n} x_i \right\| + \frac{1}{m} \sum_{k=1}^{m} \sum_{i=1}^{n} M_{ik}. \tag{6.12}$$

The case of equality holds in (6.12) if both

$$\frac{1}{m} \sum_{k=1}^{m} F_k \left(\sum_{i=1}^{n} x_i \right) = \left\| \frac{1}{m} \sum_{k=1}^{m} F_k \right\| \left\| \sum_{i=1}^{n} x_i \right\| \tag{6.13}$$

and

$$\frac{1}{m} \sum_{k=1}^{m} F_k \left(\sum_{i=1}^{n} x_i \right) = \sum_{i=1}^{n} \|x_i\| - \frac{1}{m} \sum_{k=1}^{m} \sum_{j=1}^{n} M_{ik}. \tag{6.14}$$

Proof. If we sum (6.11) over i from 1 to n, then we deduce

$$\sum_{i=1}^{n}\|x_i\| - \operatorname{Re} F_k\left(\sum_{i=1}^{n} x_i\right) \leq \sum_{i=1}^{n} M_{ik}$$

for each $k \in \{1,\dots,m\}$.

Summing these inequalities over k from 1 to m, we deduce

$$\sum_{i=1}^{n}\|x_i\| \leq \frac{1}{m}\sum_{k=1}^{m}\operatorname{Re} F_k\left(\sum_{i=1}^{n} x_i\right) + \frac{1}{m}\sum_{k=1}^{m}\sum_{i=1}^{n} M_{ik}. \tag{6.15}$$

Utilising the continuity property of the functionals F_k and the properties of the modulus, we have

$$\begin{aligned}\sum_{k=1}^{m}\operatorname{Re} F_k\left(\sum_{i=1}^{n} x_i\right) &\leq \left|\sum_{k=1}^{m}\operatorname{Re} F_k\left(\sum_{i=1}^{n} x_i\right)\right| \\ &\leq \left|\sum_{k=1}^{m} F_k\left(\sum_{i=1}^{n} x_i\right)\right| \leq \left\|\sum_{k=1}^{m} F_k\right\|\left\|\sum_{i=1}^{n} x_i\right\|.\end{aligned} \tag{6.16}$$

Now, by (6.15) and (6.16), we deduce (6.12).

Obviously, if (6.13) and (6.14) hold true, then the case of equality is valid in (6.12).

Conversely, if the case of equality holds in (6.12), then it must hold in all the inequalities used to prove (6.12). Therefore we have

$$\sum_{i=1}^{n}\|x_i\| = \frac{1}{m}\sum_{k=1}^{m}\operatorname{Re} F_k\left(\sum_{i=1}^{n} x_i\right) + \frac{1}{m}\sum_{k=1}^{m}\sum_{i=1}^{n} M_{ik},$$

$$\sum_{k=1}^{m}\operatorname{Re} F_k\left(\sum_{i=1}^{n} x_i\right) = \left\|\sum_{k=1}^{m} F_k\right\|\left\|\sum_{i=1}^{n} x_i\right\|$$

and

$$\sum_{k=1}^{m}\operatorname{Im} F_k\left(\sum_{i=1}^{n} x_i\right) = 0.$$

These imply that (6.13) and (6.14) hold true, and the theorem is completely proved. ∎

Remark 4. *If F_k, $k \in \{1,\dots,m\}$ are of unit norm, then, from (6.12), we deduce the inequality*

$$\sum_{i=1}^{n}\|x_i\| \leq \left\|\sum_{i=1}^{n} x_i\right\| + \frac{1}{m}\sum_{k=1}^{m}\sum_{i=1}^{n} M_{ik}, \tag{6.17}$$

which is obviously coarser than (6.12), but perhaps more useful for applications.

6.3. The Case of Inner Product Spaces. The case of inner product spaces, in which we may provide a simpler condition of equality, is of interest in applications [**6**].

THEOREM 12 (Dragomir, 2004). *Let $(X, \|\cdot\|)$ be an inner product space over the real or complex number field $\mathbb{K}$, e_k, $x_i \in H\setminus\{0\}$, $k \in \{1, \dots, m\}$, $i \in \{1, \dots, n\}$. If $M_{ik} \geq 0$ for $i \in \{1, \dots, n\}$, $\{1, \dots, n\}$ such that*

$$\|x_i\| - \operatorname{Re}\langle x_i, e_k\rangle \leq M_{ik} \tag{6.18}$$

for each $i \in \{1, \dots, n\}$, $k \in \{1, \dots, m\}$, then we have the inequality

$$\sum_{i=1}^{n} \|x_i\| \leq \left\| \frac{1}{m} \sum_{k=1}^{m} e_k \right\| \left\| \sum_{i=1}^{n} x_i \right\| + \frac{1}{m} \sum_{k=1}^{m} \sum_{i=1}^{n} M_{ik}. \tag{6.19}$$

The case of equality holds in (6.19) if and only if

$$\sum_{i=1}^{n} \|x_i\| \geq \frac{1}{m} \sum_{k=1}^{m} \sum_{i=1}^{n} M_{ik} \tag{6.20}$$

and

$$\sum_{i=1}^{n} x_i = \frac{m\left(\sum_{i=1}^{n} \|x_i\| - \frac{1}{m}\sum_{k=1}^{m}\sum_{i=1}^{n} M_{ik}\right)}{\left\|\sum_{k=1}^{m} e_k\right\|^2} \sum_{k=1}^{m} e_k. \tag{6.21}$$

PROOF. As in the proof of Theorem 11, we have

$$\sum_{i=1}^{n} \|x_i\| \leq \operatorname{Re}\left\langle \frac{1}{m} \sum_{k=1}^{m} e_k, \sum_{i=1}^{n} x_i \right\rangle + \frac{1}{m} \sum_{k=1}^{m} \sum_{i=1}^{n} M_{ik}, \tag{6.22}$$

and $\sum_{k=1}^{m} e_k \neq 0$.

On utilising the Schwarz inequality in the inner product space $(H; \langle\cdot,\cdot\rangle)$ for $\sum_{i=1}^{n} x_i$, $\sum_{k=1}^{m} e_k$, we have

$$\begin{aligned} \left\| \sum_{i=1}^{n} x_i \right\| \left\| \sum_{k=1}^{m} e_k \right\| &\geq \left| \left\langle \sum_{i=1}^{n} x_i, \sum_{k=1}^{m} e_k \right\rangle \right| \\ &\geq \left| \operatorname{Re}\left\langle \sum_{i=1}^{n} x_i, \sum_{k=1}^{m} e_k \right\rangle \right| \\ &\geq \operatorname{Re}\left\langle \sum_{i=1}^{n} x_i, \sum_{k=1}^{m} e_k \right\rangle. \end{aligned} \tag{6.23}$$

By (6.22) and (6.23) we deduce (6.19).

Taking the norm in (6.21) and using (6.20), we have

$$\left\| \sum_{i=1}^{n} x_i \right\| = \frac{m\left(\sum_{i=1}^{n} \|x_i\| - \frac{1}{m}\sum_{k=1}^{m}\sum_{i=1}^{n} M_{ik}\right)}{\left\|\sum_{k=1}^{m} e_k\right\|},$$

showing that the equality holds in (6.19).

Conversely, if the case of equality holds in (6.19), then it must hold in all the inequalities used to prove (6.19). Therefore we have

$$\|x_i\| = \operatorname{Re}\langle x_i, e_k\rangle + M_{ik} \tag{6.24}$$

for each $i \in \{1, \dots, n\}$, $k \in \{1, \dots, m\}$,

$$\left\| \sum_{i=1}^{n} x_i \right\| \left\| \sum_{k=1}^{m} e_k \right\| = \left| \left\langle \sum_{i=1}^{n} x_i, \sum_{k=1}^{m} e_k \right\rangle \right| \tag{6.25}$$

and

$$\operatorname{Im}\left\langle \sum_{i=1}^{n} x_i, \sum_{k=1}^{m} e_k \right\rangle = 0. \tag{6.26}$$

From (6.24), on summing over i and k, we get

$$\operatorname{Re}\left\langle \sum_{i=1}^{n} x_i, \sum_{k=1}^{m} e_k \right\rangle = m\sum_{i=1}^{n} \|x_i\| - \sum_{k=1}^{m}\sum_{i=1}^{n} M_{ik}. \tag{6.27}$$

On the other hand, by the use of the identity (3.22), the relation (6.25) holds if and only if

$$\sum_{i=1}^{n} x_i = \frac{\left\langle \sum_{i=1}^{n} x_i, \sum_{k=1}^{m} e_k \right\rangle}{\left\| \sum_{k=1}^{m} e_k \right\|^2} \sum_{k=1}^{m} e_k,$$

giving, from (6.26) and (6.27), that

$$\sum_{i=1}^{n} x_i = \frac{m\sum_{i=1}^{n} \|x_i\| - \sum_{k=1}^{m}\sum_{i=1}^{n} M_{ik}}{\left\| \sum_{k=1}^{m} e_k \right\|^2} \sum_{k=1}^{m} e_k.$$

If the inequality holds in (6.19), then obviously (6.20) is valid, and the theorem is proved. ∎

Remark 5. *If in the above theorem the vectors $\{e_k\}_{k=\overline{1,m}}$ are assumed to be orthogonal, then (6.19) becomes:*

$$\sum_{i=1}^{n} \|x_i\| \le \frac{1}{m}\left(\sum_{k=1}^{m} \|e_k\|^2\right)^{\frac{1}{2}} \left\| \sum_{i=1}^{n} x_i \right\| + \frac{1}{m}\sum_{k=1}^{m}\sum_{i=1}^{n} M_{ik}. \tag{6.28}$$

Moreover, if $\{e_k\}_{k=\overline{1,m}}$ is an orthonormal family, then (6.28) becomes

$$\sum_{i=1}^{n} \|x_i\| \le \frac{\sqrt{m}}{m} \left\| \sum_{i=1}^{n} x_i \right\| + \frac{1}{m}\sum_{k=1}^{m}\sum_{i=1}^{n} M_{ik}, \tag{6.29}$$

which has been obtained in [**12**].

Before we provide some natural consequences of Theorem 12, we need some preliminary results concerning another reverse of Schwarz's inequality in inner product spaces (see for instance [**4**, p. 27]).

Lemma 3 (Dragomir, 2004). *Let $(X, \|\cdot\|)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $x, a \in H$, $r > 0$. If $\|x - a\| \le r$, then we have the inequality*

$$\|x\|\,\|a\| - \operatorname{Re}\langle x, a\rangle \le \frac{1}{2}r^2. \tag{6.30}$$

The case of equality holds in (6.30) if and only if

$$\|x - a\| = r \quad \text{and} \quad \|x\| = \|a\|. \tag{6.31}$$

Proof. The condition $\|x - a\| \le r$ is clearly equivalent to

$$\|x\|^2 + \|a\|^2 \le 2\operatorname{Re}\langle x, a\rangle + r^2. \tag{6.32}$$

Since

$$2\|x\|\,\|a\| \le \|x\|^2 + \|a\|^2, \tag{6.33}$$

with equality if and only if $\|x\| = \|a\|$, hence by (6.32) and (6.33) we deduce (6.30).

The case of equality is obvious. ∎

Utilising the above lemma we may state the following corollary of Theorem 12 [**6**].

Corollary 6. *Let $(H;\langle\cdot,\cdot\rangle)$, e_k, x_i be as in Theorem 12. If $r_{ik} > 0$, $i \in \{1,\dots,n\}$, $k \in \{1,\dots,m\}$ such that*

$$\|x_i - e_k\| \le r_{ik} \quad \text{for each } i \in \{1,\dots,n\} \text{ and } k \in \{1,\dots,m\}, \tag{6.34}$$

then we have the inequality

$$\sum_{i=1}^{n} \|x_i\| \le \left\| \frac{1}{m} \sum_{k=1}^{m} e_k \right\| \left\| \sum_{i=1}^{n} x_i \right\| + \frac{1}{2m} \sum_{k=1}^{m} \sum_{i=1}^{n} r_{ik}^2. \tag{6.35}$$

The equality holds in (6.35) if and only if

$$\sum_{i=1}^{n} \|x_i\| \ge \frac{1}{2m} \sum_{k=1}^{m} \sum_{i=1}^{n} r_{ik}^2$$

and

$$\sum_{i=1}^{n} x_i = \frac{m\left(\sum_{i=1}^{n}\|x_i\| - \frac{1}{2m}\sum_{k=1}^{m}\sum_{i=1}^{n} r_{ik}^2\right)}{\left\|\sum_{k=1}^{m} e_k\right\|^2} \sum_{k=1}^{m} e_k.$$

The following lemma may provide another sufficient condition for (6.18) to hold (see also [**4**, p. 28]).

Lemma 4 (Dragomir, 2004). *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $x, y \in H$, $M \ge m > 0$. If either*

$$\operatorname{Re}\langle My - x, x - my\rangle \ge 0 \tag{6.36}$$

or, equivalently,

$$\left\| x - \frac{m+M}{2} y \right\| \le \frac{1}{2}(M-m)\|y\|, \tag{6.37}$$

holds, then

$$\|x\|\|y\| - \operatorname{Re}\langle x, y\rangle \le \frac{1}{4} \cdot \frac{(M-m)^2}{m+M} \|y\|^2. \tag{6.38}$$

The case of equality holds in (6.38) if and only if the equality case is realised in (6.36) and

$$\|x\| = \frac{M+m}{2}\|y\|.$$

The proof is obvious by Lemma 3 for $a = \frac{M+m}{2} y$ and $r = \frac{1}{2}(M-m)\|y\|$.

Finally, the following corollary of Theorem 12 may be stated [**6**].

Corollary 7. *Assume that $(H,\langle\cdot,\cdot\rangle)$, e_k, x_i are as in Theorem 12. If $M_{ik} \ge m_{ik} > 0$ satisfy the condition*

$$\operatorname{Re}\langle M_k e_k - x_i, x_i - \mu_k e_k\rangle \ge 0$$

for each $i \in \{1,\dots,n\}$ and $k \in \{1,\dots,m\}$, then

$$\sum_{i=1}^{n} \|x_i\| \le \left\| \frac{1}{m} \sum_{k=1}^{m} e_k \right\| \left\| \sum_{i=1}^{n} x_i \right\| + \frac{1}{4m} \sum_{k=1}^{m} \sum_{i=1}^{n} \frac{(M_{ik} - m_{ik})^2}{M_{ik} + m_{ik}} \|e_k\|^2.$$

7. Other Additive Reverses for m Functionals

A different approach in obtaining other additive reverses for the generalised triangle inequality is incorporated in the following new result:

Theorem 13. *Let $(X, \|\cdot\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$. Assume F_k , $k \in \{1, \dots, m\}$, are bounded linear functionals on the normed linear space X and $x_i \in X$, $i \in \{1, \dots, n\}$, $M_{ik} \geq 0$, $i \in \{1, \dots, n\}$, $k \in \{1, \dots, m\}$ are such that*

$$\|x_i\| - \operatorname{Re} F_k(x_i) \leq M_{ik} \tag{7.1}$$

for each $i \in \{1, \dots, n\}$ and $k \in \{1, \dots, m\}$.

(i) *If c_∞ is defined by (c_∞), then we have the inequality*

$$\sum_{i=1}^{n} \|x_i\| \leq c_\infty \left\| \sum_{i=1}^{n} x_i \right\| + \frac{1}{m} \sum_{k=1}^{m} \sum_{i=1}^{n} M_{ik}. \tag{7.2}$$

(ii) *If c_p is defined by (c_p) for $p \geq 1$, then we have the inequality:*

$$\sum_{i=1}^{n} \|x_i\| \leq \frac{1}{m^{\frac{1}{p}}} c_p \left\| \sum_{i=1}^{n} x_i \right\| + \frac{1}{m} \sum_{k=1}^{m} \sum_{i=1}^{n} M_{ik}. \tag{7.3}$$

Proof. (i) Since

$$\max_{1 \leq k \leq m} \|F_k(x)\| \leq c_\infty \|x\| \quad \text{for any } x \in X,$$

then we have

$$\sum_{k=1}^{m} \left| F_k\left(\sum_{i=1}^{n} x_i \right) \right| \leq m \max_{1 \leq k \leq m} \left| F_k\left(\sum_{i=1}^{n} x_i \right) \right| \leq m c_\infty \left\| \sum_{i=1}^{n} x_i \right\|.$$

Using (6.16), we may state that

$$\frac{1}{m} \sum_{k=1}^{m} \operatorname{Re} F_k\left(\sum_{i=1}^{n} x_i \right) \leq c_\infty \left\| \sum_{i=1}^{n} x_i \right\|,$$

which, together with (6.15) imply the desired inequality (7.2).
(ii) Using the fact that, obviously

$$\left(\sum_{k=1}^{m} |F_k(x)|^p \right)^{\frac{1}{p}} \leq c_p \|x\| \quad \text{for any } x \in X,$$

then, by Hölder's inequality for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\begin{aligned} \sum_{k=1}^{m} \left| F_k\left(\sum_{i=1}^{n} x_i \right) \right| &\leq m^{\frac{1}{q}} \left(\sum_{k=1}^{m} \left| F_k\left(\sum_{i=1}^{n} x_i \right) \right|^p \right)^{\frac{1}{p}} \\ &\leq c_p m^{\frac{1}{q}} \left\| \sum_{i=1}^{n} x_i \right\|, \end{aligned}$$

which, combined with (6.15) and (6.16) will give the desired inequality (7.3).

The case $p = 1$ goes likewise and we omit the details. ∎

REMARK 6. *Since, obviously* $c_\infty \leq \max_{1\leq k\leq m} \|F_k\|$, *then from (7.2) we have*

$$\sum_{i=1}^{n}\|x_i\| \leq \max_{1\leq k\leq m}\{\|F_k\|\}\cdot\left\|\sum_{i=1}^{n}x_i\right\| + \frac{1}{m}\sum_{k=1}^{m}\sum_{i=1}^{n}M_{ik}. \tag{7.4}$$

Finally, since $c_p \leq \left(\sum_{k=1}^{m}\|F_k\|^p\right)^{\frac{1}{p}}, p \geq 1$, *hence by (7.3) we have*

$$\sum_{i=1}^{n}\|x_i\| \leq \left(\frac{\sum_{k=1}^{m}\|F_k\|^p}{m}\right)^{\frac{1}{p}}\left\|\sum_{i=1}^{n}x_i\right\| + \frac{1}{m}\sum_{k=1}^{m}\sum_{i=1}^{n}M_{ik}. \tag{7.5}$$

The following corollary for semi-inner products may be stated as well.

COROLLARY 8. *Let* $(X, \|\cdot\|)$ *be a real or complex normed space and* $[\cdot,\cdot] : X \times X \to \mathbb{K}$ *a semi-inner product generating the norm* $\|\cdot\|$. *Assume* e_k, $x_i \in H$ *and* $M_{ik} \geq 0$, $i \in \{1,\dots,n\}$, $k \in \{1,\dots,m\}$ *are such that*

$$\|x_i\| - \operatorname{Re}[x_i, e_k] \leq M_{ik}, \tag{7.6}$$

for any $i \in \{1,\dots,n\}$, $k \in \{1,\dots,m\}$.

(i) *If*

$$d_\infty := \sup_{x\neq 0}\left\{\frac{\max_{1\leq k\leq n}|[x,e_k]|}{\|x\|}\right\}\left(\leq \max_{1\leq k\leq n}\|e_k\|\right),$$

then

$$\sum_{i=1}^{n}\|x_i\| \leq d_\infty\left\|\sum_{i=1}^{n}x_i\right\| + \frac{1}{m}\sum_{k=1}^{m}\sum_{i=1}^{n}M_{ik} \tag{7.7}$$
$$\left(\leq \max_{1\leq k\leq n}\|e_k\|\cdot\left\|\sum_{i=1}^{n}x_i\right\| + \frac{1}{m}\sum_{k=1}^{m}\sum_{i=1}^{n}M_{ik}\right);$$

(ii) *If*

$$d_p := \sup_{x\neq 0}\left\{\frac{\sum_{k=1}^{m}|[x,e_k]|^p}{\|x\|^p}\right\}^{\frac{1}{p}}\left(\leq\left(\sum_{k=1}^{m}\|e_k\|^p\right)^{\frac{1}{p}}\right),$$

where $p \geq 1$, *then*

$$\sum_{i=1}^{n}\|x_i\| \leq \frac{1}{m^{\frac{1}{p}}}d_p\left\|\sum_{i=1}^{n}x_i\right\| + \frac{1}{m}\sum_{k=1}^{m}\sum_{i=1}^{n}M_{ik} \tag{7.8}$$
$$\left(\leq\left(\frac{\sum_{k=1}^{m}\|e_k\|^p}{m}\right)^{\frac{1}{p}}\left\|\sum_{i=1}^{n}x_i\right\| + \frac{1}{m}\sum_{k=1}^{m}\sum_{i=1}^{n}M_{ik}\right).$$

8. Applications for Complex Numbers

Let $\mathbb{C}$ be the field of complex numbers. If $z = \operatorname{Re} z + i\operatorname{Im} z$, then by $|\cdot|_p : \mathbb{C} \to [0,\infty)$, $p \in [1,\infty]$ we define the $p-$*modulus* of z as

$$|z|_p := \begin{cases} \max\{|\operatorname{Re} z|, |\operatorname{Im} z|\} & \text{if } p = \infty, \\ \\ (|\operatorname{Re} z|^p + |\operatorname{Im} z|^p)^{\frac{1}{p}} & \text{if } p \in [1,\infty), \end{cases}$$

where $|a|$, $a \in \mathbb{R}$ is the usual modulus of the real number a.

For $p=2$, we recapture the usual modulus of a complex number, i.e.,

$$|z|_2 = \sqrt{|\operatorname{Re} z|^2 + |\operatorname{Im} z|^2} = |z|, \quad z \in \mathbb{C}.$$

It is well known that $\left(\mathbb{C}, |\cdot|_p\right)$, $p \in [1,\infty]$ is a Banach space over the *real number field* $\mathbb{R}$.

Consider the Banach space $(\mathbb{C}, |\cdot|_1)$ and $F : \mathbb{C} \to \mathbb{C}$, $F(z) = az$ with $a \in \mathbb{C}$, $a \neq 0$. Obviously, F is linear on $\mathbb{C}$. For $z \neq 0$, we have

$$\frac{|F(z)|}{|z|_1} = \frac{|a|\,|z|}{|z|_1} = \frac{|a|\sqrt{|\operatorname{Re} z|^2 + |\operatorname{Im} z|^2}}{|\operatorname{Re} z| + |\operatorname{Im} z|} \le |a|.$$

Since, for $z_0 = 1$, we have $|F(z_0)| = |a|$ and $|z_0|_1 = 1$, hence

$$\|F\|_1 := \sup_{z \neq 0} \frac{|F(z)|}{|z|_1} = |a|,$$

showing that F is a bounded linear functional on $(\mathbb{C}, |\cdot|_1)$ and $\|F\|_1 = |a|$.

We can apply Theorem 5 to state the following reverse of the generalised triangle inequality for complex numbers [**5**].

Proposition 1. *Let a_k, $x_j \in \mathbb{C}$, $k \in \{1,\dots,m\}$ and $j \in \{1,\dots,n\}$. If there exist the constants $r_k \ge 0$, $k \in \{1,\dots,m\}$ with $\sum_{k=1}^m r_k > 0$ and*

$$r_k\left[|\operatorname{Re} x_j| + |\operatorname{Im} x_j|\right] \le \operatorname{Re} a_k \cdot \operatorname{Re} x_j - \operatorname{Im} a_k \cdot \operatorname{Im} x_j \tag{8.1}$$

for each $j \in \{1,\dots,n\}$ and $k \in \{1,\dots,m\}$, then

$$\sum_{j=1}^n \left[|\operatorname{Re} x_j| + |\operatorname{Im} x_j|\right] \le \frac{\left|\sum_{k=1}^m a_k\right|}{\sum_{k=1}^m r_k}\left[\left|\sum_{j=1}^n \operatorname{Re} x_j\right| + \left|\sum_{j=1}^n \operatorname{Im} x_j\right|\right]. \tag{8.2}$$

The case of equality holds in (8.2) if both

$$\begin{aligned}
&\operatorname{Re}\left(\sum_{k=1}^m a_k\right)\operatorname{Re}\left(\sum_{j=1}^n x_j\right) - \operatorname{Im}\left(\sum_{k=1}^m a_k\right)\operatorname{Im}\left(\sum_{j=1}^n x_j\right)\\
&= \left(\sum_{k=1}^m r_k\right)\sum_{j=1}^n \left[|\operatorname{Re} x_j| + |\operatorname{Im} x_j|\right]\\
&= \left|\sum_{k=1}^m a_k\right|\left[\left|\sum_{j=1}^n \operatorname{Re} x_j\right| + \left|\sum_{j=1}^n \operatorname{Im} x_j\right|\right].
\end{aligned}$$

The proof follows by Theorem 5 applied for the Banach space $(\mathbb{C}, |\cdot|_1)$ and $F_k(z) = a_k z$, $k \in \{1,\dots,m\}$ on taking into account that:

$$\left\|\sum_{k=1}^m F_k\right\|_1 = \left|\sum_{k=1}^m a_k\right|.$$

Now, consider the Banach space $(\mathbb{C}, |\cdot|_\infty)$. If $F(z) = dz$, then for $z \neq 0$ we have

$$\frac{|F(z)|}{|z|_\infty} = \frac{|d|\,|z|}{|z|_\infty} = \frac{|d|\sqrt{|\operatorname{Re} z|^2 + |\operatorname{Im} z|^2}}{\max\{|\operatorname{Re} z|, |\operatorname{Im} z|\}} \le \sqrt{2}\,|d|.$$

Since, for $z_0 = 1 + i$, we have $|F(z_0)| = \sqrt{2}\,|d|$, $|z_0|_\infty = 1$, hence

$$\|F\|_\infty := \sup_{z\neq 0} \frac{|F(z)|}{|z|_\infty} = \sqrt{2}\,|d|,$$

showing that F is a bounded linear functional on $(\mathbb{C}, |\cdot|_\infty)$ and $\|F\|_\infty = \sqrt{2}\,|d|$.

If we apply Theorem 5, then we can state the following reverse of the generalised triangle inequality for complex numbers [**5**].

Proposition 2. *Let $a_k,\ x_j \in \mathbb{C}$, $k \in \{1,\dots,m\}$ and $j \in \{1,\dots,n\}$. If there exist the constants $r_k \geq 0$, $k \in \{1,\dots,m\}$ with $\sum_{k=1}^m r_k > 0$ and*

$$r_k \max\{|\operatorname{Re} x_j|, |\operatorname{Im} x_j|\} \leq \operatorname{Re} a_k \cdot \operatorname{Re} x_j - \operatorname{Im} a_k \cdot \operatorname{Im} x_j$$

for each $j \in \{1,\dots,n\}$ and $k \in \{1,\dots,m\}$, then

$$\sum_{j=1}^n \max\{|\operatorname{Re} x_j|, |\operatorname{Im} x_j|\} \leq \sqrt{2}\cdot \frac{\left|\sum_{k=1}^m a_k\right|}{\sum_{k=1}^m r_k} \max\left\{\left|\sum_{j=1}^n \operatorname{Re} x_j\right|, \left|\sum_{j=1}^n \operatorname{Im} x_j\right|\right\}. \tag{8.3}$$

The case of equality holds in (8.3) if both

$$\begin{aligned}
&\operatorname{Re}\left(\sum_{k=1}^m a_k\right)\operatorname{Re}\left(\sum_{j=1}^n x_j\right) - \operatorname{Im}\left(\sum_{k=1}^m a_k\right)\operatorname{Im}\left(\sum_{j=1}^n x_j\right)\\
&= \left(\sum_{k=1}^m r_k\right)\sum_{j=1}^n \max\{|\operatorname{Re} x_j|, |\operatorname{Im} x_j|\}\\
&= \sqrt{2}\left|\sum_{k=1}^m a_k\right| \max\left\{\left|\sum_{j=1}^n \operatorname{Re} x_j\right|, \left|\sum_{j=1}^n \operatorname{Im} x_j\right|\right\}.
\end{aligned}$$

Finally, consider the Banach space $\left(\mathbb{C}, |\cdot|_{2p}\right)$ with $p \geq 1$.

Let $F : \mathbb{C} \to \mathbb{C}$, $F(z) = cz$. By Hölder's inequality, we have

$$\frac{|F(z)|}{|z|_{2p}} = \frac{|c|\sqrt{|\operatorname{Re} z|^2 + |\operatorname{Im} z|^2}}{\left(|\operatorname{Re} z|^{2p} + |\operatorname{Im} z|^{2p}\right)^{\frac{1}{2p}}} \leq 2^{\frac{1}{2}-\frac{1}{2p}}\,|c|.$$

Since, for $z_0 = 1 + i$ we have $|F(z_0)| = 2^{\frac{1}{2}}\,|c|$, $|z_0|_{2p} = 2^{\frac{1}{2p}}$ $(p \geq 1)$, hence

$$\|F\|_{2p} := \sup_{z\neq 0} \frac{|F(z)|}{|z|_{2p}} = 2^{\frac{1}{2}-\frac{1}{2p}}\,|c|,$$

showing that F is a bounded linear functional on $\left(\mathbb{C}, |\cdot|_{2p}\right)$, $p \geq 1$ and $\|F\|_{2p} = 2^{\frac{1}{2}-\frac{1}{2p}}\,|c|$.

If we apply Theorem 5, then we can state the following proposition [**5**].

Proposition 3. *Let* $a_k,\ x_j \in \mathbb{C},\ k \in \{1,\dots,m\}$ *and* $j \in \{1,\dots,n\}$. *If there exist the constants* $r_k \geq 0,\ k \in \{1,\dots,m\}$ *with* $\sum_{k=1}^m r_k > 0$ *and*

$$r_k \left[|\operatorname{Re} x_j|^{2p} + |\operatorname{Im} x_j|^{2p} \right]^{\frac{1}{2p}} \leq \operatorname{Re} a_k \cdot \operatorname{Re} x_j - \operatorname{Im} a_k \cdot \operatorname{Im} x_j$$

for each $j \in \{1,\dots,n\}$ *and* $k \in \{1,\dots,m\}$, *then*

$$\sum_{j=1}^n \left[|\operatorname{Re} x_j|^{2p} + |\operatorname{Im} x_j|^{2p} \right]^{\frac{1}{2p}} \leq 2^{\frac{1}{2}-\frac{1}{2p}} \frac{\left|\sum_{k=1}^m a_k\right|}{\sum_{k=1}^m r_k} \left[\left|\sum_{j=1}^n \operatorname{Re} x_j\right|^{2p} + \left|\sum_{j=1}^n \operatorname{Im} x_j\right|^{2p} \right]^{\frac{1}{2p}}. \tag{8.4}$$

The case of equality holds in (8.4) if both:

$$\begin{aligned} &\operatorname{Re}\left(\sum_{k=1}^m a_k\right) \operatorname{Re}\left(\sum_{j=1}^n x_j\right) - \operatorname{Im}\left(\sum_{k=1}^m a_k\right) \operatorname{Im}\left(\sum_{j=1}^n x_j\right) \\ &= \left(\sum_{k=1}^m r_k\right) \sum_{j=1}^n \left[|\operatorname{Re} x_j|^{2p} + |\operatorname{Im} x_j|^{2p} \right]^{\frac{1}{2p}} \\ &= 2^{\frac{1}{2}-\frac{1}{2p}} \left|\sum_{k=1}^m a_k\right| \left[\left|\sum_{j=1}^n \operatorname{Re} x_j\right|^{2p} + \left|\sum_{j=1}^n \operatorname{Im} x_j\right|^{2p} \right]^{\frac{1}{2p}}. \end{aligned}$$

Remark 7. *If in the above proposition we choose* $p = 1$, *then we have the following reverse of the generalised triangle inequality for complex numbers*

$$\sum_{j=1}^n |x_j| \leq \frac{\left|\sum_{k=1}^m a_k\right|}{\sum_{k=1}^m r_k} \left|\sum_{j=1}^n x_j\right|$$

provided $x_j, a_k,\ j \in \{1,\dots,n\},\ k \in \{1,\dots,m\}$ *satisfy the assumption*

$$r_k |x_j| \leq \operatorname{Re} a_k \cdot \operatorname{Re} x_j - \operatorname{Im} a_k \cdot \operatorname{Im} x_j$$

for each $j \in \{1,\dots,n\},\ k \in \{1,\dots,m\}$. *Here* $|\cdot|$ *is the usual modulus of a complex number and* $r_k > 0,\ k \in \{1,\dots,m\}$ *are given.*

We can apply Theorem 11 to state the following reverse of the generalised triangle inequality for complex numbers [**6**].

Proposition 4. *Let* $a_k,\ x_j \in \mathbb{C},\ k \in \{1,\dots,m\}$ *and* $j \in \{1,\dots,n\}$. *If there exist the constants* $M_{jk} \geq 0,\ k \in \{1,\dots,m\},\ j \in \{1,\dots,n\}$ *such that*

$$|\operatorname{Re} x_j| + |\operatorname{Im} x_j| \leq \operatorname{Re} a_k \cdot \operatorname{Re} x_j - \operatorname{Im} a_k \cdot \operatorname{Im} x_j + M_{jk} \tag{8.5}$$

for each $j \in \{1, \ldots, n\}$ *and* $k \in \{1, \ldots, m\}$, *then*

$$\sum_{j=1}^{n} [|\operatorname{Re} x_j| + |\operatorname{Im} x_j|] \le \frac{1}{m} \left| \sum_{k=1}^{m} a_k \right| \left[\left| \sum_{j=1}^{n} \operatorname{Re} x_j \right| + \left| \sum_{j=1}^{n} \operatorname{Im} x_j \right| \right] + \frac{1}{m} \sum_{k=1}^{m} \sum_{j=1}^{n} M_{jk}. \tag{8.6}$$

The proof follows by Theorem 11 applied for the Banach space $(\mathbb{C}, |\cdot|_1)$ and $F_k(z) = a_k z$, $k \in \{1, \ldots, m\}$ on taking into account that:

$$\left\| \sum_{k=1}^{m} F_k \right\|_1 = \left| \sum_{k=1}^{m} a_k \right|.$$

If we apply Theorem 11 for the Banach space $(\mathbb{C}, |\cdot|_\infty)$, then we can state the following reverse of the generalised triangle inequality for complex numbers [**6**].

PROPOSITION 5. *Let* a_k, $x_j \in \mathbb{C}$, $k \in \{1, \ldots, m\}$ *and* $j \in \{1, \ldots, n\}$. *If there exist the constants* $M_{jk} \ge 0$, $k \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$ *such that*

$$\max\{|\operatorname{Re} x_j|, |\operatorname{Im} x_j|\} \le \operatorname{Re} a_k \cdot \operatorname{Re} x_j - \operatorname{Im} a_k \cdot \operatorname{Im} x_j + M_{jk}$$

for each $j \in \{1, \ldots, n\}$ *and* $k \in \{1, \ldots, m\}$, *then*

$$\sum_{j=1}^{n} \max\{|\operatorname{Re} x_j|, |\operatorname{Im} x_j|\} \le \frac{\sqrt{2}}{m} \left| \sum_{k=1}^{m} a_k \right| \max\left\{ \left| \sum_{j=1}^{n} \operatorname{Re} x_j \right|, \left| \sum_{j=1}^{n} \operatorname{Im} x_j \right| \right\} + \frac{1}{m} \sum_{k=1}^{m} \sum_{j=1}^{n} M_{jk}. \tag{8.7}$$

Finally, if we apply Theorem 11, for the Banach space $\left(\mathbb{C}, |\cdot|_{2p}\right)$ with $p \ge 1$, then we can state the following proposition [**6**].

PROPOSITION 6. *Let* a_k, x_j, M_{jk} *be as in Proposition 5. If*

$$\left[|\operatorname{Re} x_j|^{2p} + |\operatorname{Im} x_j|^{2p} \right]^{\frac{1}{2p}} \le \operatorname{Re} a_k \cdot \operatorname{Re} x_j - \operatorname{Im} a_k \cdot \operatorname{Im} x_j + M_{jk}$$

for each $j \in \{1, \ldots, n\}$ *and* $k \in \{1, \ldots, m\}$, *then*

$$\sum_{j=1}^{n} \left[|\operatorname{Re} x_j|^{2p} + |\operatorname{Im} x_j|^{2p} \right]^{\frac{1}{2p}} \le \frac{2^{\frac{1}{2} - \frac{1}{2p}}}{m} \left| \sum_{k=1}^{m} a_k \right| \left[\left| \sum_{j=1}^{n} \operatorname{Re} x_j \right|^{2p} + \left| \sum_{j=1}^{n} \operatorname{Im} x_j \right|^{2p} \right]^{\frac{1}{2p}} + \frac{1}{m} \sum_{k=1}^{m} \sum_{j=1}^{n} M_{jk}. \tag{8.8}$$

where $p \ge 1$.

REMARK 8. *If in the above proposition we choose* $p = 1$, *then we have the following reverse of the generalised triangle inequality for complex numbers*

$$\sum_{j=1}^{n} |x_j| \le \left| \frac{1}{m} \sum_{k=1}^{m} a_k \right| \left| \sum_{j=1}^{n} x_j \right| + \frac{1}{m} \sum_{k=1}^{m} \sum_{j=1}^{n} M_{jk}$$

provided x_j, a_k, $j \in \{1, \ldots, n\}$, $k \in \{1, \ldots, m\}$ *satisfy the assumption*

$$|x_j| \leq \operatorname{Re} a_k \cdot \operatorname{Re} x_j - \operatorname{Im} a_k \cdot \operatorname{Im} x_j + M_{jk}$$

for each $j \in \{1, \ldots, n\}$, $k \in \{1, \ldots, m\}$. *Here* $|\cdot|$ *is the usual modulus of a complex number and* $M_{jk} > 0, j \in \{1, \ldots, n\}$, $k \in \{1, \ldots, m\}$ *are given.*

9. Karamata Type Inequalities in Hilbert Spaces

Let $f : [a, b] \to \mathbb{K}$, $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ be a Lebesgue integrable function. The following inequality, which is the continuous version of the *triangle inequality*

$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx, \tag{9.1}$$

plays a fundamental role in Mathematical Analysis and its applications.

It appears, see [**20**, p. 492], that the first reverse inequality for (9.1) was obtained by J. Karamata in his book from 1949, [**14**]. It can be stated as

$$\cos\theta \int_a^b |f(x)|\,dx \leq \left| \int_a^b f(x)\,dx \right| \tag{9.2}$$

provided

$$-\theta \leq \arg f(x) \leq \theta, \quad x \in [a, b]$$

for given $\theta \in \left(0, \frac{\pi}{2}\right)$.

This result has recently been extended by the author for the case of Bochner integrable functions with values in a Hilbert space H (see also [**10**]):

Theorem 14 (Dragomir, 2004). *If* $f \in L([a, b]; H)$ *(this means that* $f : [a, b] \to H$ *is Bochner measurable on* $[a, b]$ *and the Lebesgue integral* $\int_a^b \|f(t)\|\,dt$ *is finite), then*

$$\int_a^b \|f(t)\|\,dt \leq K \left\| \int_a^b f(t)\,dt \right\|, \tag{9.3}$$

provided that f *satisfies the condition*

$$\|f(t)\| \leq K \operatorname{Re}\langle f(t), e\rangle \quad \text{for a.e. } t \in [a, b], \tag{9.4}$$

where $e \in H$, $\|e\| = 1$ *and* $K \geq 1$ *are given.*

The case of equality holds in (9.4) if and only if

$$\int_a^b f(t)\,dt = \frac{1}{K}\left(\int_a^b \|f(t)\|\,dt\right) e. \tag{9.5}$$

As some natural consequences of the above results, we have noticed in [**10**] that, if $\rho \in [0, 1)$ and $f \in L([a, b]; H)$ are such that

$$\|f(t) - e\| \leq \rho \quad \text{for a.e. } t \in [a, b], \tag{9.6}$$

then

$$\sqrt{1 - \rho^2} \int_a^b \|f(t)\|\,dt \leq \left\| \int_a^b f(t)\,dt \right\| \tag{9.7}$$

with equality if and only if

$$\int_a^b f(t)\,dt = \sqrt{1-\rho^2}\left(\int_a^b \|f(t)\|\,dt\right)\cdot e.$$

Also, for e as above and if $M \geq m > 0$, $f \in L([a,b];H)$ such that either

$$\operatorname{Re}\langle Me - f(t), f(t) - me\rangle \geq 0 \tag{9.8}$$

or, equivalently,

$$\left\| f(t) - \frac{M+m}{2}e \right\| \leq \frac{1}{2}(M-m) \tag{9.9}$$

for a.e. $t \in [a,b]$, then

$$\int_a^b \|f(t)\|\,dt \leq \frac{M+m}{2\sqrt{mM}} \left\| \int_a^b f(t)\,dt \right\|, \tag{9.10}$$

with equality if and only if

$$\int_a^b f(t)\,dt = \frac{2\sqrt{mM}}{M+m}\left(\int_a^b \|f(t)\|\,dt\right)\cdot e.$$

The main aim of the following sections is to extend the integral inequalities mentioned above for the case of Banach spaces. Applications for Hilbert spaces and for complex-valued functions are given as well.

10. Multiplicative Reverses of the Continuous Triangle Inequality

10.1. The Case of One Functional. Let $(X, \|\cdot\|)$ be a Banach space over the real or complex number field. Then one has the following reverse of the continuous triangle inequality [**11**].

THEOREM 15 (Dragomir, 2004). *Let F be a continuous linear functional of unit norm on X. Suppose that the function $f : [a,b] \to X$ is Bochner integrable on $[a,b]$ and there exists a $r \geq 0$ such that*

$$r\|f(t)\| \leq \operatorname{Re} F(f(t)) \quad \text{for a.e. } t \in [a,b]. \tag{10.1}$$

Then

$$r\int_a^b \|f(t)\|\,dt \leq \left\| \int_a^b f(t)\,dt \right\|, \tag{10.2}$$

where equality holds in (10.2) if and only if both

$$F\left(\int_a^b f(t)\,dt\right) = r\int_a^b \|f(t)\|\,dt \tag{10.3}$$

and

$$F\left(\int_a^b f(t)\,dt\right) = \left\| \int_a^b f(t)\,dt \right\|. \tag{10.4}$$

Proof. Since the norm of F is one, then

$$|F(x)| \leq \|x\| \quad \text{for any } \; x \in X.$$

Applying this inequality for the vector $\int_a^b f(t)\,dt$, we get

$$\left\|\int_a^b f(t)\,dt\right\| \geq \left|F\left(\int_a^b f(t)\,dt\right)\right| \geq \left|\operatorname{Re} F\left(\int_a^b f(t)\,dt\right)\right| = \left|\int_a^b \operatorname{Re} F(f(t))\,dt\right|. \tag{10.5}$$

Now, by integration of (10.1), we obtain

$$\int_a^b \operatorname{Re} F(f(t))\,dt \geq r \int_a^b \|f(t)\|\,dt, \tag{10.6}$$

and by (10.5) and (10.6) we deduce the desired inequality (10.1).

Obviously, if (10.3) and (10.4) hold true, then the equality case holds in (10.2).

Conversely, if the case of equality holds in (10.2), then it must hold in all the inequalities used before in proving this inequality. Therefore, we must have

$$r\|f(t)\| = \operatorname{Re} F(f(t)) \quad \text{for a.e. } \; t \in [a,b], \tag{10.7}$$

$$\operatorname{Im} F\left(\int_a^b f(t)\,dt\right) = 0 \tag{10.8}$$

and

$$\left\|\int_a^b f(t)\,dt\right\| = \operatorname{Re} F\left(\int_a^b f(t)\,dt\right). \tag{10.9}$$

Integrating (10.7) on $[a,b]$, we get

$$r\int_a^b \|f(t)\|\,dt = \operatorname{Re} F\left(\int_a^b f(t)\,dt\right). \tag{10.10}$$

On utilising (10.10) and (10.8), we deduce (10.3) while (10.9) and (10.10) would imply (10.4), and the theorem is proved. ∎

Corollary 9. *Let $(X, \|\cdot\|)$ be a Banach space, $[\cdot,\cdot] : X \times X \to \mathbb{R}$ a semi-inner product generating the norm $\|\cdot\|$ and $e \in X$, $\|e\| = 1$. Suppose that the function $f : [a,b] \to X$ is Bochner integrable on $[a,b]$ and there exists a $r \geq 0$ such that*

$$r\|f(t)\| \leq \operatorname{Re}[f(t), e] \quad \textit{for a.e. } \; t \in [a,b]. \tag{10.11}$$

Then

$$r\int_a^b \|f(t)\|\,dt \leq \left\|\int_a^b f(t)\,dt\right\| \tag{10.12}$$

where equality holds in (10.12) if and only if both

$$\left[\int_a^b f(t)\,dt, e\right] = r\int_a^b \|f(t)\|\,dt \tag{10.13}$$

and

$$\left[\int_a^b f(t)\,dt, e\right] = \left\|\int_a^b f(t)\,dt\right\|. \tag{10.14}$$

The proof follows from Theorem 15 for the continuous linear functional $F(x) = [x, e]$, $x \in X$, and we omit the details.

The following corollary of Theorem 15 may be stated [**8**].

COROLLARY 10. *Let $(X, \|\cdot\|)$ be a strictly convex Banach space, $[\cdot,\cdot] : X \times X \to \mathbb{K}$ a semi-inner product generating the norm $\|\cdot\|$ and $e \in X$, $\|e\| = 1$. If $f : [a,b] \to X$ is Bochner integrable on $[a,b]$ and there exists a $r \geq 0$ such that (10.11) holds true, then (10.12) is valid. The case of equality holds in (10.12) if and only if*

$$\int_a^b f(t)\,dt = r\left(\int_a^b \|f(t)\|\,dt\right)e. \tag{10.15}$$

PROOF. If (10.15) holds true, then, obviously

$$\left\|\int_a^b f(t)\,dt\right\| = r\left(\int_a^b \|f(t)\|\,dt\right)\|e\| = r\int_a^b \|f(t)\|\,dt,$$

which is the equality case in (10.12).

Conversely, if the equality holds in (10.12), then, by Corollary 9, we must have (10.13) and (10.14). Utilising Theorem 7, by (10.14) we can conclude that there exists a $\mu > 0$ such that

$$\int_a^b f(t)\,dt = \mu e. \tag{10.16}$$

Replacing this in (10.13), we get

$$\mu\|e\|^2 = r\int_a^b \|f(t)\|\,dt,$$

giving

$$\mu = r\int_a^b \|f(t)\|\,dt. \tag{10.17}$$

Utilising (10.16) and (10.17) we deduce (10.15) and the proof is completed. ∎

10.2. The Case of m Functionals. The following result may be stated [**8**]:

THEOREM 16 (Dragomir, 2004). *Let $(X, \|\cdot\|)$ be a Banach space over the real or complex number field $\mathbb{K}$ and $F_k : X \to \mathbb{K}$, $k \in \{1, \dots, m\}$ continuous linear functionals on X. If $f : [a,b] \to X$ is a Bochner integrable function on $[a,b]$ and there exists $r_k \geq 0$, $k \in \{1, \dots, m\}$ with $\sum_{k=1}^m r_k > 0$ and*

$$r_k\|f(t)\| \leq \operatorname{Re} F_k(f(t)) \tag{10.18}$$

for each $k \in \{1, \dots, m\}$ and a.e. $t \in [a,b]$, then

$$\int_a^b \|f(t)\|\,dt \leq \frac{\left\|\sum_{k=1}^m F_k\right\|}{\sum_{k=1}^m r_k}\left\|\int_a^b f(t)\,dt\right\|. \tag{10.19}$$

The case of equality holds in (10.19) if both

$$\left(\sum_{k=1}^{m} F_k\right)\left(\int_a^b f(t)\,dt\right) = \left(\sum_{k=1}^{m} r_k\right)\int_a^b \|f(t)\|\,dt \tag{10.20}$$

and

$$\left(\sum_{k=1}^{m} F_k\right)\left(\int_a^b f(t)\,dt\right) = \left\|\sum_{k=1}^{m} F_k\right\|\left\|\int_a^b f(t)\,dt\right\|. \tag{10.21}$$

Proof. Utilising the hypothesis (10.18), we have

$$\begin{aligned} I := \left|\sum_{k=1}^{m} F_k\left(\int_a^b f(t)\,dt\right)\right| &\geq \left|\operatorname{Re}\left[\sum_{k=1}^{m} F_k\left(\int_a^b f(t)\,dt\right)\right]\right| \\ &\geq \operatorname{Re}\left[\sum_{k=1}^{m} F_k\left(\int_a^b f(t)\,dt\right)\right] = \sum_{k=1}^{m}\left(\int_a^b \operatorname{Re} F_k f(t)\,dt\right) \\ &\geq \left(\sum_{k=1}^{m} r_k\right)\cdot\int_a^b \|f(t)\|\,dt. \end{aligned} \tag{10.22}$$

On the other hand, by the continuity property of F_k, $k \in \{1,\dots,m\}$, we obviously have

$$I = \left|\left(\sum_{k=1}^{m} F_k\right)\left(\int_a^b f(t)\,dt\right)\right| \leq \left\|\sum_{k=1}^{m} F_k\right\|\left\|\int_a^b f(t)\,dt\right\|. \tag{10.23}$$

Making use of (10.22) and (10.23), we deduce (10.19).

Now, obviously, if (10.20) and (10.21) are valid, then the case of equality holds true in (10.19).

Conversely, if the equality holds in the inequality (10.19), then it must hold in all the inequalities used to prove (10.19), therefore we have

$$r_k \|f(t)\| = \operatorname{Re} F_k(f(t)) \tag{10.24}$$

for each $k \in \{1,\dots,m\}$ and a.e. $t \in [a,b]$,

$$\operatorname{Im}\left(\sum_{k=1}^{m} F_k\right)\left(\int_a^b f(t)\,dt\right) = 0, \tag{10.25}$$

$$\operatorname{Re}\left(\sum_{k=1}^{m} F_k\right)\left(\int_a^b f(t)\,dt\right) = \left\|\sum_{k=1}^{m} F_k\right\|\left\|\int_a^b f(t)\,dt\right\|. \tag{10.26}$$

Note that, by (10.24), on integrating on $[a,b]$ and summing over $k \in \{1,\dots,m\}$, we get

$$\operatorname{Re}\left(\sum_{k=1}^{m} F_k\right)\left(\int_a^b f(t)\,dt\right) = \left(\sum_{k=1}^{m} r_k\right)\int_a^b \|f(t)\|\,dt. \tag{10.27}$$

Now, (10.25) and (10.27) imply (10.20) while (10.25) and (10.26) imply (10.21), therefore the theorem is proved. ∎

The following new results may be stated as well:

THEOREM 17. *Let $(X, \|\cdot\|)$ be a Banach space over the real or complex number field $\mathbb{K}$ and $F_k : X \to \mathbb{K}$, $k \in \{1, \dots, m\}$ continuous linear functionals on X. Also, assume that $f : [a, b] \to X$ is a Bochner integrable function on $[a, b]$ and there exists $r_k \geq 0$, $k \in \{1, \dots, m\}$ with $\sum_{k=1}^{m} r_k > 0$ and*

$$r_k \|f(t)\| \leq \operatorname{Re} F_k(f(t))$$

for each $k \in \{1, \dots, m\}$ and a.e. $t \in [a, b]$.

(i) If c_∞ is defined by (c_∞), then we have the inequality

$$(1 \leq) \frac{\int_a^b \|f(t)\| \, dt}{\left\| \int_a^b f(t) \, dt \right\|} \leq \frac{c_\infty}{\max_{1 \leq k \leq m} \{r_k\}} \left(\leq \frac{\max_{1 \leq k \leq m} \|F_k\|}{\max_{1 \leq k \leq m} \{r_k\}} \right) \tag{10.28}$$

with equality if and only if

$$\operatorname{Re}(F_k) \left(\int_a^b f(t) \, dt \right) = r_k \int_a^b \|f(t)\| \, dt$$

for each $k \in \{1, \dots, m\}$ and

$$\max_{1 \leq k \leq m} \left[\operatorname{Re}(F_k) \left(\int_a^b f(t) \, dt \right) \right] = c_\infty \int_a^b \|f(t)\| \, dt.$$

(ii) If $c_p, p \geq 1$, is defined by (c_p), then we have the inequality

$$(1 \leq) \frac{\int_a^b \|f(t)\| \, dt}{\left\| \int_a^b f(t) \, dt \right\|} \leq \frac{c_p}{\left(\sum_{k=1}^{m} r_k^p \right)^{\frac{1}{p}}} \left(\leq \frac{\sum_{k=1}^{m} \|F_k\|^p}{\sum_{k=1}^{m} r_k^p} \right)^{\frac{1}{p}}$$

with equality if and only if

$$\operatorname{Re}(F_k) \left(\int_a^b f(t) \, dt \right) = r_k \int_a^b \|f(t)\| \, dt$$

for each $k \in \{1, \dots, m\}$ and

$$\sum_{k=1}^{m} \left[\operatorname{Re} F_k \left(\int_a^b f(t) \, dt \right) \right]^p = c_p^p \left\| \int_a^b f(t) \, dt \right\|^p$$

where $p \geq 1$.

The proof is similar to the ones from Theorems 8, 9 and 16 and we omit the details.

The case of Hilbert spaces which provides a simpler condition for equality is of interest for applications [**8**].

THEOREM 18 (Dragomir, 2004). *Let $(X, \|\cdot\|)$ be a Hilbert space over the real or complex number field $\mathbb{K}$ and $e_k \in H \setminus \{0\}$, $k \in \{1, \dots, m\}$. If $f : [a, b] \to H$ is a Bochner integrable function and $r_k \geq 0$, $k \in \{1, \dots, m\}$ and $\sum_{k=1}^{m} r_k > 0$ satisfy*

$$r_k \|f(t)\| \leq \operatorname{Re} \langle f(t), e_k \rangle \tag{10.29}$$

for each $k \in \{1, \dots, m\}$ and for a.e. $t \in [a, b]$, then

$$\int_a^b \|f(t)\| \, dt \leq \frac{\left\| \sum_{k=1}^{m} e_k \right\|}{\sum_{k=1}^{m} r_k} \left\| \int_a^b f(t) \, dt \right\|. \tag{10.30}$$

The case of equality holds in (10.30) for $f \neq 0$ a.e. on $[a,b]$ if and only if

$$\int_a^b f(t)\,dt = \frac{\left(\sum_{k=1}^m r_k\right)\int_a^b \|f(t)\|\,dt}{\left\|\sum_{k=1}^m e_k\right\|^2}\sum_{k=1}^m e_k. \tag{10.31}$$

Proof. Utilising the hypothesis (10.29) and the modulus properties, we have

$$\begin{aligned} \left|\left\langle \int_a^b f(t)\,dt, \sum_{k=1}^m e_k \right\rangle\right| &\geq \left|\sum_{k=1}^m \operatorname{Re}\left\langle \int_a^b f(t)\,dt, e_k \right\rangle\right| \\ &\geq \sum_{k=1}^m \operatorname{Re}\left\langle \int_a^b f(t)\,dt, e_k \right\rangle \\ &= \sum_{k=1}^m \int_a^b \operatorname{Re}\langle f(t), e_k\rangle\,dt \\ &\geq \left(\sum_{k=1}^m r_k\right)\int_a^b \|f(t)\|\,dt. \end{aligned} \tag{10.32}$$

By Schwarz's inequality in Hilbert spaces applied for $\int_a^b f(t)\,dt$ and $\sum_{k=1}^m e_k$, we have

$$\left\|\int_a^b f(t)\,dt\right\|\left\|\sum_{k=1}^m e_k\right\| \geq \left|\left\langle \int_a^b f(t)\,dt, \sum_{k=1}^m e_k \right\rangle\right|. \tag{10.33}$$

Making use of (10.32) and (10.33), we deduce (10.30).

Now, if $f \neq 0$ a.e. on $[a,b]$, then $\int_a^b \|f(t)\|\,dt \neq 0$ and by (10.32) $\sum_{k=1}^m e_k \neq 0$. Obviously, if (10.31) is valid, then taking the norm we have

$$\begin{aligned} \left\|\int_a^b f(t)\,dt\right\| &= \frac{\left(\sum_{k=1}^m r_k\right)\int_a^b \|f(t)\|\,dt}{\left\|\sum_{k=1}^m e_k\right\|^2}\left\|\sum_{k=1}^m e_k\right\| \\ &= \frac{\sum_{k=1}^m r_k}{\left\|\sum_{k=1}^m e_k\right\|}\int_a^b \|f(t)\|\,dt, \end{aligned}$$

i.e., the case of equality holds true in (10.30).

Conversely, if the equality case holds true in (10.30), then it must hold in all the inequalities used to prove (10.30), therefore we have

$$\operatorname{Re}\langle f(t), e_k\rangle = r_k \|f(t)\| \tag{10.34}$$

for each $k \in \{1,\dots,m\}$ and a.e. $t \in [a,b]$,

$$\left\|\int_a^b f(t)\,dt\right\|\left\|\sum_{k=1}^m e_k\right\| = \left|\left\langle \int_a^b f(t)\,dt, \sum_{k=1}^m e_k \right\rangle\right|, \tag{10.35}$$

and

$$\operatorname{Im}\left\langle \int_a^b f(t)\,dt, \sum_{k=1}^m e_k \right\rangle = 0. \tag{10.36}$$

From (10.34) on integrating on $[a,b]$ and summing over k from 1 to m, we get

$$\operatorname{Re}\left\langle \int_a^b f(t)\,dt, \sum_{k=1}^m e_k \right\rangle = \left(\sum_{k=1}^m r_k\right)\int_a^b \|f(t)\|\,dt, \tag{10.37}$$

and then, by (10.36) and (10.37), we have

$$\left\langle \int_a^b f(t)\,dt, \sum_{k=1}^m e_k \right\rangle = \left(\sum_{k=1}^m r_k\right) \int_a^b \|f(t)\|\,dt. \tag{10.38}$$

On the other hand, by the use of the identity (3.22), the relation (10.35) holds true if and only if

$$\int_a^b f(t)\,dt = \frac{\left\langle \int_a^b f(t)\,dt, \sum_{k=1}^m e_k \right\rangle}{\left\|\sum_{k=1}^m e_k\right\|} \sum_{k=1}^m e_k. \tag{10.39}$$

Finally, by (10.38) and (10.39) we deduce that (10.31) is also necessary for the equality case in (10.30) and the theorem is proved. ∎

Remark 9. *If $\{e_k\}_{k\in\{1,\dots,m\}}$ are orthogonal, then (10.30) can be replaced by*

$$\int_a^b \|f(t)\|\,dt \le \frac{\left(\sum_{k=1}^m \|e_k\|^2\right)^{\frac{1}{2}}}{\sum_{k=1}^m r_k} \left\|\int_a^b f(t)\,dt\right\|, \tag{10.40}$$

with equality if and only if

$$\int_a^b f(t)\,dt = \frac{\left(\sum_{k=1}^m r_k\right)\int_a^b \|f(t)\|\,dt}{\sum_{k=1}^m \|e_k\|^2} \sum_{k=1}^m e_k. \tag{10.41}$$

Moreover, if $\{e_k\}_{k\in\{1,\dots,m\}}$ are orthonormal, then (10.40) becomes

$$\int_a^b \|f(t)\|\,dt \le \frac{\sqrt{m}}{\sum_{k=1}^m r_k} \left\|\int_a^b f(t)\,dt\right\|, \tag{10.42}$$

with equality if and only if

$$\int_a^b f(t)\,dt = \frac{1}{m}\left(\sum_{k=1}^m r_k\right)\left(\int_a^b \|f(t)\|\,dt\right)\sum_{k=1}^m e_k. \tag{10.43}$$

The following corollary of Theorem 18 may be stated as well [**8**].

Corollary 11. *Let $(H;\langle\cdot,\cdot\rangle)$ be a Hilbert space over the real or complex number field $\mathbb{K}$ and $e_k \in H\setminus\{0\}$, $k \in \{1,\dots,m\}$. If $f : [a,b] \to H$ is a Bochner integrable function on $[a,b]$ and $\rho_k > 0$, $k \in \{1,\dots,m\}$ with*

$$\|f(t) - e_k\| \le \rho_k < \|e_k\| \tag{10.44}$$

for each $k \in \{1,\dots,m\}$ and a.e. $t \in [a,b]$, then

$$\int_a^b \|f(t)\|\,dt \le \frac{\left\|\sum_{k=1}^m e_k\right\|}{\sum_{k=1}^m \left(\|e_k\|^2 - \rho_k^2\right)^{\frac{1}{2}}} \left\|\int_a^b f(t)\,dt\right\|. \tag{10.45}$$

The case of equality holds in (10.45) if and only if

$$\int_a^b f(t)\,dt = \frac{\sum_{k=1}^m \left(\|e_k\|^2 - \rho_k^2\right)^{\frac{1}{2}}}{\left\|\sum_{k=1}^m e_k\right\|^2}\left(\int_a^b \|f(t)\|\,dt\right)\sum_{k=1}^m e_k. \tag{10.46}$$

Proof. Utilising Lemma 1, we have from (10.44) that

$$\|f(t)\|\left(\|e_k\|^2-\rho_k^2\right)^{\frac{1}{2}}\leq \operatorname{Re}\langle f(t),e_k\rangle$$

for any $k\in\{1,\ldots,m\}$ and a.e. $t\in[a,b]$.

Applying Theorem 18 for

$$r_k:=\left(\|e_k\|^2-\rho_k^2\right)^{\frac{1}{2}},\quad k\in\{1,\ldots,m\},$$

we deduce the desired result. ∎

Remark 10. *If $\{e_k\}_{k\in\{1,\ldots,m\}}$ are orthogonal, then (10.45) becomes*

$$\int_a^b\|f(t)\|\,dt\leq\frac{\left(\sum_{k=1}^m\|e_k\|^2\right)^{\frac{1}{2}}}{\sum_{k=1}^m\left(\|e_k\|^2-\rho_k^2\right)^{\frac{1}{2}}}\left\|\int_a^b f(t)\,dt\right\|,\tag{10.47}$$

with equality if and only if

$$\int_a^b f(t)\,dt=\frac{\sum_{k=1}^m\left(\|e_k\|^2-\rho_k^2\right)^{\frac{1}{2}}}{\sum_{k=1}^m\|e_k\|^2}\left(\int_a^b\|f(t)\|\,dt\right)\sum_{k=1}^m e_k.\tag{10.48}$$

Moreover, if $\{e_k\}_{k\in\{1,\ldots,m\}}$ is assumed to be orthonormal and

$$\|f(t)-e_k\|\leq\rho_k\quad\text{for a.e. } t\in[a,b],$$

where $\rho_k\in[0,1)$, $k\in\{1,\ldots,m\}$, then

$$\int_a^b\|f(t)\|\,dt\leq\frac{\sqrt{m}}{\sum_{k=1}^m\left(1-\rho_k^2\right)^{\frac{1}{2}}}\left\|\int_a^b f(t)\,dt\right\|,\tag{10.49}$$

with equality iff

$$\int_a^b f(t)\,dt=\frac{\sum_{k=1}^m\left(1-\rho_k^2\right)^{\frac{1}{2}}}{m}\left(\int_a^b\|f(t)\|\,dt\right)\sum_{k=1}^m e_k.\tag{10.50}$$

Finally, we may state the following corollary of Theorem 18 [**11**].

Corollary 12. *Let $(H;\langle\cdot,\cdot\rangle)$ be a Hilbert space over the real or complex number field $\mathbb{K}$ and $e_k\in H\setminus\{0\}$, $k\in\{1,\ldots,m\}$. If $f:[a,b]\to H$ is a Bochner integrable function on $[a,b]$ and $M_k\geq\mu_k>0$, $k\in\{1,\ldots,m\}$ are such that either*

$$\operatorname{Re}\langle M_ke_k-f(t),f(t)-\mu_ke_k\rangle\geq 0\tag{10.51}$$

or, equivalently

$$\left\|f(t)-\frac{M_k+\mu_k}{2}e_k\right\|\leq\frac{1}{2}(M_k-\mu_k)\|e_k\|\tag{10.52}$$

for each $k\in\{1,\ldots,m\}$ and a.e. $t\in[a,b]$, then

$$\int_a^b\|f(t)\|\,dt\leq\frac{\left\|\sum_{k=1}^m e_k\right\|}{\sum_{k=1}^m\frac{2\cdot\sqrt{\mu_kM_k}}{\mu_k+M_k}\|e_k\|}\left\|\int_a^b f(t)\,dt\right\|.\tag{10.53}$$

The case of equality holds if and only if

$$\int_a^b f(t)\,dt = \frac{\sum_{k=1}^m \frac{2\cdot\sqrt{\mu_k M_k}}{\mu_k+M_k}\|e_k\|}{\left\|\sum_{k=1}^m e_k\right\|^2}\left(\int_a^b \|f(t)\|\,dt\right)\cdot\sum_{k=1}^m e_k.$$

Proof. Utilising Lemma 2, by (10.51) we deduce

$$\|f(t)\|\frac{2\cdot\sqrt{\mu_k M_k}}{\mu_k+M_k}\|e_k\| \le \operatorname{Re}\langle f(t), e_k\rangle$$

for each $k\in\{1,\dots,m\}$ and a.e. $t\in[a,b]$.

Applying Theorem 18 for

$$r_k := \frac{2\cdot\sqrt{\mu_k M_k}}{\mu_k+M_k}\|e_k\|,\quad k\in\{1,\dots,m\}$$

we deduce the desired result. ∎

11. Additive Reverses of the Continuous Triangle Inequality

11.1. The Case of One Functional. The aim of this section is to provide a different approach to the problem of reversing the continuous triangle inequality. Namely, we are interested in finding upper bounds for the positive difference

$$\int_a^b \|f(t)\|\,dt - \left\|\int_a^b f(t)\,dt\right\|$$

under various assumptions for the Bochner integrable function $f:[a,b]\to X$.

In the following we provide an additive reverse for the continuous triangle inequality that has been established in [**8**].

Theorem 19 (Dragomir, 2004). *Let $(X,\|\cdot\|)$ be a Banach space over the real or complex number field $\mathbb{K}$ and $F:X\to\mathbb{K}$ be a continuous linear functional of unit norm on X. Suppose that the function $f:[a,b]\to X$ is Bochner integrable on $[a,b]$ and there exists a Lebesgue integrable function $k:[a,b]\to[0,\infty)$ such that*

$$\|f(t)\| - \operatorname{Re} F[f(t)] \le k(t) \tag{11.1}$$

for a.e. $t\in[a,b]$. Then we have the inequality

$$(0\le)\int_a^b \|f(t)\|\,dt - \left\|\int_a^b f(t)\,dt\right\| \le \int_a^b k(t)\,dt. \tag{11.2}$$

The equality holds in (11.2) if and only if both

$$F\left(\int_a^b f(t)\,dt\right) = \left\|\int_a^b f(t)\,dt\right\| \tag{11.3}$$

and

$$F\left(\int_a^b f(t)\,dt\right) = \int_a^b \|f(t)\|\,dt - \int_a^b k(t)\,dt. \tag{11.4}$$

Proof. Since the norm of F is unity, then

$$|F(x)| \leq \|x\| \quad \text{for any} \quad x \in X.$$

Applying this inequality for the vector $\int_a^b f(t)\,dt$, we get

$$\begin{aligned} \left\| \int_a^b f(t)\,dt \right\| &\geq \left| F\left(\int_a^b f(t)\,dt \right) \right| \\ &\geq \left| \operatorname{Re} F\left(\int_a^b f(t)\,dt \right) \right| \\ &= \left| \int_a^b \operatorname{Re} F[f(t)]\,dt \right| \geq \int_a^b \operatorname{Re} F[f(t)]\,dt. \end{aligned} \tag{11.5}$$

Integrating (11.1), we have

$$\int_a^b \|f(t)\|\,dt - \operatorname{Re} F\left(\int_a^b f(t)\,dt \right) \leq \int_a^b k(t)\,dt. \tag{11.6}$$

Now, making use of (11.5) and (11.6), we deduce (11.2).

Obviously, if the equality hold in (11.3) and (11.4), then it holds in (11.2) as well. Conversely, if the equality holds in (11.2), then it must hold in all the inequalities used to prove (11.2). Therefore, we have

$$\int_a^b \|f(t)\|\,dt = \operatorname{Re}\left[F\left(\int_a^b f(t)\,dt \right) \right] + \int_a^b k(t)\,dt.$$

and

$$\operatorname{Re}\left[F\left(\int_a^b f(t)\,dt \right) \right] = \left| F\left(\int_a^b f(t)\,dt \right) \right| = \left\| \int_a^b f(t)\,dt \right\|$$

which imply (11.3) and (11.4). ∎

Corollary 13. *Let $(X, \|\cdot\|)$ be a Banach space, $[\cdot,\cdot] : X \times X \to \mathbb{K}$ a semi-inner product which generates its norm. If $e \in X$ is such that $\|e\| = 1$, $f : [a,b] \to X$ is Bochner integrable on $[a,b]$ and there exists a Lebesgue integrable function $k : [a,b] \to [0,\infty)$ such that*

$$(0 \leq)\ \|f(t)\| - \operatorname{Re}[f(t), e] \leq k(t), \tag{11.7}$$

for a.e. $t \in [a,b]$, then

$$(0 \leq) \int_a^b \|f(t)\|\,dt - \left\| \int_a^b f(t)\,dt \right\| \leq \int_a^b k(t)\,dt. \tag{11.8}$$

where equality holds in (11.8) if and only if both

$$\begin{aligned} \left[\int_a^b f(t)\,dt, e \right] &= \left\| \int_a^b f(t)\,dt \right\| \quad \text{and} \\ \left[\int_a^b f(t)\,dt, e \right] &= \left\| \int_a^b f(t)\,dt \right\| - \int_a^b k(t)\,dt. \end{aligned} \tag{11.9}$$

The proof is obvious by Theorem 19 applied for the continuous linear functional of unit norm $F_e : X \to \mathbb{K}$, $F_e(x) = [x, e]$.

The following corollary may be stated.

Corollary 14. *Let $(X, \|\cdot\|)$ be a strictly convex Banach space, and $[\cdot,\cdot]$, e, f, k as in Corollary 13. Then the case of equality holds in (11.8) if and only if*

$$\int_a^b \|f(t)\| \, dt \geq \int_a^b k(t) \, dt \tag{11.10}$$

and

$$\int_a^b f(t) \, dt = \left(\int_a^b \|f(t)\| \, dt - \int_a^b k(t) \, dt \right) e. \tag{11.11}$$

Proof. Suppose that (11.10) and (11.11) are valid. Taking the norm on (11.11) we have

$$\left\| \int_a^b f(t) \, dt \right\| = \left| \int_a^b \|f(t)\| \, dt - \int_a^b k(t) \, dt \right| \|e\| = \int_a^b \|f(t)\| \, dt - \int_a^b k(t) \, dt,$$

and the case of equality holds true in (11.8).

Now, if the equality case holds in (11.8), then obviously (11.10) is valid, and by Corollary 13,

$$\left[\int_a^b f(t) \, dt, e \right] = \left\| \int_a^b f(t) \, dt \right\| \|e\|.$$

Utilising Theorem 7, we get

$$\int_a^b f(t) \, dt = \lambda e \quad \text{with} \quad \lambda > 0. \tag{11.12}$$

Replacing $\int_a^b f(t) \, dt$ with λe in the second equation of (11.9) we deduce

$$\lambda = \int_a^b \|f(t)\| \, dt - \int_a^b k(t) \, dt, \tag{11.13}$$

and by (11.12) and (11.13) we deduce (11.11). ∎

Remark 11. *If $X = H$, $(H; \langle \cdot, \cdot \rangle)$ is a Hilbert space, then from Corollary 14 we deduce the additive reverse inequality obtained in* [**7**]. *For further similar results in Hilbert spaces, see* [**7**] *and* [**9**].

11.2. The Case of m Functionals. The following result may be stated [**8**]:

Theorem 20 (Dragomir, 2004). *Let $(X, \|\cdot\|)$ be a Banach space over the real or complex number field $\mathbb{K}$ and $F_k : X \to \mathbb{K}$, $k \in \{1, \dots, m\}$ continuous linear functionals on X. If $f : [a,b] \to X$ is a Bochner integrable function on $[a,b]$ and $M_k : [a,b] \to [0,\infty)$, $k \in \{1,\dots,m\}$ are Lebesgue integrable functions such that*

$$\|f(t)\| - \operatorname{Re} F_k [f(t)] \leq M_k(t) \tag{11.14}$$

for each $k \in \{1,\dots,m\}$ and a.e. $t \in [a,b]$, then

$$\int_a^b \|f(t)\| \, dt \leq \left\| \frac{1}{m} \sum_{k=1}^m F_k \right\| \left\| \int_a^b f(t) \, dt \right\| + \frac{1}{m} \sum_{k=1}^m \int_a^b M_k(t) \, dt. \tag{11.15}$$

The case of equality holds in (11.15) if and only if both

$$\frac{1}{m}\sum_{k=1}^{m}F_k\left(\int_a^b f(t)\,dt\right)=\left\|\frac{1}{m}\sum_{k=1}^{m}F_k\right\|\left\|\int_a^b f(t)\,dt\right\| \tag{11.16}$$

and

$$\frac{1}{m}\sum_{k=1}^{m}F_k\left(\int_a^b f(t)\,dt\right)=\int_a^b\|f(t)\|\,dt-\frac{1}{m}\sum_{k=1}^{m}\int_a^b M_k(t)\,dt. \tag{11.17}$$

PROOF. If we integrate on $[a,b]$ and sum over k from 1 to m, we deduce

$$\int_a^b\|f(t)\|\,dt\le\frac{1}{m}\sum_{k=1}^{m}\operatorname{Re}\left[F_k\left(\int_a^b f(t)\,dt\right)\right]+\frac{1}{m}\sum_{k=1}^{m}\int_a^b M_k(t)\,dt. \tag{11.18}$$

Utilising the continuity property of the functionals F_k and the properties of the modulus, we have:

$$\begin{aligned}\sum_{k=1}^{m}\operatorname{Re}F_k\left(\int_a^b f(t)\,dt\right)&\le\left|\sum_{k=1}^{m}\operatorname{Re}\left[F_k\left(\int_a^b f(t)\,dt\right)\right]\right|\\&\le\left|\sum_{k=1}^{m}F_k\left(\int_a^b f(t)\,dt\right)\right|\\&\le\left\|\sum_{k=1}^{m}F_k\right\|\left\|\int_a^b f(t)\,dt\right\|.\end{aligned} \tag{11.19}$$

Now, by (11.18) and (11.19) we deduce (11.15).

Obviously, if (11.16) and (11.17) hold true, then the case of equality is valid in (11.15).

Conversely, if the case of equality holds in (11.15), then it must hold in all the inequalities used to prove (11.15). Therefore, we have

$$\int_a^b\|f(t)\|\,dt=\frac{1}{m}\sum_{k=1}^{m}\operatorname{Re}\left[F_k\left(\int_a^b f(t)\,dt\right)\right]+\frac{1}{m}\sum_{k=1}^{m}\int_a^b M_k(t)\,dt,$$

$$\sum_{k=1}^{m}\operatorname{Re}\left[F_k\left(\int_a^b f(t)\,dt\right)\right]=\left\|\int_a^b f(t)\,dt\right\|\left\|\sum_{k=1}^{m}F_k\right\|$$

and

$$\sum_{k=1}^{m}\operatorname{Im}\left[F_k\left(\int_a^b f(t)\,dt\right)\right]=0.$$

These imply that (11.16) and (11.17) hold true, and the theorem is completely proved. ∎

REMARK 12. *If F_k, $k\in\{1,\ldots,m\}$ are of unit norm, then, from (11.15) we deduce the inequality*

$$\int_a^b\|f(t)\|\,dt\le\left\|\int_a^b f(t)\,dt\right\|+\frac{1}{m}\sum_{k=1}^{m}\int_a^b M_k(t)\,dt, \tag{11.20}$$

which is obviously coarser than (11.15) but, perhaps more useful for applications.

The following new result may be stated as well:

THEOREM 21. *Let $(X, \|\cdot\|)$ be a Banach space over the real or complex number field $\mathbb{K}$ and $F_k : X \to \mathbb{K}$, $k \in \{1, \dots, m\}$ continuous linear functionals on X. Assume also that $f : [a, b] \to X$ is a Bochner integrable function on $[a, b]$ and $M_k : [a, b] \to [0, \infty)$, $k \in \{1, \dots, m\}$ are Lebesgue integrable functions such that*

$$\|f(t)\| - \operatorname{Re} F_k[f(t)] \leq M_k(t) \tag{11.21}$$

for each $k \in \{1, \dots, m\}$ and a.e. $t \in [a, b]$.

(i) If c_∞ is defined by (c_∞), then we have the inequality

$$\int_a^b \|f(t)\| \, dt \leq c_\infty \left\| \int_a^b f(t) \, dt \right\| + \frac{1}{m} \sum_{k=1}^{m} \int_a^b M_k(t) \, dt. \tag{11.22}$$

(ii) If $c_p, p \geq 1$, is defined by (c_p), then we have the inequality

$$\int_a^b \|f(t)\| \, dt \leq \frac{c_p}{m^{1/p}} \left\| \int_a^b f(t) \, dt \right\| + \frac{1}{m} \sum_{k=1}^{m} \int_a^b M_k(t) \, dt.$$

The proof is similar to the ones from Theorem 13 and 20 and we omit the details.

The case of Hilbert spaces, in which one may provide a simpler condition for equality, is of interest in applications [**8**].

THEOREM 22 (Dragomir, 2004). *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space over the real or complex number field $\mathbb{K}$ and $e_k \in H$, $k \in \{1, \dots, m\}$. If $f : [a, b] \to H$ is a Bochner integrable function on $[a, b]$, $f(t) \neq 0$ for a.e. $t \in [a, b]$ and $M_k : [a, b] \to [0, \infty)$, $k \in \{1, \dots, m\}$ is a Lebesgue integrable function such that*

$$\|f(t)\| - \operatorname{Re} \langle f(t), e_k \rangle \leq M_k(t) \tag{11.23}$$

for each $k \in \{1, \dots, m\}$ and for a.e. $t \in [a, b]$, then

$$\int_a^b \|f(t)\| \, dt \leq \left\| \frac{1}{m} \sum_{k=1}^{m} e_k \right\| \left\| \int_a^b f(t) \, dt \right\| + \frac{1}{m} \sum_{k=1}^{m} \int_a^b M_k(t) \, dt. \tag{11.24}$$

The case of equality holds in (11.24) if and only if

$$\int_a^b \|f(t)\| \, dt \geq \frac{1}{m} \sum_{k=1}^{m} \int_a^b M_k(t) \, dt \tag{11.25}$$

and

$$\int_a^b f(t) \, dt = \frac{m \left(\int_a^b \|f(t)\| \, dt - \frac{1}{m} \sum_{k=1}^{m} \int_a^b M_k(t) \, dt \right)}{\left\| \sum_{k=1}^{m} e_k \right\|^2} \sum_{k=1}^{m} e_k. \tag{11.26}$$

PROOF. As in the proof of Theorem 20, we have

$$\int_a^b \|f(t)\| \, dt \leq \operatorname{Re} \left\langle \frac{1}{m} \sum_{k=1}^{m} e_k, \int_a^b f(t) \, dt \right\rangle + \frac{1}{m} \sum_{k=1}^{m} \int_a^b M_k(t) \, dt \tag{11.27}$$

and $\sum_{k=1}^{m} e_k \neq 0$.

On utilising Schwarz's inequality in Hilbert space $(H, \langle \cdot, \cdot \rangle)$ for $\int_a^b f(t)\,dt$ and $\sum_{k=1}^m e_k$, we have

$$\left\| \int_a^b f(t)\,dt \right\| \left\| \sum_{k=1}^m e_k \right\| \geq \left| \left\langle \int_a^b f(t)\,dt, \sum_{k=1}^m e_k \right\rangle \right| \geq \left| \operatorname{Re} \left\langle \int_a^b f(t)\,dt, \sum_{k=1}^m e_k \right\rangle \right| \geq \operatorname{Re} \left\langle \int_a^b f(t)\,dt, \sum_{k=1}^m e_k \right\rangle . \tag{11.28}$$

By (11.27) and (11.28), we deduce (11.24).

Taking the norm on (11.26) and using (11.25), we have

$$\left\| \int_a^b f(t)\,dt \right\| = \frac{m \left(\int_a^b \|f(t)\|\,dt - \frac{1}{m} \sum_{k=1}^m \int_a^b M_k(t)\,dt \right)}{\left\| \sum_{k=1}^m e_k \right\|},$$

showing that the equality holds in (11.24).

Conversely, if the equality case holds in (11.24), then it must hold in all the inequalities used to prove (11.24). Therefore we have

$$\|f(t)\| = \operatorname{Re} \langle f(t), e_k \rangle + M_k(t) \tag{11.29}$$

for each $k \in \{1, \dots, m\}$ and for a.e. $t \in [a, b]$,

$$\left\| \int_a^b f(t)\,dt \right\| \left\| \sum_{k=1}^m e_k \right\| = \left| \left\langle \int_a^b f(t)\,dt, \sum_{k=1}^m e_k \right\rangle \right| \tag{11.30}$$

and

$$\operatorname{Im} \left\langle \int_a^b f(t)\,dt, \sum_{k=1}^m e_k \right\rangle = 0. \tag{11.31}$$

From (11.29) on integrating on $[a, b]$ and summing over k, we get

$$\operatorname{Re} \left\langle \int_a^b f(t)\,dt, \sum_{k=1}^m e_k \right\rangle = m \int_a^b \|f(t)\|\,dt - \sum_{k=1}^m \int_a^b M_k(t)\,dt. \tag{11.32}$$

On the other hand, by the use of the identity (3.22), the relation (11.30) holds if and only if

$$\int_a^b f(t)\,dt = \frac{\left\langle \int_a^b f(t)\,dt, \sum_{k=1}^m e_k \right\rangle}{\left\| \sum_{k=1}^m e_k \right\|^2} \sum_{k=1}^m e_k,$$

giving, from (11.31) and (11.32), that (11.26) holds true.

If the equality holds in (11.24), then obviously (11.25) is valid and the theorem is proved. ∎

Remark 13. *If in the above theorem, the vectors $\{e_k\}_{k \in \{1, \dots, m\}}$ are assumed to be orthogonal, then (11.24) becomes*

$$\int_a^b \|f(t)\|\,dt \leq \frac{1}{m} \left(\sum_{k=1}^m \|e_k\|^2 \right)^{\frac{1}{2}} \left\| \int_a^b f(t)\,dt \right\| + \frac{1}{m} \sum_{k=1}^m \int_a^b M_k(t)\,dt. \tag{11.33}$$

Moreover, if $\{e_k\}_{k\in\{1,\dots,m\}}$ *is an orthonormal family, then (11.33) becomes*

$$\int_a^b \|f(t)\|\,dt \leq \frac{1}{\sqrt{m}}\left\|\int_a^b f(t)\,dt\right\| + \frac{1}{m}\sum_{k=1}^m \int_a^b M_k(t)\,dt \tag{11.34}$$

which has been obtained in [**4**].

The following corollaries are of interest.

Corollary 15. *Let* $(H;\langle\cdot,\cdot\rangle)$, e_k, $k\in\{1,\dots,m\}$ *and* f *be as in Theorem 22. If* $r_k:[a,b]\to[0,\infty)$, $k\in\{1,\dots,m\}$ *are such that* $r_k\in L^2[a,b]$, $k\in\{1,\dots,m\}$ *and*

$$\|f(t)-e_k\| \leq r_k(t), \tag{11.35}$$

for each $k\in\{1,\dots,m\}$ *and a.e.* $t\in[a,b]$, *then*

$$\int_a^b \|f(t)\|\,dt \leq \left\|\frac{1}{m}\sum_{k=1}^m e_k\right\|\left\|\int_a^b f(t)\,dt\right\| + \frac{1}{2m}\sum_{k=1}^m \int_a^b r_k^2(t)\,dt. \tag{11.36}$$

The case of equality holds in (11.36) if and only if

$$\int_a^b \|f(t)\|\,dt \geq \frac{1}{2m}\sum_{k=1}^m \int_a^b r_k^2(t)\,dt$$

and

$$\int_a^b f(t)\,dt = \frac{m\left(\int_a^b \|f(t)\|\,dt - \frac{1}{2m}\sum_{k=1}^m \int_a^b r_k^2(t)\,dt\right)}{\left\|\sum_{k=1}^m e_k\right\|^2}\sum_{k=1}^m e_k.$$

Finally, the following corollary may be stated.

Corollary 16. *Let* $(H;\langle\cdot,\cdot\rangle)$, e_k, $k\in\{1,\dots,m\}$ *and* f *be as in Theorem 22. If* $M_k,\mu_k:[a,b]\to\mathbb{R}$ *are such that* $M_k\geq\mu_k>0$ *a.e. on* $[a,b]$, $\frac{(M_k-\mu_k)^2}{M_k+\mu_k}\in L[a,b]$ *and*

$$\operatorname{Re}\langle M_k(t)e_k - f(t), f(t)-\mu_k(t)e_k\rangle \geq 0$$

for each $k\in\{1,\dots,m\}$ *and for a.e.* $t\in[a,b]$, *then*

$$\begin{aligned}\int_a^b \|f(t)\|\,dt &\leq \left\|\frac{1}{m}\sum_{k=1}^m e_k\right\|\left\|\int_a^b f(t)\,dt\right\| \\ &\quad + \frac{1}{4m}\sum_{k=1}^m \|e_k\|^2 \int_a^b \frac{[M_k(t)-\mu_k(t)]^2}{M_k(t)+\mu_k(t)}dt.\end{aligned}$$

12. Applications for Complex-Valued Functions

We now give some examples of inequalities for complex-valued functions that are Lebesgue integrable on using the general result obtained in Section 10.

Consider the Banach space $(\mathbb{C},|\cdot|_1)$ and $F:\mathbb{C}\to\mathbb{C}$, $F(z)=ez$ with $e=\alpha+i\beta$ and $|e|^2=\alpha^2+\beta^2=1$, then F is linear on $\mathbb{C}$. For $z\neq 0$, we have

$$\frac{|F(z)|}{|z|_1} = \frac{|e|\,|z|}{|z|_1} = \frac{\sqrt{|\operatorname{Re} z|^2+|\operatorname{Im} z|^2}}{|\operatorname{Re} z|+|\operatorname{Im} z|} \leq 1.$$

Since, for $z_0 = 1$, we have $|F(z_0)| = 1$ and $|z_0|_1 = 1$, hence

$$\|F\|_1 := \sup_{z \neq 0} \frac{|F(z)|}{|z|_1} = 1,$$

showing that F is a bounded linear functional on $(\mathbb{C}, |\cdot|_1)$.

Therefore we can apply Theorem 15 to state the following result for complex-valued functions.

Proposition 7. *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha^2 + \beta^2 = 1$, $f : [a,b] \to \mathbb{C}$ be a Lebesgue integrable function on $[a,b]$ and $r \geq 0$ such that*

$$r[|\operatorname{Re} f(t)| + |\operatorname{Im} f(t)|] \leq \alpha \operatorname{Re} f(t) - \beta \operatorname{Im} f(t) \tag{12.1}$$

for a.e. $t \in [a,b]$. Then

$$r\left[\int_a^b |\operatorname{Re} f(t)|\, dt + \int_a^b |\operatorname{Im} f(t)|\, dt\right] \leq \left|\int_a^b \operatorname{Re} f(t)\, dt\right| + \left|\int_a^b \operatorname{Im} f(t)\, dt\right|. \tag{12.2}$$

The equality holds in (12.2) if and only if both

$$\alpha \int_a^b \operatorname{Re} f(t)\, dt - \beta \int_a^b \operatorname{Im} f(t)\, dt = r\left[\int_a^b |\operatorname{Re} f(t)|\, dt + \int_a^b |\operatorname{Im} f(t)|\, dt\right]$$

and

$$\alpha \int_a^b \operatorname{Re} f(t)\, dt - \beta \int_a^b \operatorname{Im} f(t)\, dt = \left|\int_a^b \operatorname{Re} f(t)\, dt\right| + \left|\int_a^b \operatorname{Im} f(t)\, dt\right|.$$

Now, consider the Banach space $(\mathbb{C}, |\cdot|_\infty)$. If $F(z) = dz$ with $d = \gamma + i\delta$ and $|d| = \frac{\sqrt{2}}{2}$, i.e., $\gamma^2 + \delta^2 = \frac{1}{2}$, then F is linear on $\mathbb{C}$. For $z \neq 0$ we have

$$\frac{|F(z)|}{|z|_\infty} = \frac{|d|\,|z|}{|z|_\infty} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{|\operatorname{Re} z|^2 + |\operatorname{Im} z|^2}}{\max\{|\operatorname{Re} z|, |\operatorname{Im} z|\}} \leq 1.$$

Since, for $z_0 = 1 + i$, we have $|F(z_0)| = 1$, $|z_0|_\infty = 1$, hence

$$\|F\|_\infty := \sup_{z \neq 0} \frac{|F(z)|}{|z|_\infty} = 1,$$

showing that F is a bounded linear functional of unit norm on $(\mathbb{C}, |\cdot|_\infty)$.

Therefore, we can apply Theorem 15, to state the following result for complex-valued functions.

Proposition 8. *Let $\gamma, \delta \in \mathbb{R}$ with $\gamma^2 + \delta^2 = \frac{1}{2}$, $f : [a,b] \to \mathbb{C}$ be a Lebesgue integrable function on $[a,b]$ and $r \geq 0$ such that*

$$r \max\{|\operatorname{Re} f(t)|, |\operatorname{Im} f(t)|\} \leq \gamma \operatorname{Re} f(t) - \delta \operatorname{Im} f(t)$$

for a.e. $t \in [a,b]$. Then

$$r \int_a^b \max\{|\operatorname{Re} f(t)|, |\operatorname{Im} f(t)|\}\, dt \leq \max\left\{\left|\int_a^b \operatorname{Re} f(t)\, dt\right|, \left|\int_a^b \operatorname{Im} f(t)\, dt\right|\right\}. \tag{12.3}$$

The equality holds in (12.3) if and only if both

$$\gamma \int_a^b \operatorname{Re} f(t)\,dt - \delta \int_a^b \operatorname{Im} f(t)\,dt = r \int_a^b \max\{|\operatorname{Re} f(t)|, |\operatorname{Im} f(t)|\}\,dt$$

and

$$\gamma \int_a^b \operatorname{Re} f(t)\,dt - \delta \int_a^b \operatorname{Im} f(t)\,dt = \max\left\{\left|\int_a^b \operatorname{Re} f(t)\,dt\right|, \left|\int_a^b \operatorname{Im} f(t)\,dt\right|\right\}.$$

Now, consider the Banach space $\left(\mathbb{C}, |\cdot|_{2p}\right)$ with $p \geq 1$. Let $F : \mathbb{C} \to \mathbb{C}$, $F(z) = cz$ with $|c| = 2^{\frac{1}{2p}-\frac{1}{2}}$ $(p \geq 1)$. Obviously, F is linear and by Hölder's inequality

$$\frac{|F(z)|}{|z|_{2p}} = \frac{2^{\frac{1}{2p}-\frac{1}{2}}\sqrt{|\operatorname{Re} z|^2 + |\operatorname{Im} z|^2}}{\left(|\operatorname{Re} z|^{2p} + |\operatorname{Im} z|^{2p}\right)^{\frac{1}{2p}}} \leq 1.$$

Since, for $z_0 = 1 + i$ we have $|F(z_0)| = 2^{\frac{1}{p}}$, $|z_0|_{2p} = 2^{\frac{1}{2p}}$ $(p \geq 1)$, hence

$$\|F\|_{2p} := \sup_{z \neq 0} \frac{|F(z)|}{|z|_{2p}} = 1,$$

showing that F is a bounded linear functional of unit norm on $\left(\mathbb{C}, |\cdot|_{2p}\right)$, $(p \geq 1)$. Therefore on using Theorem 15, we may state the following result.

Proposition 9. *Let $\varphi, \phi \in \mathbb{R}$ with $\varphi^2 + \phi^2 = 2^{\frac{1}{2p}-\frac{1}{2}}$ $(p \geq 1)$, $f : [a,b] \to \mathbb{C}$ be a Lebesgue integrable function on $[a,b]$ and $r \geq 0$ such that*

$$r\left[|\operatorname{Re} f(t)|^{2p} + |\operatorname{Im} f(t)|^{2p}\right]^{\frac{1}{2p}} \leq \varphi \operatorname{Re} f(t) - \phi \operatorname{Im} f(t)$$

for a.e. $t \in [a,b]$, then

$$r \int_a^b \left[|\operatorname{Re} f(t)|^{2p} + |\operatorname{Im} f(t)|^{2p}\right]^{\frac{1}{2p}} dt \leq \left[\left|\int_a^b \operatorname{Re} f(t)\,dt\right|^{2p} + \left|\int_a^b \operatorname{Im} f(t)\,dt\right|^{2p}\right]^{\frac{1}{2p}}, \quad (p \geq 1) \tag{12.4}$$

where equality holds in (12.4) if and only if both

$$\varphi \int_a^b \operatorname{Re} f(t)\,dt - \phi \int_a^b \operatorname{Im} f(t)\,dt = r \int_a^b \left[|\operatorname{Re} f(t)|^{2p} + |\operatorname{Im} f(t)|^{2p}\right]^{\frac{1}{2p}} dt$$

and

$$\varphi \int_a^b \operatorname{Re} f(t)\,dt - \phi \int_a^b \operatorname{Im} f(t)\,dt = \left[\left|\int_a^b \operatorname{Re} f(t)\,dt\right|^{2p} + \left|\int_a^b \operatorname{Im} f(t)\,dt\right|^{2p}\right]^{\frac{1}{2p}}.$$

Remark 14. *If $p = 1$ above, and*

$$r|f(t)| \leq \varphi \operatorname{Re} f(t) - \psi \operatorname{Im} f(t) \quad \text{for a.e. } t \in [a,b],$$

provided $\varphi,\ \psi \in \mathbb{R}$ *and* $\varphi^2+\psi^2=1, r\geq 0$, *then we have a reverse of the classical continuous triangle inequality for modulus:*

$$r\int_a^b |f(t)|\,dt \leq \left|\int_a^b f(t)\,dt\right|,$$

with equality iff

$$\varphi\int_a^b \operatorname{Re} f(t)\,dt-\psi\int_a^b \operatorname{Im} f(t)\,dt=r\int_a^b |f(t)|\,dt$$

and

$$\varphi\int_a^b \operatorname{Re} f(t)\,dt-\psi\int_a^b \operatorname{Im} f(t)\,dt=\left|\int_a^b f(t)\,dt\right|.$$

If we apply Theorem 19, then, in a similar manner we can prove the following result for complex-valued functions.

Proposition 10. *Let* $\alpha,\beta\in\mathbb{R}$ *with* $\alpha^2+\beta^2=1,\ f,k:[a,b]\to\mathbb{C}$ *Lebesgue integrable functions such that*

$$|\operatorname{Re} f(t)|+|\operatorname{Im} f(t)|\leq \alpha\operatorname{Re} f(t)-\beta\operatorname{Im} f(t)+k(t)$$

for a.e. $t\in[a,b]$. *Then*

$$(0\leq)\int_a^b |\operatorname{Re} f(t)|\,dt+\int_a^b |\operatorname{Im} f(t)|\,dt-\left[\left|\int_a^b \operatorname{Re} f(t)\,dt\right|+\left|\int_a^b \operatorname{Im} f(t)\,dt\right|\right] \leq \int_a^b k(t)\,dt.$$

Applying Theorem 19, for $(\mathbb{C},|\cdot|_\infty)$ we may state:

Proposition 11. *Let* $\gamma,\delta\in\mathbb{R}$ *with* $\gamma^2+\delta^2=\frac{1}{2},\ f,k:[a,b]\to\mathbb{C}$ *Lebesgue integrable functions on* $[a,b]$ *such that*

$$\max\{|\operatorname{Re} f(t)|,|\operatorname{Im} f(t)|\}\leq \gamma\operatorname{Re} f(t)-\delta\operatorname{Im} f(t)+k(t)$$

for a.e. $t\in[a,b]$. *Then*

$$(0\leq)\int_a^b \max\{|\operatorname{Re} f(t)|,|\operatorname{Im} f(t)|\}\,dt-\max\left\{\left|\int_a^b \operatorname{Re} f(t)\,dt\right|,\left|\int_a^b \operatorname{Im} f(t)\,dt\right|\right\} \leq \int_a^b k(t)\,dt.$$

Finally, utilising Theorem 19, for $\left(\mathbb{C},|\cdot|_{2p}\right)$ with $p\geq 1$, we may state that:

Proposition 12. *Let* $\varphi,\phi\in\mathbb{R}$ *with* $\varphi^2+\phi^2=2^{\frac{1}{2p}-\frac{1}{2}}\ (p\geq 1),\ f,k:[a,b]\to\mathbb{C}$ *be Lebesgue integrable functions such that*

$$\left[|\operatorname{Re} f(t)|^{2p}+|\operatorname{Im} f(t)|^{2p}\right]^{\frac{1}{2p}}\leq \varphi\operatorname{Re} f(t)-\phi\operatorname{Im} f(t)+k(t)$$

for a.e. $t \in [a,b]$. *Then*

$$(0 \leq) \int_a^b \left[|\operatorname{Re} f(t)|^{2p} + |\operatorname{Im} f(t)|^{2p} \right]^{\frac{1}{2p}} dt - \left[\left| \int_a^b \operatorname{Re} f(t)\, dt \right|^{2p} + \left| \int_a^b \operatorname{Im} f(t)\, dt \right|^{2p} \right]^{\frac{1}{2p}} \leq \int_a^b k(t)\, dt.$$

Remark 15. *If* $p = 1$ *in the above proposition, then, from*

$$|f(t)| \leq \varphi \operatorname{Re} f(t) - \psi \operatorname{Im} f(t) + k(t) \quad \text{for a.e. } t \in [a,b],$$

provided $\varphi, \psi \in \mathbb{R}$ *and* $\varphi^2 + \psi^2 = 1$, *we have the additive reverse of the classical continuous triangle inequality*

$$(0 \leq) \int_a^b |f(t)|\, dt - \left| \int_a^b f(t)\, dt \right| \leq \int_a^b k(t)\, dt.$$

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