REVERSE INEQUALITIES FOR THE NUMERICAL RADIUS OF LINEAR OPERATORS IN HILBERT SPACES

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Abstract. Some elementary inequalities providing upper bounds for the difference of the norm and the numerical radius of a bounded linear operator on Hilbert spaces under appropriate conditions are given.

1. Introduction

Let $(H; \langle \cdot, \cdot \rangle)$ be a complex Hilbert space. The *numerical range* of an operator T is the subset of the complex numbers $\mathbb{C}$ given by [1, p. 1]:

$$W(T) = \{\langle Tx, x \rangle, \ x \in H, \ \|x\| = 1\}.$$

The following properties of $W(T)$ are immediate:

(i) $W(\alpha I + \beta T) = \alpha + \beta W(T)$ for $\alpha, \beta \in \mathbb{C}$;
(ii) $W(T^*) = \{\bar{\lambda}, \lambda \in W(T)\}$, where T^* is the *adjoint operator* of T;
(iii) $W(U^*TU) = W(T)$ for any *unitary* operator U.

The following classical fact about the geometry of the numerical range [1, p. 4] may be stated:

Theorem 1 (Toeplitz-Hausdorff). *The numerical range of an operator is convex.*

An important use of $W(T)$ is to bound the *spectrum* $\sigma(T)$ of the operator T [1, p. 6]:

Theorem 2 (Spectral inclusion). *The spectrum of an operator is contained in the closure of its numerical range.*

The self-adjoint operators have their spectra bounded sharply by the numerical range [1, p. 7]:

Theorem 3. *The following statements hold true:*

(i) *T is self-adjoint iff $W(T)$ is real;*
(ii) *If T is self-adjoint and $W(T) = [m, M]$ (the closed interval of real numbers m, M), then $\|T\| = \max\{|m|, |M|\}$.*
(iii) *If $W(T) = [m, M]$, then $m, M \in \sigma(T)$.*

The *numerical radius* $w(T)$ of an operator T on H is given by [1, p. 8]:

$$w(T) = \sup\{|\lambda|, \lambda \in W(T)\} = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\}. \tag{1.1}$$

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Obviously, by (1.1), for any $x \in H$ one has

$$|\langle Tx, x\rangle| \leq w(T)\|x\|^2. \tag{1.2}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T : H \to H$, i.e.,

(i) $w(T) \geq 0$ for any $T \in B(H)$ and $w(T) = 0$ if and only if $T = 0$;
(ii) $w(\lambda T) = |\lambda| w(T)$ for any $\lambda \in \mathbb{C}$ and $T \in B(H)$;
(iii) $w(T + V) \leq w(T) + w(V)$ for any $T, V \in B(H)$.

This norm is equivalent with the operator norm. In fact, the following more precise result holds [1, p. 9]:

Theorem 4 (Equivalent norm)**.** *For any $T \in B(H)$ one has*

$$w(T) \leq \|T\| \leq 2w(T). \tag{1.3}$$

Let us now look at two extreme cases of the inequality (1.3). In the following $r(t) := \sup\{|\lambda|, \lambda \in \sigma(T)\}$ will denote the *spectral radius* of T and $\sigma_p(T) = \{\lambda \in \sigma(T),\ Tf = \lambda f \text{ for some } f \in H\}$ the *point spectrum* of T.

The following results hold [1, p.10]:

Theorem 5. *We have*

(i) *If $w(T) = \|T\|$, then $r(T) = \|T\|$.*
(ii) *If $\lambda \in W(T)$ and $|\lambda| = \|T\|$, then $\lambda \in \sigma_p(T)$.*

To address the other extreme case $w(T) = \frac{1}{2}\|T\|$, we can state the following sufficient condition in terms of (see [1, p. 11])

$$R(T) := \{Tf,\ f \in H\} \quad \text{and} \quad R(T^*) := \{T^*f,\ f \in H\}.$$

Theorem 6. *If $R(T) \perp R(T^*)$, then $w(T) = \frac{1}{2}\|T\|$.*

It is well-known that the two-dimensional shift

$$S_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

has the property that $w(T) = \frac{1}{2}\|T\|$.

The following theorem shows that some operators T with $w(T) = \frac{1}{2}\|T\|$ have S_2 as a component [1, p. 11]:

Theorem 7. *If $w(T) = \frac{1}{2}\|T\|$ and T attains its norm, then T has a two-dimensional reducing subspace on which it is the shift S_2.*

For other results on numerical radius, see [2], Chapter 11.

The main aim of the present paper is to point out some upper bounds for the nonnegative difference

$$\|T\| - w(T) \qquad \left(\|T\|^2 - (W(T))^2\right)$$

under appropriate assumptions for the bounded linear operator $T : H \to H$.

2. The Results

The following results may be stated:

Theorem 8. *Let $T : H \to H$ be a bounded linear operator on the complex Hilbert space H. If $\lambda \in \mathbb{C} \setminus \{0\}$ and $r > 0$ are such that*

$$\|T - \lambda I\| \leq r, \tag{2.1}$$

where $I : H \to H$ is the identity operator on H, then

$$(0 \leq) \|T\| - w(T) \leq \frac{1}{2} \cdot \frac{r^2}{|\lambda|}. \tag{2.2}$$

Proof. For $x \in H$ with $\|x\| = 1$, we have from (2.1) that

$$\|Tx - \lambda x\| \leq \|T - \lambda I\| \leq r,$$

giving

$$\|Tx\|^2 + |\lambda|^2 \leq 2 \operatorname{Re}\left[\overline{\lambda} \langle Tx, x\rangle\right] + r^2 \leq 2|\lambda| |\langle Tx, x\rangle| + r^2. \tag{2.3}$$

Taking the supremum over $x \in H$, $\|x\| = 1$ in (2.3) we get the following inequality that is of interest in itself:

$$\|T\|^2 + |\lambda|^2 \leq 2w(T)|\lambda| + r^2. \tag{2.4}$$

Since, obviously,

$$\|T\|^2 + |\lambda|^2 \geq 2\|T\| |\lambda|, \tag{2.5}$$

hence by (2.4) and (2.5) we deduce the desired inequality (2.2). ∎

Remark 1. *If the operator $T : H \to H$ is such that $R(T) \perp R(T^*)$, $\|T\| = 1$ and $\|T - I\| \leq 1$, then the equality case holds in (2.2). Indeed, by Theorem 6, we have in this case $w(T) = \frac{1}{2}\|T\| = \frac{1}{2}$ and since we can choose in Theorem 8, $\lambda = 1$, $r = 1$, then we get in both sides of (2.2) the same quantity $\frac{1}{2}$.*

Problem 1. *Find the bounded linear operators $T : H \to H$ with $\|T\| = 1$, $R(T) \perp R(T^*)$ and $\|T - \lambda I\| \leq |\lambda|^{\frac{1}{2}}$.*

The following corollary may be stated:

Corollary 1. *Let $A : H \to H$ be a bounded linear operator and $\varphi, \phi \in \mathbb{C}$ with $\phi \neq -\varphi, \varphi$. If*

$$\operatorname{Re}\langle \phi x - Ax, Ax - \varphi x\rangle \geq 0 \quad \text{for any} \quad x \in H, \ \|x\| = 1 \tag{2.6}$$

then

$$(0 \leq) \|A\| - w(A) \leq \frac{1}{4} \cdot \frac{|\phi - \varphi|^2}{|\phi + \varphi|}. \tag{2.7}$$

Proof. Utilising the fact that in any Hilbert space the following two statements are equivalent:

(i) $\operatorname{Re}\langle Z - x, x - z\rangle \geq 0$, $x, z, Z \in H$;
(ii) $\left\|x - \frac{z+Z}{2}\right\| \leq \frac{1}{2}\|Z - z\|$,

we deduce that (2.6) is equivalent to

$$\left\| Ax - \frac{\phi + \varphi}{2} \cdot Ix \right\| \le \frac{1}{2} |\phi - \varphi| \tag{2.8}$$

for any $x \in H$, $\|x\| = 1$, which in its turn is equivalent with the operator norm inequality:

$$\left\| A - \frac{\phi + \varphi}{2} \cdot I \right\| \le \frac{1}{2} |\phi - \varphi| . \tag{2.9}$$

Now, applying Theorem 8 for $T = A$, $\lambda = \frac{\varphi + \phi}{2}$ and $r = \frac{1}{2} |\Gamma - \gamma|$, we deduce the desired result (2.7). ∎

Remark 2. *Following* [1, p. 25]*, we say that an operator $B : H \to H$ is accreative, if $\operatorname{Re} \langle Bx, x \rangle \ge 0$ for any $x \in H$. One may observe that the assumption (2.6) above is then equivalent with the fact that the operator $(A^* - \bar{\varphi} I)(\phi I - A)$ is accreative.*

Perhaps a more convenient sufficient condition in terms of positive operators is the following one:

Corollary 2. *Let $\varphi, \phi \in \mathbb{C}$ with $\phi \ne -\varphi, \varphi$ and $A : H \to H$ a bounded linear operator in H. If $(A^* - \bar{\varphi} I)(\phi I - A)$ is self-adjoint and*

$$(A^* - \bar{\varphi} I)(\phi I - A) \ge 0 \tag{2.10}$$

in the operator order, then

$$(0 \le) \|A\| - w(A) \le \frac{1}{4} \cdot \frac{|\phi - \varphi|^2}{|\phi + \varphi|} . \tag{2.11}$$

The following result may be stated as well:

Corollary 3. *Assume that T, λ, r are as in Theorem 8. If, in addition, there exists $\rho \ge 0$ such that*

$$||\lambda| - w(T)| \ge \rho, \tag{2.12}$$

then

$$(0 \le) \|T\|^2 - w^2(T) \le r^2 - \rho^2 . \tag{2.13}$$

Proof. From (2.4) of Theorem 8, we have

$$\begin{aligned} \|T\|^2 - w^2(T) &\le r^2 - w^2(T) + 2w(T)|\lambda| - |\lambda|^2 \\ &= r^2 - (|\lambda| - w(T))^2 . \end{aligned} \tag{2.14}$$

On utilising (2.4) and (2.12) we deduce the desired inequality (2.13). ∎

Remark 3. *In particular, if $\|T - \lambda I\| \le r$ and $|\lambda| = w(T)$, $\lambda \in \mathbb{C}$, then*

$$(0 \le) \|T\|^2 - w^2(T) \le r^2 . \tag{2.15}$$

The following result may be stated as well.

Theorem 9. *Let $T : H \to H$ be a nonzero bounded linear operator on H and $\lambda \in \mathbb{C} \setminus \{0\}$, $r > 0$ with $|\lambda| > r$. If*

$$\|T - \lambda I\| \le r, \tag{2.16}$$

then

$$\sqrt{1-\frac{r^2}{|\lambda|^2}} \le \frac{w(T)}{\|T\|} \quad (\le 1). \tag{2.17}$$

Proof. From (2.4) of Theorem 8, we have

$$\|T\|^2 + |\lambda|^2 - r^2 \le 2|\lambda| w(T),$$

which implies, on dividing with $\sqrt{|\lambda|^2 - r^2} > 0$ that

$$\frac{\|T\|^2}{\sqrt{|\lambda|^2 - r^2}} + \sqrt{|\lambda|^2 - r^2} \le \frac{2|\lambda| w(T)}{\sqrt{|\lambda|^2 - r^2}}. \tag{2.18}$$

By the elementary inequality

$$2\|T\| \le \frac{\|T\|^2}{\sqrt{|\lambda|^2 - r^2}} + \sqrt{|\lambda|^2 - r^2} \tag{2.19}$$

and by (2.18) we deduce

$$\|T\| \le \frac{w(T)|\lambda|}{\sqrt{|\lambda|^2 - r^2}},$$

which is equivalent to (2.17). ∎

Remark 4. *Squaring (2.17), we get the inequality*

$$(0 \le) \|T\|^2 - w^2(T) \le \frac{r^2}{|\lambda|^2} \|T\|^2. \tag{2.20}$$

Remark 5. *Since for any bounded linear operator $T : H \to H$ we have that $w(T) \ge \frac{1}{2}\|T\|$, hence (2.17) would produce a refinement of this classic fact only in the case when*

$$\frac{1}{2} \le \left(1 - \frac{r^2}{|\lambda|^2}\right)^{\frac{1}{2}},$$

which is equivalent to $r/|\lambda| \le \sqrt{3}/2$.

The following corollary holds.

Corollary 4. *Let $\varphi, \phi \in \mathbb{C}$ with $\operatorname{Re}(\phi\bar{\varphi}) > 0$. If $T : H \to H$ is a bounded linear operator such that either (2.6) or (2.10) holds true, then:*

$$\frac{2\sqrt{\operatorname{Re}(\phi\bar{\varphi})}}{|\phi + \varphi|} \le \frac{w(T)}{\|T\|} \ (\le 1) \tag{2.21}$$

and

$$(0 \le) \|T\|^2 - w^2(T) \le \left|\frac{\phi - \varphi}{\phi + \varphi}\right|^2 \|T\|^2. \tag{2.22}$$

Proof. If we consider $\lambda = \frac{\phi+\varphi}{2}$ and $r = \frac{1}{2}|\phi - \varphi|$, then $|\lambda|^2 - r^2 = \left|\frac{\phi+\varphi}{2}\right|^2 - \left|\frac{\phi-\varphi}{2}\right|^2 = \operatorname{Re}(\phi\bar{\varphi}) > 0$. Now, on applying Theorem 9, we deduce the desired result. ∎

Remark 6. *If $|\phi - \varphi| \le \frac{\sqrt{3}}{2}|\phi + \varphi|$, $\operatorname{Re}(\phi\bar{\varphi}) > 0$, then (2.21) is a refinement of the inequality $w(T) \ge \frac{1}{2}\|T\|$.*

The following result may be of interest as well.

Theorem 10. *Let* $T : H \to H$ *be a nonzero bounded linear operator on* H *and* $\lambda \in \mathbb{C} \backslash \{0\}$, $r > 0$ *with* $|\lambda| > r$. *If*

$$\|T - \lambda I\| \leq r, \tag{2.23}$$

then

$$(0 \leq) \|T\|^2 - w^2(T) \leq \frac{2r^2}{|\lambda| + \sqrt{|\lambda|^2 - r^2}} w(T). \tag{2.24}$$

Proof. From the proof of Theorem 8, we have

$$\|Tx\|^2 + |\lambda|^2 \leq 2\operatorname{Re}\left[\overline{\lambda}\langle Tx, x\rangle\right] + r^2 \tag{2.25}$$

for any $x \in H$, $\|x\| = 1$.

If we divide (2.25) by $|\lambda| |\langle Tx, x\rangle|$, (which, by (2.25), is positive) then we obtain

$$\frac{\|Tx\|^2}{|\lambda| |\langle Tx, x\rangle|} \leq \frac{2\operatorname{Re}\left[\overline{\lambda}\langle Tx, x\rangle\right]}{|\lambda| |\langle Tx, x\rangle|} + \frac{r^2}{|\lambda| |\langle Tx, x\rangle|} - \frac{|\lambda|}{|\langle Tx, x\rangle|} \tag{2.26}$$

for any $x \in H$, $\|x\| = 1$.

If we subtract in (2.26) the same quantity $\frac{|\langle Tx,x\rangle|}{|\lambda|}$ from both sides, then we get

$$\begin{aligned} &\frac{\|Tx\|^2}{|\lambda| |\langle Tx, x\rangle|} - \frac{|\langle Tx, x\rangle|}{|\lambda|} \\ &\leq \frac{2\operatorname{Re}\left[\overline{\lambda}\langle Tx, x\rangle\right]}{|\lambda| |\langle Tx, x\rangle|} + \frac{r^2}{|\lambda| |\langle Tx, x\rangle|} - \frac{|\langle Tx, x\rangle|}{|\lambda|} - \frac{|\lambda|}{|\langle Tx, x\rangle|} \\ &= \frac{2\operatorname{Re}\left[\overline{\lambda}\langle Tx, x\rangle\right]}{|\lambda| |\langle Tx, x\rangle|} - \frac{|\lambda|^2 - r^2}{|\lambda| |\langle Tx, x\rangle|} - \frac{|\langle Tx, x\rangle|}{|\lambda|} \\ &= \frac{2\operatorname{Re}\left[\overline{\lambda}\langle Tx, x\rangle\right]}{|\lambda| |\langle Tx, x\rangle|} - \left(\frac{\sqrt{|\lambda|^2 - r^2}}{\sqrt{|\lambda| |\langle Tx, x\rangle|}} - \frac{\sqrt{|\langle Tx, x\rangle|}}{\sqrt{|\lambda|}}\right)^2 - 2\frac{\sqrt{|\lambda|^2 - r^2}}{|\lambda|}. \end{aligned} \tag{2.27}$$

Since

$$\operatorname{Re}\left[\overline{\lambda}\langle Tx, x\rangle\right] \leq |\lambda| |\langle Tx, x\rangle|$$

and

$$\left(\frac{\sqrt{|\lambda|^2 - r^2}}{\sqrt{|\lambda| |\langle Tx, x\rangle|}} - \frac{\sqrt{|\langle Tx, x\rangle|}}{\sqrt{|\lambda|}}\right)^2 \geq 0$$

hence by (2.27) we get

$$\frac{\|Tx\|^2}{|\lambda| |\langle Tx, x\rangle|} - \frac{|\langle Tx, x\rangle|}{|\lambda|} \leq \frac{2\left(|\lambda| - \sqrt{|\lambda|^2 - r^2}\right)}{|\lambda|}$$

which gives the inequality

$$\|Tx\|^2 \leq |\langle Tx, x\rangle|^2 + 2|\langle Tx, x\rangle| \left(|\lambda| - \sqrt{|\lambda|^2 - r^2}\right) \tag{2.28}$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum over $x \in H$, $\|x\| = 1$, we get

$$\begin{aligned} \|T\|^2 &\leq \sup \left\{ |\langle Tx, x\rangle|^2 + 2\,|\langle Tx, x\rangle| \left(|\lambda| - \sqrt{|\lambda|^2 - r^2} \right) \right\} \\ &\leq \sup \left\{ |\langle Tx, x\rangle|^2 \right\} + 2 \left(|\lambda| - \sqrt{|\lambda|^2 - r^2} \right) \sup \left\{ |\langle Tx, x\rangle| \right\} \\ &= w^2(T) + 2 \left(|\lambda| - \sqrt{|\lambda|^2 - r^2} \right) w(T), \end{aligned}$$

which is clearly equivalent to (2.24). ∎

Corollary 5. *Let* $\varphi, \phi \in \mathbb{C}$ *with* $\operatorname{Re}(\phi\bar{\varphi}) > 0$. *If* $A : H \to H$ *is a bounded linear operator such that either (2.6) or (2.10) hold true, then:*

$$(0 \leq)\, \|A\|^2 - w^2(A) \leq \left[|\phi + \varphi| - 2\sqrt{\operatorname{Re}(\phi\bar{\varphi})} \right] w(A). \tag{2.29}$$

Remark 7. *If* $M \geq m > 0$ *are such that either* $(A^* - mI)(MI - A)$ *is accreative, or, sufficiently,* $(A^* - mI)(MI - A)$ *is self-adjoint and*

$$(A^* - mI)(MI - A) \geq 0 \quad \text{in the operator order,} \tag{2.30}$$

then, by (2.21) we have:

$$(1 \leq)\, \frac{\|A\|}{w(A)} \leq \frac{M + m}{2\sqrt{mM}}, \tag{2.31}$$

which is equivalent to

$$(0 \leq)\, \|A\| - w(A) \leq \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} w(A), \tag{2.32}$$

while from (2.24) we have

$$(0 \leq)\, \|A\|^2 - w^2(A) \leq \left(\sqrt{M} - \sqrt{m}\right)^2 w(A). \tag{2.33}$$

Also, the inequality (2.7) becomes

$$(0 \leq)\, \|A\| - w(A) \leq \frac{1}{4} \cdot \frac{(M - m)^2}{M + m}. \tag{2.34}$$

References

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