SOME INEQUALITIES FOR NORMAL OPERATORS IN HILBERT SPACES

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Abstract. Some inequalities for normal operators in Hilbert spaces are given. For this purpose, some results for vectors in inner product spaces due to Buzano, Dunkl-Williams, Hile, Goldstein-Ryff-Clarke, Dragomir-Sándor and the author are employed.

1. Introduction

Let $(H; \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $T : H \to H$ a bounded linear operator on H. Recall that T is a *normal operator* if $T^*T = TT^*$. Normal operators may be regarded as a generalisation of self-adjoint operator T in which T^* need not be exactly T but commutes with T [8, p. 15].

The *numerical range* of an operator T is the subset of the complex numbers $\mathbb{C}$ given by [8, p. 1]:

$$W(T) = \{\langle Tx, x \rangle, \ x \in H, \ \|x\| = 1\}.$$

For various properties of the numerical range see [8].

We recall here some of the ones related to normal operators.

Theorem 1. *If $W(T)$ is a line segment, then T is normal.*

We denote by $r(T)$ the operator *spectral radius* [8, p. 10] and by $w(T)$ its *numerical radius* [8, p. 8]. The following result may be stated as well [8, p. 15].

Theorem 2. *If T is normal, then $\|T^n\| = \|T\|^n$, $n = 1, \ldots$. Moreover, we have:*

$$r(T) = w(T) = \|T\|. \tag{1.1}$$

An important fact about the normal operators that will be used frequently in the sequel is the following one [9, p. 42]:

Theorem 3. *A necessary and sufficient condition that an operator T be normal is that $\|Tx\| = \|T^*x\|$ for every vector $x \in H$.*

The aim of this paper is to establish some inequalities for normal operators in Hilbert spaces. For this purpose, some inequalities for vectors in inner product spaces due to Buzano, Dunkl-Williams, Hile, Goldstein-Ryff-Clarke, Dragomir-Sándor and the author are employed.

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2. Inequalities for Vectors

The following result may be stated.

Theorem 4. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be a Hilbert space and* $T:H\to H$ *a normal linear operator on* H*. Then*

$$\frac{1}{2}\left(\|Tx\|^2+\left|\left\langle T^2x,x\right\rangle\right|\right)\geq|\langle Tx,x\rangle|^2, \tag{2.1}$$

for any $x\in H$, $\|x\|=1$*. The constant* $\frac{1}{2}$ *is best possible in (2.1).*

Proof. We need the following refinement of Schwarz's inequality obtained by the author in 1985 [2, Theorem 2] (see also [5] and [4]):

$$\|a\|\,\|b\|\geq|\langle a,b\rangle-\langle a,e\rangle\langle e,b\rangle|+|\langle a,e\rangle\langle e,b\rangle|\geq|\langle a,b\rangle|, \tag{2.2}$$

provided a,b,e are vectors in H and $\|e\|=1$.

Observing that

$$|\langle a,b\rangle-\langle a,e\rangle\langle e,b\rangle|\geq|\langle a,e\rangle\langle e,b\rangle|-|\langle a,b\rangle|,$$

then by the first inequality in (2.2) we deduce

$$\frac{1}{2}\left(\|a\|\,\|b\|+|\langle a,b\rangle|\right)\geq|\langle a,e\rangle\langle e,b\rangle|. \tag{2.3}$$

This inequality was obtained in a different way earlier by M.L. Buzano in [1].

Now, choose in (2.3), $e=x$, $\|x\|=1$, $a=Tx$ and $b=T^*x$ to get

$$\frac{1}{2}\left(\|Tx\|\,\|T^*x\|+\left|\left\langle T^2x,x\right\rangle\right|\right)\geq|\langle Tx,x\rangle|^2 \tag{2.4}$$

for any $x\in H$, $\|x\|=1$. Since T is normal, then $\|Tx\|=\|T^*x\|$, and by (2.4) we deduce the desire result (2.1).

The fact that, the constant $\frac{1}{2}$ is best possible in (2.1) is obvious since for $T=I$, the identity operator, we get equality in (2.1). ∎

From a different perspective, we can state the followin reverse result:

Theorem 5. *Let* $T:H\to H$ *be a normal operator on the Hilbert space* $(H;\langle\cdot,\cdot\rangle)$*. If* $\lambda\in\mathbb{C}$*, then*

$$(0\leq)\,\|Tx\|^2-\left|\left\langle T^2x,x\right\rangle\right|\leq\frac{2}{(1+|\lambda|)^2}\|Tx-\lambda T^*x\|^2 \tag{2.5}$$

for any $x\in H$, $\|x\|=1$.

Proof. We use the following inequality [6]:

$$\|a-b\|\geq\frac{1}{2}\left(\|a\|+\|b\|\right)\left\|\frac{a}{\|a\|}-\frac{b}{\|b\|}\right\|,\quad a,b\in H\backslash\{0\},$$

which is well known in the literature as the *Dunkl-Williams inequality.*

This inequality, by taking the square, is clearly equivalent to

$$\frac{4\|a-b\|^2}{(\|a\|+\|b\|)^2}\geq\left\|\frac{a}{\|a\|}-\frac{b}{\|b\|}\right\|^2=2-2\cdot\frac{\operatorname{Re}\langle a,b\rangle}{\|a\|\,\|b\|},$$

which shows that (see [3, Eq. (2.5)])

$$\frac{\|a\|\,\|b\|-|\langle a,b\rangle|}{\|a\|\,\|b\|}\leq\frac{2\|a-b\|^2}{(\|a\|+\|b\|)^2}.$$

Now, for $x \in H \setminus \ker(T)$, $\|x\| = 1$, choose $a = Tx$ and $b = \lambda T^* x$ $(\lambda \neq 0)$ to obtain

$$\|Tx\| \|T^*x\| - \left|\left\langle T^2 x, x \right\rangle\right| \leq \frac{2\|Tx\| \|T^*x\|}{(\|Tx\| + |\lambda| \|T^*x\|)^2} \|Tx - \lambda T^* x\|^2 . \tag{2.6}$$

Since $\|Tx\| = \|T^*x\|$, T being a normal operator, we get from (2.6) that (2.5) holds true for any $x \in H \setminus \ker(T)$, $\|x\| = 1$.

For $\lambda = 0$ the inequality (2.5) is obvious.

Since for normal operators $\ker(T) = \ker(T^*)$ then for $x \in \ker(T)$, $\|x\| = 1$ the inequality (2.5) also holds true. ∎

The following result which provides a different bound for the nonnegative quantity

$$\|Tx\|^2 - \left|\left\langle T^2 x, x \right\rangle\right|, x \in H, \|x\| = 1$$

may be stated as well:

Theorem 6. *Let $T : H \to H$ be a normal operator on the Hilbert space H and $\alpha, \lambda \in \mathbb{C} \setminus \{0\}$. Then*

$$\begin{aligned} &(0 \leq) \|Tx\|^2 - \left|\left\langle T^2 x, x \right\rangle\right| \\ \leq\ & \frac{1}{2} \cdot \frac{\left[|\operatorname{Re}\alpha| \left\|Tx - \frac{\alpha}{\bar{\alpha}} \lambda T^* x\right\| + |\operatorname{Im}\alpha| \left\|Tx + \frac{\alpha}{\bar{\alpha}} \lambda T^* x\right\|\right]^2}{|\lambda| |\alpha|^2} \end{aligned} \tag{2.7}$$

for any $x \in H$, $\|x\| = 1$.

Proof. We use the following inequality (see [3, Theorem 2.11]):

$$\|a\| \|b\| - \operatorname{Re}\left[\frac{\alpha^2}{|\alpha|^2} \langle a, b \rangle\right] \leq \frac{1}{2} \cdot \frac{[|\operatorname{Re}\alpha| \|a - b\| + |\operatorname{Im}\alpha| \|a + b\|]^2}{|\alpha|^2} \tag{2.8}$$

for the choices:

$$a = \frac{Tx}{\alpha}, \quad b = \frac{\lambda}{\bar{\alpha}} T^* x, \quad x \in H$$

to obtain:

$$\begin{aligned} &\frac{|\lambda| \|Tx\| \|T^*x\|}{|\alpha|^2} - \operatorname{Re}\left[\frac{\alpha^2}{|\alpha|^2} \cdot \frac{\bar{\lambda}}{\alpha^2} \langle Tx, T^* x \rangle\right] \\ \leq\ & \frac{1}{2} \cdot \frac{\left[|\operatorname{Re}\alpha| \left\|\frac{Tx}{\alpha} - \frac{\lambda}{\alpha} T^* x\right\| + |\operatorname{Im}\alpha| \left\|\frac{Tx}{\alpha} + \frac{\lambda}{\alpha} T^* x\right\|\right]^2}{|\alpha|^2} . \end{aligned} \tag{2.9}$$

Since T is normal, we get from (2.9) the desired result (2.7). The details are omitted. ∎

Another result of this type is incorporated in:

Theorem 7. *Let $T : H \to H$ be a normal operator on the Hilbert space H, $s \in [0, 1]$ and $t \in \mathbb{R}$. Then*

$$\begin{aligned} &(0 \leq) \|Tx\|^4 - \left|\left\langle T^2 x, x \right\rangle\right|^2 \\ &\leq \|Tx\|^2 \left[s \|tT^* x - Tx\|^2 + (1 - s) \|T^* x - tTx\|^2\right] . \end{aligned} \tag{2.10}$$

In particular

$$\begin{aligned}&(0\le)\|Tx\|^4-\left|\left\langle T^2x,x\right\rangle\right|^2\\ \le\ &\frac{1}{2}\|Tx\|^2\inf_{t\in\mathbb{R}}\left[\|tT^*x-Tx\|^2+\|T^*x-tTx\|^2\right].\end{aligned}$$

Proof. We use the inequality obtained in [4, Theorem 2], to state that

$$\begin{aligned}&\left[(1-s)\|a\|^2+s\|b\|^2\right]\left[(1-s)\|b\|^2+s\|a\|^2\right]-|\langle a,b\rangle|^2\\ \le\ &\left[(1-s)\|a\|^2+s\|b\|^2\right]\left[(1-s)\|b-ta\|^2+s\|tb-a\|^2\right]\end{aligned}\tag{2.11}$$

for any $s\in[0,1]$, $t\in\mathbb{R}$ and $a,b\in H$.

If in (2.11) we choose $a=Tx$, $b=T^*x$, $x\in H$ and $\|x\|=1$, then we get

$$\begin{aligned}&\|Tx\|^4-\left|\left\langle T^2x,x\right\rangle\right|^2\\ \le\ &\|Tx\|^2\left[s\|tT^*x-Tx\|^2+(1-s)\|T^*x-tTx\|^2\right]\end{aligned}$$

for any $s\in[0,1]$, $t\in\mathbb{R}$, from where we deduce the desired inequality (2.10). ∎

From a different perspective, we can state the following result as well.

Theorem 8. *Let $T:H\to H$ be a normal operator on the Hilbert space $(H;\langle\cdot,\cdot\rangle)$. If $\lambda\in\mathbb{C}\setminus\{0\}$ and $r>0$ are such that*

$$\|T-\lambda T^*\|\le r,\tag{2.12}$$

then:

$$(0\le)\|Tx\|^4-\left|\left\langle T^2x,x\right\rangle\right|^2\le\frac{r^2}{|\lambda|^2}\|Tx\|^2\tag{2.13}$$

for any $x\in H$, $\|x\|=1$.

Proof. We use the following reverse of the quadratic Schwarz inequality obtained by the author in [4]

$$(0\le)\|a\|^2\|b\|^2-|\langle a,b\rangle|^2\le\frac{1}{|\alpha|^2}\|a\|^2\|a-\alpha b\|^2\tag{2.14}$$

provided $a,b\in H$ and $\alpha\in\mathbb{C}\setminus\{0\}$.

Choosing in (2.14) $a=Tx$, $\alpha=\lambda$, $b=T^*x$, we get

$$\begin{aligned}\|Tx\|^4&\le\left|\left\langle T^2x,x\right\rangle\right|^2+\frac{1}{|\lambda|^2}\|Tx\|^2\|Tx-\lambda T^*x\|^2\\ &\le\left|\left\langle T^2x,x\right\rangle\right|^2+\frac{1}{|\lambda|^2}r^2\|Tx\|^2\end{aligned}\tag{2.15}$$

which is the desired result (2.13). ∎

Finally, on utilising the following result obtained in [4]:

Lemma 1. *Let $a,b\in H\setminus\{0\}$ and $\varepsilon\in\left(0,\frac{1}{2}\right]$. If*

$$(0\le)\,1-\varepsilon-\sqrt{1-2\varepsilon}\le\frac{\|a\|}{\|b\|}\le1-\varepsilon+\sqrt{1-2\varepsilon},\tag{2.16}$$

then

$$(0\le)\|a\|\|b\|-\operatorname{Re}\langle a,b\rangle\le\varepsilon\|a-b\|^2,\tag{2.17}$$

we can state:

Theorem 9. *Let $T : H \to H$ be a normal operator on H. If $\lambda \in \mathbb{C}$ is such that*

$$(0 \leq)\, 1-\varepsilon-\sqrt{1-2\varepsilon} \leq |\lambda| \leq 1-\varepsilon+\sqrt{1-2\varepsilon}, \quad \varepsilon \in \left(0, \frac{1}{2}\right] \tag{2.18}$$

then

$$(0 \leq)\, \|Tx\|^2 - \left|\left\langle T^2x, x\right\rangle\right| \leq \frac{\varepsilon}{|\lambda|} \|Tx - \lambda T^* x\|^2 \tag{2.19}$$

for any $x \in H$, $\|x\| = 1$.

Proof. Utilising Lemma 1 for $a = \lambda T^* x$, $b = Tx$, $x \in H \setminus \ker(T)$, $\|x\| = 1$, we have

$$|\lambda| \|Tx\|^2 - |\lambda| \left|\left\langle T^2x, x\right\rangle\right| \leq \varepsilon \|Tx - \lambda T^* x\|^2 . \tag{2.20}$$

For $x \in \ker(T)$, $\|x\| = 1$ the inequality (2.19) also holds, and the proof is completed. ∎

3. Inequalities for Operator Norm

The following result concerning the operator norm can be stated:

Theorem 10. *Let $T : H \to H$ be a normal operator on the Hilbert space H. If $\lambda \in \mathbb{C}$, $|\lambda| \neq 1$, then:*

$$\left\|T - |\lambda|^{v+1} T^*\right\| \leq \frac{1-|\lambda|^{v+1}}{1-|\lambda|} \|T - \lambda T^*\| , \tag{3.1}$$

for any $v > 0$.

Proof. We use the following inequality:

$$\left\|\|a\|^v a - \|b\|^v b\right\| \leq \frac{\|a\|^{v+1} - \|b\|^{v+1}}{\|a\| - \|b\|} \|a - b\| \tag{3.2}$$

provided $v > 0$ and $\|a\| \neq \|b\|$, which is known in the literature as the *Hile inequality* [10].

Now, if we choose in (3.2) $a = Tx$, $b = \lambda T^* x$, since T is normal, we have $\|a\| = \|Tx\|$, $\|b\| = |\lambda| \|Tx\|$ and by (3.2) we get

$$\|Tx\|^v \left\|Tx - |\lambda|^{v+1} T^* x\right\| \leq \|Tx\|^v \frac{\left(1-|\lambda|^{v+1}\right)}{1-|\lambda|} \|Tx - \lambda T^* x\| \tag{3.3}$$

for any $x \in H \setminus \ker(T)$.

If $x \notin \ker(T)$, then from (3.3) we get

$$\left\|Tx - |\lambda|^{v+1} T^* x\right\| \leq \frac{1-|\lambda|^{v+1}}{1-|\lambda|} \|Tx - \lambda T^* x\| . \tag{3.4}$$

If $x \in \ker(T)$ and since $\ker(T) = \ker(T^*)$, T being normal, then the inequality (3.4) is also valid. Therefore, (3.4) holds for any $x \in H$.

Taking the supremum over $x \in H$, $\|x\| = 1$, we get the desired inequality (3.1). ∎

Remark 1. *For $v = 1$, we get the inequality:*

$$\left\|T - |\lambda|^2 T^*\right\| \leq (1 + |\lambda|) \|T - \lambda T^*\| . \tag{3.5}$$

Utilising the second inequality due to Hile (see [10, Eq. (5.2)]):

$$\left\| \frac{a}{\|a\|^{v+2}} - \frac{b}{\|b\|^{v+2}} \right\| \le \frac{\|a\|^{v+2} - \|b\|^{v+2}}{\|a\| - \|b\|} \cdot \frac{\|a-b\|}{\|a\|^{v+1} \cdot \|b\|^{v+1}}$$

for $a, b \in H$, $a, b \neq 0$ and $\|a\| \neq \|b\|$, and making use of an argument similar to the one in the proof of the above theorem, we can state the following result:

Theorem 11. *Let $T : H \to H$ be a normal operator on the Hilbert space H. If $\lambda \in \mathbb{C}$, $|\lambda| \neq 0, 1$, then:*

$$\left\| T - \frac{\lambda}{|\lambda|^{v+2}} T^* \right\| \le \frac{1 - |\lambda|^{v+1}}{(1 - |\lambda|) |\lambda|^{v+1}} \|T - \lambda T^*\|, \tag{3.6}$$

where $v > 0$.

The following result may be stated as well.

Theorem 12. *Let $T : H \to H$ be a normal operator on the Hilbert space H. If $|\lambda| \le 1$, then*

$$(1 - |\lambda|^\rho)^2 \|T\|^2 \le \begin{cases} \rho^2 \|T - \lambda T^*\|^2 & \text{if } \rho \ge 1, \\ |\lambda|^{2\rho-2} \|T - \lambda T^*\|^2 & \text{if } \rho < 1. \end{cases} \tag{3.7}$$

Proof. We use the following inequality due to Goldstein, Ryff and Clarke [7]

$$\|a\|^{2\rho} + \|b\|^{2\rho} - 2\|a\|^{\rho-1}\|b\|^{\rho-1}\operatorname{Re}\langle a, b\rangle \le \begin{cases} \rho^2 \|a\|^{2\rho-2} \|a-b\|^2 & \text{if } \rho \ge 1, \\ \|b\|^{2\rho-2} \|a-b\|^2 & \text{if } \rho < 1, \end{cases} \tag{3.8}$$

provided $\rho \in \mathbb{R}$ and $a, b \in H$ with $\|a\| \ge \|b\|$.

Since $\operatorname{Re}\langle a, b\rangle \le |\langle a, b\rangle|$, then, from (3.8), we have the inequality

$$\|a\|^{2\rho} + \|b\|^{2\rho} \le 2\|a\|^{\rho-1}\|b\|^{\rho-1}|\langle a, b\rangle| + \begin{cases} \rho^2 \|a\|^{2\rho-2} \|a-b\|^2 & \text{if } \rho \ge 1, \\ \|b\|^{2\rho-2} \|a-b\|^2 & \text{if } \rho < 1. \end{cases} \tag{3.9}$$

We choose $a = Tx$, $b = \lambda T^* x$ and since $|\lambda| \le 1$, we have $\|a\| \ge \|b\|$. From (3.9), on taking into account that $\|Tx\| = \|T^* x\|$, we deduce

$$\|Tx\|^{2\rho} + |\lambda|^{2\rho}\|Tx\|^{2\rho} \le 2\|Tx\|^{2\rho-2}|\lambda|^\rho \left|\left\langle T^2 x, x\right\rangle\right| + \begin{cases} \rho^2 \|Tx\|^{2\rho-2} \|Tx - \lambda T^* x\|^2 & \text{if } \rho \ge 1, \\ |\lambda|^{2\rho-2}\|Tx\|^{2\rho-2} \|Tx - \lambda T^* x\|^2 & \text{if } \rho < 1, \end{cases}$$

which implies that:

$$\left(1 + |\lambda|^{2\rho}\right)\|Tx\|^2 \le 2|\lambda|^\rho \left|\left\langle T^2 x, x\right\rangle\right| + \begin{cases} \rho^2 \|Tx - \lambda T^* x\|^2 & \text{if } \rho \ge 1, \\ |\lambda|^{2\rho-2} \|Tx - \lambda T^* x\|^2 & \text{if } \rho < 1, \end{cases} \tag{3.10}$$

for any $x \in H$, $\|x\| = 1$.

This inequality is of interest in itself.

Taking the supremum over $x \in H$, $\|x\| = 1$, and using the fact that

$$\sup_{\|x\|=1} \left|\left\langle T^2 x, x\right\rangle\right| = w\left(T^2\right) = \|T\|^2,$$

we get the desired inequality (3.7). ∎

Remark 2. *If $|\lambda| > 1$, on choosing in (3.9) $a = \lambda T^* x$, $b = Tx$ we get:*

$$\left(|\lambda|^{2\rho} + 1\right) \|Tx\|^2 \leq 2\,|\lambda|^\rho \left|\left\langle T^2 x, x\right\rangle\right| + \begin{cases} \rho^2 |\lambda|^{2\rho-2} \|Tx - \lambda T^* x\|^2 & \text{if } \rho \geq 1, \\ \|Tx - \lambda T^* x\|^2 & \text{if } \rho < 1, \end{cases}$$

which implies the "dual" inequality:

$$(1 - |\lambda|^\rho)^2 \|T\|^2 \leq \begin{cases} \rho^2 |\lambda|^{2\rho-2} \|T - \lambda T^*\|^2 & \text{if } \rho \geq 1, \\ \|T - \lambda T^*\|^2 & \text{if } \rho < 1, \end{cases} \tag{3.11}$$

for any $\lambda \in \mathbb{C}$, $|\lambda| > 1$.

The following result concerning operator norm inequalities may be stated as well:

Theorem 13. *Let $T : H \to H$ be a normal operator on the Hilbert space $(H; \langle\cdot,\cdot\rangle)$ and $\alpha, \beta \in \mathbb{C}$. Then:*

$$\|T\|^p \left[(|\alpha| + |\beta|)^p + ||\alpha| - |\beta||^p\right] \leq \|\alpha T + \beta T^*\|^p + \|\alpha T - \beta T^*\|^p \tag{3.12}$$

if $p \in (1,2)$ and

$$\|\alpha T + \beta T^*\|^p + \|\alpha T - \beta T^*\|^p \geq 2\left(|\alpha|^p + |\beta|^p\right)\|T\|^p \tag{3.13}$$

if $p \geq 2$.

Proof. We use the following result obtained by Dragomir and Sándor in [5]:

$$\|a + b\|^p + \|a - b\|^p \geq (\|a\| + \|b\|)^p + |\|a\| - \|b\||^p \tag{3.14}$$

if $p \in (1,2)$ and

$$\|a + b\|^p + \|a - b\|^p \geq 2\left(\|a\|^p + \|b\|^p\right) \tag{3.15}$$

if $p \geq 2$, where a, b are arbitrary vectors in the inner product space $(H; \langle\cdot,\cdot\rangle)$.

We choose $a = \alpha Tx$, $b = \beta T^* x$ to get:

$$\begin{aligned} &\|(\alpha T + \beta T^*)(x)\|^p + \|(\alpha T - \beta T^*)(x)\|^p \\ &\geq (|\alpha| + |\beta|)^p \|Tx\|^p + ||\alpha| - |\beta||^p \|Tx\|^p \\ &= \left[(|\alpha| + |\beta|)^p + ||\alpha| - |\beta||^p\right] \|Tx\|^p \end{aligned} \tag{3.16}$$

if $p \in (1,2)$ and

$$\|(\alpha T + \beta T^*)(x)\|^p + \|(\alpha T - \beta T^*)(x)\|^p \geq 2\left(|\alpha|^p + |\beta|^p\right)\|Tx\|^p \tag{3.17}$$

if $p \geq 2$.

Taking the supremum over $x \in H$, $\|x\| = 1$, we deduce (3.12) and (3.13). ∎

Remark 3. *The case $p = 2$ produces the following inequality:*

$$\|\alpha T + \beta T^*\|^2 + \|\alpha T - \beta T^*\|^2 \geq 2\left(|\alpha|^2 + |\beta|^2\right)\|T\|^2,$$

that can also be obtained by utilising the parallelogram identity.

The following general result may be stated as well:

Theorem 14. *Let $T : H \to H$ be a normal operator on the Hilbert space H. If $\alpha, \beta \in \mathbb{C}$ and $r, \rho > 0$ are such that*

$$\|T - \bar{\alpha} I\| \leq r \quad \text{and} \quad \|T^* - \beta I\| \leq \rho, \tag{3.18}$$

then

$$\|T\|^2 + \frac{1}{2}\left(|\alpha|^2 + |\beta|^2\right) \leq \frac{1}{2}\left(r^2 + \rho^2\right) + \|\alpha T + \beta T^*\|. \tag{3.19}$$

Proof. The condition (3.18) obviously implies that

$$\|Tx\|^2 + |\alpha|^2 \leq 2\operatorname{Re}\langle(\alpha T)x, x\rangle + r^2 \tag{3.20}$$

and

$$\|T^* x\|^2 + |\beta|^2 \leq 2\operatorname{Re}\left\langle(\beta T)^* x, x\right\rangle + \rho^2 \tag{3.21}$$

for any $x \in H$, $\|x\| = 1$.

Adding (3.20) and (3.21) and taking into account that $\|Tx\| = \|T^* x\|$, we obtain

$$\begin{aligned} 2\|Tx\|^2 + |\alpha|^2 + |\beta|^2 &\leq 2\operatorname{Re}\langle(\alpha T + \beta T^*)x, x\rangle + r^2 + \rho^2 \\ &\leq 2\left|\langle(\alpha T + \beta T^*)x, x\rangle\right| + r^2 + \rho^2. \end{aligned} \tag{3.22}$$

Taking the supremum on (3.22) over $x \in H$, $\|x\| = 1$, and utilising the fact that for the normal operator T we have

$$w(\alpha T + \beta T^*) = \|\alpha T + \beta T^*\|$$

then we get the desired inequality (3.19). ∎

Remark 4. *If $\alpha, \beta \in \mathbb{C}$ and $r, \rho > 0$ are such that $|\alpha|^2 + |\beta|^2 = \rho^2 + r^2$, then from (3.19) we have:*

$$\|T\|^2 \leq \|\alpha T + \beta T^*\|. \tag{3.23}$$

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