INEQUALITIES FOR THE NORM AND NUMERICAL RADIUS OF COMPOSITE OPERATORS IN HILBERT SPACES

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Abstract. Some new inequalities for the norm and the numerical radius of composite operators generated by a pair of operators are given.

1. Introduction

Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. The *numerical range* of an operator T is the subset of the complex numbers $\mathbb{C}$ given by [4, p. 1]:

$$W(T)=\{\langle Tx,x\rangle,\ x\in H,\ \|x\|=1\}. \tag{1.1}$$

It is well known that (see [4]):

(i) The numerical range of an operator is convex;
(ii) The spectrum of an operator is contained in the closure of its numerical range;
(iii) T is self-adjoint if and only if $W(T)$ is real.

The *numerical radius* $w(T)$ of an operator T on H is defined by [4, p. 8]

$$w(T):=\sup\{|\lambda|,\lambda\in W(T)\}=\sup\{|\langle Tx,x\rangle|,\|x\|=1\}. \tag{1.2}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators acting on H and the following inequality holds true:

$$w(T)\leq\|T\|\leq 2w(T). \tag{1.3}$$

We recall some classical results involving the numerical radius of two linear operators A,B.

The following general result for the product of two operators holds [4, p. 37]:

Theorem 1. *If A,B are two bounded linear operators on the Hilbert space $(H,\langle\cdot,\cdot\rangle)$, then*

$$w(AB)\leq 4w(A)\,w(B). \tag{1.4}$$

In the case that $AB=BA$, then

$$w(AB)\leq 2w(A)\,w(B). \tag{1.5}$$

The following results are also well known [4, p. 38].

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Theorem 2. *If A is a unitary operator that commutes with another operator B, then*

$$w(AB) \leq w(B). \tag{1.6}$$

If A is an isometry and $AB = BA$, then (1.6) also holds true.

We say that A and B *double commute* if $AB = BA$ and $AB^* = B^*A$.

The following result holds [4, p. 38].

Theorem 3 (Double commute)**.** *If the operators A and B double commute, then*

$$w(AB) \leq w(B)\|A\|. \tag{1.7}$$

As a consequence of the above, we have [4, p. 39]:

Corollary 1. *Let A be a normal operator commuting with B. Then*

$$w(AB) \leq w(A)\,w(B). \tag{1.8}$$

For other results and historical comments on the above see [4, p. 39–41]. For more results on the numerical radius, see [5].

The main aim of this paper is to establish some new inequalities for composite operators generated by a pair of operators (A, B) under various assumptions. Namely, in one side, several inequalities involving the norm

$$\left\| \frac{A^*A + B^*B}{2} \right\|$$

and the numerical radius $w(B^*A)$ are established. On the other side, upper bounds for the nonnegative quantities

$$\|A\|\|B\| - w(B^*A) \text{ and } \|A\|^2\|B\|^2 - w^2(B^*A)$$

under special conditions for the operators involved are also given.

2. The Results

The following result may be stated:

Theorem 4. *Let $A, B : H \to H$ be two bounded linear operators on the Hilbert space $(H, \langle \cdot, \cdot \rangle)$. If $r > 0$ and*

$$\|A - B\| \leq r, \tag{2.1}$$

then

$$\left\| \frac{A^*A + B^*B}{2} \right\| \leq w(B^*A) + \frac{1}{2}r^2. \tag{2.2}$$

Proof. For any $x \in H$, $\|x\| = 1$, we have from (2.1) that

$$\|Ax\|^2 + \|Bx\|^2 \leq 2\operatorname{Re}\langle Ax, Bx\rangle + r^2. \tag{2.3}$$

However

$$\begin{aligned} \|Ax\|^2 + \|Bx\|^2 &= \langle (A^*A)x, x\rangle + \langle (B^*B)x, x\rangle \\ &= \langle (A^*A + B^*B)x, x\rangle \end{aligned}$$

and by (2.3) we obtain

$$\langle (A^*A + B^*B)x, x\rangle \leq 2|\langle (B^*A)x, x\rangle| + r^2 \tag{2.4}$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum over $x \in H$, $\|x\| = 1$ in (2.4) we get

$$w(A^*A + B^*B) \leq 2w(B^*A) + r^2 \tag{2.5}$$

and since the operator $A^*A + B^*B$ is self-adjoint, hence

$$w(A^*A + B^*B) = \|A^*A + B^*B\|$$

and by (2.5) we deduce the desired inequality (2.2). ∎

Remark 1. *We observe that, from the proof of the above theorem, we have the inequalities*

$$0 \leq \left\|\frac{A^*A + B^*B}{2}\right\| - w(B^*A) \leq \frac{1}{2}\|A - B\|^2, \tag{2.6}$$

provided that A, B are bounded linear operators in H.

The second inequality in (2.6) is obvious while the first inequality follows by the fact that

$$\begin{aligned} \langle (A^*A + B^*B) x, x\rangle &= \|Ax\|^2 + \|Bx\|^2 \\ &\geq 2\|Ax\|\|Bx\| \geq 2|\langle (B^*A) x, x\rangle| \end{aligned}$$

for any $x \in H$.

The inequality (2.2) is obviously a reach source of particular inequalities of interest.

Indeed, if we assume, for $\lambda \in \mathbb{C}$ and a bounded linear operator T, that we have

$$\|T - \lambda T^*\| \leq r, \tag{2.7}$$

for a given positive number r, then by (2.6) we deduce the inequality

$$0 \leq \left\|\frac{T^*T + |\lambda|^2 TT^*}{2}\right\| - |\lambda| w\left(T^2\right) \leq \frac{1}{2}r^2. \tag{2.8}$$

Now, if we assume that for $\lambda \in \mathbb{C}$ and a bounded linear operator V we have that

$$\|V - \lambda I\| \leq r, \tag{2.9}$$

where I is the identity operator on H, then by (2.2) we deduce the inequality

$$0 \leq \left\|\frac{V^*V + |\lambda|^2 I}{2}\right\| - |\lambda| w(V) \leq \frac{1}{2}r^2.$$

As a dual approach, the following result may be noted as well:

Theorem 5. *Let $A, B : H \to H$ be two bounded linear operators on the Hilbert space H. Then*

$$\left\|\frac{A + B}{2}\right\|^2 \leq \frac{1}{2}\left[\left\|\frac{A^*A + B^*B}{2}\right\| + w(B^*A)\right]. \tag{2.10}$$

Proof. We obviously have

$$\begin{aligned} \|Ax + Bx\|^2 &= \|Ax\|^2 + 2\operatorname{Re}\langle Ax, Bx\rangle + \|Bx\|^2 \\ &\leq \langle (A^*A + B^*B) x, x\rangle + 2|\langle (B^*A) x, x\rangle| \end{aligned}$$

for any $x \in H$.

Taking the supremum over $x \in H$, $\|x\| = 1$, we get

$$\begin{aligned}\|A+B\|^2 &\leq w\left(A^*A + B^*B\right) + 2w\left(B^*A\right) \\ &= \|A^*A + B^*B\| + 2w\left(B^*A\right),\end{aligned}$$

from where we get the desired inequality (2.10). ∎

Remark 2. *The inequality (2.10) can generate some interesting particular results such as the following inequality*

$$\left\|\frac{T+T^*}{2}\right\|^2 \leq \frac{1}{2}\left[\left\|\frac{T^*T+TT^*}{2}\right\| + w\left(T^2\right)\right], \tag{2.11}$$

holding for each bounded linear operator $T : H \to H$.

The following result may be stated as well.

Theorem 6. *Let $A, B : H \to H$ be two bounded linear operators on the Hilbert space H and $p \geq 2$. Then*

$$\left\|\frac{A^*A+B^*B}{2}\right\|^{\frac{p}{2}} \leq \frac{1}{4}\left[\|A-B\|^p + \|A+B\|^p\right]. \tag{2.12}$$

Proof. We use the following inequality for vectors in inner product spaces obtained by Dragomir and Sándor in [2]:

$$2\left(\|a\|^p + \|b\|^p\right) \leq \|a+b\|^p + \|a-b\|^p \tag{2.13}$$

for any $a, b \in H$ and $p \geq 2$.

Utilising (2.13) we may write

$$2\left(\|Ax\|^p + \|Bx\|^p\right) \leq \|Ax+Bx\|^p + \|Ax-Bx\|^p \tag{2.14}$$

for any $x \in H$.

Now, observe that

$$\|Ax\|^p + \|Bx\|^p = \left(\|Ax\|^2\right)^{\frac{p}{2}} + \left(\|Bx\|^2\right)^{\frac{p}{2}}$$

and by the elementary inequality:

$$\frac{\alpha^q + \beta^q}{2} \geq \left(\frac{\alpha+\beta}{2}\right)^q, \quad \alpha, \beta \geq 0 \text{ and } q \geq 1$$

we have

$$\begin{aligned}\left(\|Ax\|^2\right)^{\frac{p}{2}} + \left(\|Bx\|^2\right)^{\frac{p}{2}} &\geq 2^{1-\frac{p}{2}}\left(\|Ax\|^2 + \|Bx\|^2\right)^{\frac{p}{2}} \\ &= 2^{1-\frac{p}{2}}\left[\left\langle\left(A^*A+B^*B\right)x, x\right\rangle\right]^{\frac{p}{2}}.\end{aligned} \tag{2.15}$$

Combining (2.14) with (2.15) we get

$$\frac{1}{4}\left[\|Ax-Bx\|^p + \|Ax+Bx\|^p\right] \geq \left|\left\langle\left(\frac{A^*A+B^*B}{2}\right)x, x\right\rangle\right|^{\frac{p}{2}} \tag{2.16}$$

for any $x \in H$, $\|x\| = 1$. Taking the supremum over $x \in H$, $\|x\| = 1$, and taking into account that

$$w\left(\frac{A^*A+B^*B}{2}\right) = \left\|\frac{A^*A+B^*B}{2}\right\|,$$

we deduce the desired result (2.12). ∎

Remark 3. *If $p=2$, then we have the inequality:*

$$\left\|\frac{A^*A+B^*B}{2}\right\| \le \left\|\frac{A-B}{2}\right\|^2 + \left\|\frac{A+B}{2}\right\|^2, \tag{2.17}$$

for any A, B bounded linear operators. This result can also be obtained directly on utilising the parallelogram identity.

We also should observe that for $A=T$ and $B=T^*$, T a normal operator, the inequality (2.12) becomes

$$\|T\|^p \le \frac{1}{4}\left[\|T-T^*\|^p + \|T+T^*\|^p\right],$$

where $p \ge 2$.

The following result may be stated as well.

Theorem 7. *Let $A, B : H \to H$ be two bounded linear operators on the Hilbert space H and $r \ge 1$. If $A^*A \ge B^*B$ in the operator order or, equivalently, $\|Ax\| \ge \|Bx\|$ for any $x \in H$, then:*

$$\left\|\frac{A^*A+B^*B}{2}\right\|^r \le \|A\|^{r-1}\|B\|^{r-1} w(B^*A) + \frac{1}{2}r^2\|A\|^{2r-2}\|A-B\|^2. \tag{2.18}$$

Proof. We use the following inequality for vectors in inner product spaces due to Goldstein, Ryff and Clarke [3]:

$$\|a\|^{2r} + \|b\|^{2r} \le 2\|a\|^{r-1}\|b\|^{r-1}\operatorname{Re}\langle a,b\rangle + r^2\|a\|^{2r-2}\|a-b\|^2, \tag{2.19}$$

where $r \ge 1$, $a, b \in H$ and $\|a\| \ge \|b\|$.

Utilising (2.19) we can state that:

$$\begin{aligned} &\|Ax\|^{2r} + \|Bx\|^{2r} \\ &\quad\le 2\|Ax\|^{r-1}\|Bx\|^{r-1}|\langle Ax,Bx\rangle| + r^2\|Ax\|^{2r-2}\|Ax-Bx\|^2, \end{aligned} \tag{2.20}$$

for any $x \in H$.

As in the proof of Theorem 6, we also have

$$2^{1-r}\left[\langle (A^*A+B^*B)x,x\rangle\right]^r \le \|Ax\|^{2r} + \|Bx\|^{2r}, \tag{2.21}$$

for any $x \in H$.

Therefore, by (2.20) and (2.21) we deduce

$$\begin{aligned} &\left[\left\langle\left(\frac{A^*A+B^*B}{2}\right)x,x\right\rangle\right]^r \\ &\quad\le \|Ax\|^{r-1}\|Bx\|^{r-1}|\langle Ax,Bx\rangle| + \frac{1}{2}r^2\|A\|^{2r-2}\|Ax-Bx\|^2 \end{aligned} \tag{2.22}$$

for any $x \in H$.

Taking the supremum in (2.22) we obtain the desired result (2.18). ∎

Remark 4. *Following* [4, p. 156]*, we recall that the bounded linear operator V is hyponormal, if*

$$\|V^*x\| \le \|Vx\| \text{ for all } x \in H.$$

Now, if we choose in (2.18) $A=V$ and $B=V^*$, then, on taking into account that for hyponormal operators $w(V^2) = \|V\|^2$, we get the inequality

$$\left\|\frac{V^*V+VV^*}{2}\right\|^r \le \|V\|^{2r-2}\left[\|V\|^2 + \frac{1}{2}r^2\|V-V^*\|^2\right], \tag{2.23}$$

holding for any hyponormal operator V and any $r \geq 1$.

3. Further Inequalities for an Invertible Operator

In this section we assume that $B : H \to H$ is an invertible bounded linear operator and let $B^{-1} : H \to H$ be its inverse. Then, obviously,

$$\|Bx\| \geq \frac{1}{\|B^{-1}\|} \|x\| \quad \text{for any } x \in H, \tag{3.1}$$

where $\|B^{-1}\|$ denotes the norm of the inverse B^{-1}.

The following result holds true:

Theorem 8. *Let $A, B : H \to H$ be two bounded linear operators on H and B is invertible such that, for a given $r > 0$,*

$$\|A - B\| \leq r. \tag{3.2}$$

Then:

$$\|A\| \leq \|B^{-1}\| \left[w(B^* A) + \frac{1}{2} r^2 \right]. \tag{3.3}$$

Proof. The condition (3.2) is obviously equivalent to:

$$\|Ax\|^2 + \|Bx\|^2 \leq 2 \operatorname{Re} \langle (B^* A) x, x \rangle + r^2 \tag{3.4}$$

for any $x \in H$, $\|x\| = 1$.

Since, by (3.1),

$$\|Bx\|^2 \geq \frac{1}{\|B^{-1}\|^2} \|x\|^2, \quad x \in H$$

and $\operatorname{Re} \langle (B^* A) x, x \rangle \leq |\langle (B^* A) x, x \rangle|$, hence by (3.4) we get

$$\|Ax\|^2 + \frac{\|x\|^2}{\|B^{-1}\|^2} \leq 2 |\langle (B^* A) x, x \rangle| + r^2 \tag{3.5}$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum over $x \in H$, $\|x\| = 1$ in (3.5), we have

$$\|A\|^2 + \frac{1}{\|B^{-1}\|^2} \leq 2w(B^* A) + r^2. \tag{3.6}$$

By the elementary inequality

$$\frac{2\|A\|}{\|B^{-1}\|} \leq \|A\|^2 + \frac{1}{\|B^{-1}\|^2} \tag{3.7}$$

and by (3.6) we then deduce the desired result (3.3). ∎

Remark 5. *If we choose above $B = \lambda I$, $\lambda \neq 0$, then we get the inequality*

$$(0 \leq) \|A\| - w(A) \leq \frac{1}{2|\lambda|} r^2, \tag{3.8}$$

provided $\|A - \lambda I\| \leq r$. This result has been obtained in the earlier paper [1].

Also, if we assume that $B = \lambda A^*$, A is invertible, then we obtain

$$\|A\| \leq \|A^{-1}\| \left[w(A^2) + \frac{1}{2|\lambda|} r^2 \right], \tag{3.9}$$

provided $\|A - \lambda A^*\| \leq r$, $\lambda \neq 0$.

The following result may be stated as well:

Theorem 9. *Let $A, B : H \to H$ be two bounded linear operators on H. If B is invertible and for $r > 0$,*

$$\|A - B\| \leq r, \tag{3.10}$$

then

$$(0 \leq)\, \|A\| \, \|B\| - w\left(B^* A\right) \leq \frac{1}{2} r^2 + \frac{\|B\|^2 \left\|B^{-1}\right\|^2 - 1}{\left\|B^{-1}\right\|^2}. \tag{3.11}$$

Proof. The condition (3.10) is obviously equivalent to

$$\|Ax\|^2 + \|Bx\|^2 \leq 2 \operatorname{Re} \langle Ax, Bx \rangle + r^2$$

for any $x \in H$, which is clearly equivalent to

$$\|Ax\|^2 + \|B\|^2 \leq 2 \operatorname{Re} \langle B^* Ax, x \rangle + r^2 + \|B\|^2 - \|Bx\|^2 . \tag{3.12}$$

Since

$$\operatorname{Re} \langle B^* Ax, x \rangle \leq |\langle B^* Ax, x \rangle| , \quad \|Bx\|^2 \geq \frac{1}{\left\|B^{-1}\right\|^2} \|x\|^2$$

and

$$\|Ax\|^2 + \|B\|^2 \geq 2 \|B\| \, \|Ax\|$$

for any $x \in H$, hence by (3.12) we get

$$2 \|B\| \, \|Ax\| \leq 2 |\langle B^* Ax, x \rangle| + r^2 + \frac{\|B\|^2 \left\|B^{-1}\right\|^2 - 1}{\left\|B^{-1}\right\|^2} \tag{3.13}$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum over $x \in H$, $\|x\| = 1$ we deduce the desired result (3.11). ∎

Remark 6. *If we choose in Theorem 9, $B = \lambda A^*$, $\lambda \neq 0$, A is invertible, then we get the inequality:*

$$(0 \leq)\, \|A\|^2 - w\left(A^2\right) \leq \frac{1}{2|\lambda|} r^2 + |\lambda| \cdot \frac{\|A\|^2 \left\|A^{-1}\right\|^2 - 1}{\left\|A^{-1}\right\|^2} \tag{3.14}$$

provided $\|A - \lambda A^*\| \leq r$.

The following result may be stated as well.

Theorem 10. *Let $A, B : H \to H$ be two bounded linear operators on H. If B is invertible and for $r > 0$ we have*

$$\|A - B\| \leq r < \|B\| , \tag{3.15}$$

then

$$\|A\| \leq \frac{1}{\sqrt{\|B\|^2 - r^2}} \left(w\left(B^* A\right) + \frac{\|B\|^2 \left\|B^{-1}\right\|^2 - 1}{2 \left\|B^{-1}\right\|^2} \right). \tag{3.16}$$

Proof. The first part of condition (3.15) is obviously equivalent to

$$\|Ax\|^2 + \|Bx\|^2 \leq 2 \operatorname{Re} \langle Ax, Bx \rangle + r^2$$

for any $x \in H$, which is clearly equivalent to

$$\|Ax\|^2 + \|B\|^2 - r^2 \leq 2 \operatorname{Re} \langle B^* Ax, x \rangle + \|B\|^2 - \|Bx\|^2 . \tag{3.17}$$

Since

$$\operatorname{Re}\langle B^{*}Ax,x\rangle \leq |\langle B^{*}Ax,x\rangle|,$$
$$\|Bx\|^{2} \geq \frac{1}{\|B^{-1}\|^{2}}\|x\|^{2}$$

and, by the second part of (3.15),

$$\|Ax\|^{2}+\|B\|^{2}-r^{2} \geq 2\sqrt{\|B\|^{2}-r^{2}}\,\|Ax\|,$$

for any $x\in H$, hence by (3.17) we get

$$2\|Ax\|\sqrt{\|B\|^{2}-r^{2}} \leq 2|\langle B^{*}Ax,x\rangle| + \frac{\|B\|^{2}\|B^{-1}\|^{2}-1}{\|B^{-1}\|^{2}} \tag{3.18}$$

for any $x\in H$, $\|x\|=1$.

Taking the supremum over $x\in H$, $\|x\|=1$ in (3.18), we deduce the desired inequality (3.16). ∎

Remark 7. *The above Theorem 10 has some particular cases of interest. For instance, if we choose $B=\lambda I$, with $|\lambda|>r$, then (3.15) is obviously fulfilled and by (3.16) we get*

$$\|A\| \leq \frac{w(A)}{\sqrt{1-\left(\frac{r}{|\lambda|}\right)^{2}}}, \tag{3.19}$$

provided $\|A-\lambda I\|\leq r$. This result has been obtained in the earlier paper [1].

On the other hand, if in the above we choose $B=\lambda A^{*}$ with $\|A\|\geq\frac{r}{|\lambda|}$ $(\lambda\neq 0)$, then by (3.16) we get

$$\|A\| \leq \frac{1}{\sqrt{\|A\|^{2}-\left(\frac{r}{|\lambda|}\right)^{2}}}\left[w\left(A^{2}\right)+|\lambda|\cdot\frac{\|A\|^{2}\|A^{-1}\|^{2}-1}{2\|A^{-1}\|^{2}}\right], \tag{3.20}$$

provided $\|A-\lambda A^{*}\|\leq r$.

The following result may be stated as well.

Theorem 11. *Let A,B and r be as in Theorem 8. Moreover, if*

$$\left\|B^{-1}\right\| < \frac{1}{r}, \tag{3.21}$$

then

$$\|A\| \leq \frac{\left\|B^{-1}\right\|}{\sqrt{1-r^{2}\|B^{-1}\|^{2}}}w(B^{*}A). \tag{3.22}$$

Proof. Observe that, by (3.6) we have

$$\|A\|^{2}+\frac{1-r^{2}\left\|B^{-1}\right\|^{2}}{\|B^{-1}\|^{2}} \leq 2w(B^{*}A). \tag{3.23}$$

Utilising the elementary inequality

$$2\frac{\|A\|}{\|B^{-1}\|}\sqrt{1-r^{2}\|B^{-1}\|^{2}} \leq \|A\|^{2}+\frac{1-r^{2}\left\|B^{-1}\right\|^{2}}{\|B^{-1}\|^{2}}, \tag{3.24}$$

which can be stated since (3.21) is assumed to be true, hence by (3.23) and (3.24) we deduce the desired result (3.22). ∎

Remark 8. *If we assume that $B = \lambda A^*$ with $\lambda \neq 0$ and A an invertible operator, then, by applying Theorem 11, we get the inequality:*

$$\|A\| \leq \frac{\|A^{-1}\| \, w\left(A^2\right)}{\sqrt{|\lambda|^2 - r^2 \|A^{-1}\|^2}}, \tag{3.25}$$

provided $\|A - \lambda A^*\| \leq r$ and $\|A^{-1}\| \leq \frac{|\lambda|}{r}$.

The following result may be stated as well.

Theorem 12. *Let $A, B : H \to H$ be two bounded linear operators. If $r > 0$ and B is invertible with the property that $\|A - B\| \leq r$ and*

$$\frac{1}{\sqrt{r^2+1}} \leq \|B^{-1}\| < \frac{1}{r}, \tag{3.26}$$

then

$$\|A\|^2 \leq w^2\left(B^*A\right) + 2w\left(B^*A\right) \cdot \frac{\|B^{-1}\| - \sqrt{1 - r^2 \|B^{-1}\|^2}}{\|B^{-1}\|}. \tag{3.27}$$

Proof. Let $x \in H$, $\|x\| = 1$. Then by (3.5) we have

$$\|Ax\|^2 + \frac{1}{\|B^{-1}\|^2} \leq 2\left|\langle B^*Ax, x\rangle\right| + r^2, \tag{3.28}$$

and since

$$\frac{1}{\|B^{-1}\|^2} - r^2 > 0,$$

we can conclude that $|\langle B^*Ax, x\rangle| > 0$ for any $x \in H$, $\|x\| = 1$.

Dividing in (3.28) with $|\langle B^*Ax, x\rangle| > 0$, we obtain

$$\frac{\|Ax\|^2}{|\langle B^*Ax, x\rangle|} \leq 2 + \frac{r^2}{|\langle B^*Ax, x\rangle|} - \frac{1}{\|B^{-1}\|^2 |\langle B^*Ax, x\rangle|}. \tag{3.29}$$

Subtracting $|\langle B^*Ax, x\rangle|$ from both sides of (3.29), we get

$$\begin{aligned} &\frac{\|Ax\|^2}{|\langle B^*Ax, x\rangle|} - |\langle B^*Ax, x\rangle| \\ &\leq 2 - |\langle B^*Ax, x\rangle| - \frac{1 - r^2\|B^{-1}\|^2}{|\langle B^*Ax, x\rangle| \, \|B^{-1}\|^2} \\ &= 2 - \frac{2\sqrt{1 - r^2\|B^{-1}\|^2}}{\|B^{-1}\|} - \left(\sqrt{|\langle B^*Ax, x\rangle|} - \frac{\sqrt{1 - r^2\|B^{-1}\|^2}}{\|B^{-1}\| \sqrt{|\langle B^*Ax, x\rangle|}}\right)^2 \\ &\leq 2\left(\frac{\|B^{-1}\| - \sqrt{1 - r^2\|B^{-1}\|^2}}{\|B^{-1}\|}\right), \end{aligned} \tag{3.30}$$

which gives:

$$\|Ax\|^2 \leq |\langle B^*Ax, x\rangle|^2 + 2|\langle B^*Ax, x\rangle| \frac{\|B^{-1}\| - \sqrt{1 - r^2\|B^{-1}\|^2}}{\|B^{-1}\|}. \tag{3.31}$$

We also remark that, by (3.26) the quantity

$$\left\|B^{-1}\right\| - \sqrt{1 - r^2 \left\|B^{-1}\right\|^2} \geq 0,$$

hence, on taking the supremum in (3.31) over $x \in H$, $\|x\| = 1$, we deduce the desired inequality. ∎

Remark 9. *It is interesting to remark that if we assume $\lambda \in \mathbb{C}$ with $0 < r \leq |\lambda| \leq \sqrt{r^2+1}$ and $\|A - \lambda I\| \leq r$, then by (3.2) we can state the following inequality:*

$$\|A\|^2 \leq |\lambda|^2 w\left(A^2\right) + 2|\lambda| \left(1 - \sqrt{|\lambda|^2 - r^2}\right) w(A). \tag{3.32}$$

Also, if $\|A - A^*\| \leq r$, A is invertible and $\frac{1}{\sqrt{r^2+1}} \leq \left\|A^{-1}\right\| \leq \frac{1}{r}$, then, by (3.27) we also have

$$\|A\|^2 \leq w^2\left(A^2\right) + 2w\left(A^2\right) \cdot \frac{\left\|A^{-1}\right\| - \sqrt{1 - r^2 \left\|A^{-1}\right\|^2}}{\left\|A^{-1}\right\|}. \tag{3.33}$$

One can also prove the following result.

Theorem 13. *Let $A, B : H \to H$ be two bounded linear operators. If $r > 0$ and B is invertible with the property that $\|A - B\| \leq r$ and $\left\|B^{-1}\right\| \leq \frac{1}{r}$, then*

$$\begin{aligned} &(0 \leq) \|A\|^2 \|B\|^2 - w^2(B^* A) \\ &\leq 2w(B^* A) \cdot \frac{\|B\|}{\left\|B^{-1}\right\|} \left(\|B\| \left\|B^{-1}\right\| - \sqrt{1 - r^2 \left\|B^{-1}\right\|^2}\right). \end{aligned} \tag{3.34}$$

Proof. We subtract the quantity $\frac{|\langle B^* Ax, x\rangle|}{\|B\|^2}$ from both sides of (3.29) to obtain

$$\begin{aligned} 0 &\leq \frac{\|Ax\|^2}{|\langle B^* Ax, x\rangle|} - \frac{|\langle B^* Ax, x\rangle|}{\|B\|^2} \\ &\leq 2 - \frac{|\langle B^* Ax, x\rangle|}{\|B\|^2} - \frac{1 - r^2 \left\|B^{-1}\right\|^2}{|\langle B^* Ax, x\rangle| \left\|B^{-1}\right\|^2} \\ &= 2 - 2 \cdot \frac{\sqrt{1 - r^2 \left\|B^{-1}\right\|^2}}{\|B\| \left\|B^{-1}\right\|} - \left(\frac{\sqrt{|\langle B^* Ax, x\rangle|}}{\|B\|} - \frac{\sqrt{1 - r^2 \left\|B^{-1}\right\|^2}}{\sqrt{|\langle B^* Ax, x\rangle|} \left\|B^{-1}\right\|}\right)^2 \\ &\leq 2 \cdot \frac{\left(\|B\| \left\|B^{-1}\right\| - \sqrt{1 - r^2 \left\|B^{-1}\right\|^2}\right)}{\|B\| \left\|B^{-1}\right\|}, \end{aligned} \tag{3.35}$$

which is equivalent with

$$\begin{aligned} &(0 \leq) \|Ax\|^2 \|B\|^2 - |\langle B^* Ax, x\rangle|^2 \\ &\leq 2 \frac{\|B\|}{\left\|B^{-1}\right\|} |\langle B^* Ax, x\rangle| \left(\|B\| \left\|B^{-1}\right\| - \sqrt{1 - r^2 \left\|B^{-1}\right\|^2}\right) \end{aligned} \tag{3.36}$$

for any $x \in H$, $\|x\| = 1$.

The inequality (3.36) also shows that $\|B\|\left\|B^{-1}\right\| \geq \sqrt{1-r^{2}\left\|B^{-1}\right\|^{2}}$ and then, by (3.36), we get

$$
\begin{aligned}
(3.37)\quad \|Ax\|^{2}\|B\|^{2} &\leq |\langle B^{*}Ax, x\rangle|^{2} \\
&+2\frac{\|B\|}{\left\|B^{-1}\right\|}|\langle B^{*}Ax, x\rangle|\left(\|B\|\left\|B^{-1}\right\|-\sqrt{1-r^{2}\left\|B^{-1}\right\|^{2}}\right)
\end{aligned}
$$

for any $x \in X$, $\|x\|=1$.

Taking the supremum in (3.37) we deduce the desired inequality (3.34). ∎

Remark 10. *The above Theorem 13 has some particular instances of interest as follows. If, for instance, we choose $B=\lambda I$ with $|\lambda| \geq r>0$ and $\|A-\lambda I\| \leq r$, then by (3.34) we obtain the inequality*

$$
\begin{aligned}
(3.38)\quad (0 \leq)\, \|A\|^{2}-w^{2}(A) & \\
&\leq 2|\lambda|\, w(A)\left(1-\sqrt{1-\frac{r^{2}}{|\lambda|^{2}}}\right).
\end{aligned}
$$

Also, if A is invertible, $\|A-\lambda A^{*}\| \leq r$ and $\left\|A^{-1}\right\| \leq \frac{|\lambda|}{r}$, then by (3.34) we can state:

$$
\begin{aligned}
(3.39)\quad (0 \leq)\, \|A\|^{4}-w^{2}\left(A^{2}\right) & \\
&\leq 2|\lambda|\, w\left(A^{2}\right) \cdot \frac{\|A\|}{\left\|A^{-1}\right\|}\left(\|A\|\left\|A^{-1}\right\|-\sqrt{1-\frac{r^{2}}{|\lambda|^{2}}\left\|A^{-1}\right\|^{2}}\right).
\end{aligned}
$$

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