NEW REVERSE INEQUALITIES FOR NORMAL OPERATORS IN HILBERT SPACES

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ABSTRACT. In this paper, more reverse inequalities for the class of normal operators, are established. Some of the obtained results are based on recent reverse results for the Schwarz inequality in Hilbert spaces due to the author.

1. INTRODUCTION

Let $(H; \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $T : H \to H$ a bounded linear operator on H. Recall that T is a *normal operator* if $T^*T = TT^*$. Normal operator T may be regarded as a generalisation of self-adjoint operator T in which T^* need not be exactly T but commutes with T [5, p. 15]. An equivalent condition with normality that will be extensively used in the following is that $\|Tx\| = \|T^*x\|$ for any $x \in H$.

The *numerical range* of an operator T is the subset of the complex numbers $\mathbb{C}$ given by [5, p. 1]:

$$W(T) = \{\langle Tx, x\rangle, \ x \in H, \ \|x\| = 1\}.$$

For various properties of the numerical range see [5] and [6].

For normal operators, the following results are well known:

(i) If $W(T)$ is a line segment, then T is normal;

(ii) If T is normal, then $\|T^n\| = \|T\|^n$, $n = 1, 2, \ldots$. Moreover, $r(T) = w(T) = \|T\|$; where $r(T)$ is the *spectral radius* [5, p. 10] and $w(T)$ is the *numerical radius* [5, p. 8] of T;

(iii) Let z be any complex number in the resolvent set of a normal operator T. Then

$$\|(T - zI)x\| \geq d(z, \sigma(T)) \qquad \text{for } \ x \in H, \ \|x\| = 1, \tag{1.1}$$

where $\sigma(T)$ is the *spectrum* of T [5, p. 6].

For other results, see [5, p. 16].

In the previous paper [1] we have obtained amongst others the following vector and operator norm inequalities for a normal operator $T : H \to H$:

(iv) We have the vector inequality:

$$\frac{1}{2}\left(\|Tx\|^2 + \left|\langle T^2x, x\rangle\right|\right) \geq |\langle Tx, x\rangle|^2, \tag{1.2}$$

for any $x \in H$, $\|x\| = 1$.The constant $\frac{1}{2}$ is best possible in (1.2)

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(v) If $\lambda \in \mathbb{C} \backslash \{0\}$ and $r > 0$ are such that $\|T - \lambda T^*\| \le r$, then

$$(0 \le) \|Tx\|^2 - \left|\left\langle T^2 x, x\right\rangle\right| \le \frac{2r^2}{(1 + |\lambda|)^2} \tag{1.3}$$

for any $x \in H, \|x\| = 1$.

(vi) If $\lambda \in \mathbb{C} \backslash \{0\}$ and $r > 0$ are such that $\|T - \lambda T^*\| \le r$, then

$$(0 \le) \|Tx\|^4 - \left|\left\langle T^2 x, x\right\rangle\right|^2 \le \frac{r^2}{|\lambda|^2} \|T\|^2 \tag{1.4}$$

for any $x \in H, \|x\| = 1$.

(vi) If $|\lambda| \le 1$, then

$$(1 - |\lambda|^\rho)^2 \|T\|^2 \le \begin{cases} \rho^2 \|T - \lambda T^*\|^2 & \text{if } \rho \ge 1, \\ \\ |\lambda|^{2\rho - 2} \|T - \lambda T^*\|^2 & \text{if } \rho < 1. \end{cases} \tag{1.5}$$

(vii) For each $\alpha, \beta \in \mathbb{C}$ we have:

$$\|T\|^p \left[(|\alpha| + |\beta|)^p + ||\alpha| - |\beta||^p\right] \le \|\alpha T + \beta T^*\|^p + \|\alpha T - \beta T^*\|^p \tag{1.6}$$

if $p \in (1, 2)$ and

$$\|\alpha T + \beta T^*\|^p + \|\alpha T - \beta T^*\|^p \ge 2 \left(|\alpha|^p + |\beta|^p\right) \|T\|^p \tag{1.7}$$

if $p \ge 2$.

The main aim of the present paper is to provide new upper bounds for the nonnegative quantity

$$\|Tx\|^2 - \left|\left\langle T^2 x, x\right\rangle\right| \quad \left(\|Tx\|^4 - \left|\left\langle T^2 x, x\right\rangle\right|^2\right), \ x \in H, \|x\| = 1;$$

under various assumptions on the normal operator $T : H \to H$. Some of the obtained results are based on recent reverse results for the Schwarz inequality in Hilbert spaces due to the author.

2. Some Vector Inequalities

The following result may be stated.

Theorem 1. *Let $(H; \langle \cdot, \cdot \rangle)$ be a Hilbert space and $T : H \to H$ a normal operator on H. If $\lambda \in \mathbb{C} \backslash \{0\}$ and $r > 0$ are such that*

$$\|T - \lambda T^*\| \le r, \tag{2.1}$$

then

$$\frac{1 + |\lambda|^2}{2|\lambda|} \|Tx\|^2 \le \left|\left\langle T^2 x, x\right\rangle\right| + \frac{r^2}{2|\lambda|} \tag{2.2}$$

for any $x \in H, \|x\| = 1$.

Proof. The inequality (2.1) is obviously equivalent to

$$\|Tx\|^2 + |\lambda|^2 \|T^* x\|^2 \le 2 \operatorname{Re}\left[\overline{\lambda} \langle Tx, T^* x\rangle\right] + r^2 \tag{2.3}$$

for any $x \in H$, $\|x\| = 1$.

Since T is a normal operator, then $\|Tx\| = \|T^* x\|$ for any $x \in H$ and by (2.3) we get

$$\left(1 + |\lambda|^2\right) \|Tx\|^2 \le 2 \operatorname{Re}\left[\overline{\lambda} \left\langle T^2 x, x\right\rangle\right] + r^2 \tag{2.4}$$

for any $x \in H$, $\|x\| = 1$.

Now, on observing that $\operatorname{Re}\left[\overline{\lambda}\left\langle T^2 x, x\right\rangle\right] \leq |\lambda| \left|\left\langle T^2 x, x\right\rangle\right|$, then by (2.4) we deduce (2.2). ∎

Remark 1. *Observe that, since* $|\lambda|^2 + 1 \geq 2|\lambda|$ *for any* $\lambda \in \mathbb{C}\setminus\{0\}$, *hence by (2.2) we get the simpler (yet coarser) inequality:*

$$(0 \leq) \|Tx\|^2 - \left|\left\langle T^2 x, x\right\rangle\right| \leq \frac{r^2}{2|\lambda|}, x \in H, \|x\| = 1, \tag{2.5}$$

provided $\lambda \in \mathbb{C}\setminus\{0\}$, $r > 0$ *and* T *satisfy (2.1).*

If $r > 0$ *and* $\|T - \lambda T^*\| \leq r$, *with* $|\lambda| = 1$, *then by (2.2) we have*

$$(0 \leq) \|Tx\|^2 - \left|\left\langle T^2 x, x\right\rangle\right| \leq \frac{1}{2} r^2, x \in H, \|x\| = 1. \tag{2.6}$$

The following improvement of (1.3) should be noted:

Corollary 1. *With the assumptions of Theorem 1, we have the inequality*

$$(0 \leq) \|Tx\|^2 - \left|\left\langle T^2 x, x\right\rangle\right| \leq \frac{r^2}{1 + |\lambda|^2} \left(\leq \frac{2r^2}{(1 + |\lambda|)^2}\right) \tag{2.7}$$

for any $x \in H, \|x\| = 1$.

Proof. The inequality (2.2) is obviously equivalent to:

$$\begin{aligned} \|Tx\|^2 &\leq \frac{2|\lambda|}{1 + |\lambda|^2} \left|\left\langle T^2 x, x\right\rangle\right| + \frac{r^2}{1 + |\lambda|^2} \\ &\leq \left|\left\langle T^2 x, x\right\rangle\right| + \frac{r^2}{1 + |\lambda|^2} \end{aligned}$$

and the first part of the inequality (2.7) is obtained. The second part is obvious. ∎

For a normal operator T we observe that

$$\left|\left\langle T^2 x, x\right\rangle\right| = |\langle Tx, T^* x\rangle| \leq \|Tx\| \|T^* x\| = \|Tx\|^2$$

for any $x \in H$, hence

$$\|Tx\| - |\langle Tx, T^* x\rangle|^{\frac{1}{2}} \geq 0$$

for any $x \in H$.

Define $\delta(T) := \inf_{\|x\|=1} \left[\|Tx\| - \left|\left\langle T^2 x, x\right\rangle\right|^{\frac{1}{2}}\right] \geq 0$. The following inequality may be stated:

Theorem 2. *With the assumptions of Theorem 1, we have the inequality:*

$$(0 \leq) \|Tx\|^2 - \left|\left\langle T^2 x, x\right\rangle\right| \leq r^2 - 2|\lambda| \delta(T) \mu(T), \tag{2.8}$$

for any $x \in H, \|x\| = 1$, *where* $\mu(T) = \inf_{\|x\|=1} \left|\left\langle T^2 x, x\right\rangle\right|^{\frac{1}{2}}$.

Proof. From the inequality (2.3) we obviously have

$$\|Tx\|^2 - \left|\left\langle T^2 x, x\right\rangle\right| \leq 2 \operatorname{Re}\left[\overline{\lambda}\left\langle T^2 x, x\right\rangle\right] - \left|\left\langle T^2 x, x\right\rangle\right| - |\lambda|^2 \|Tx\|^2 + r^2 \tag{2.9}$$

for any $x \in H$, $\|x\| = 1$.

Now, observe that the right hand side of (2.9) can be written as:

$$\begin{aligned} I &: = r^2 + 2\operatorname{Re}\left[\overline{\lambda}\left\langle T^2x, x\right\rangle\right] - 2|\lambda|\left|\left\langle T^2x, x\right\rangle\right|^{\frac{1}{2}}\|Tx\| \\ &\quad - \left(\left|\left\langle T^2x, x\right\rangle\right|^{\frac{1}{2}} - |\lambda|\,\|Tx\|\right)^2. \end{aligned}$$

Since, obviously,

$$\operatorname{Re}\left[\overline{\lambda}\left\langle T^2x, x\right\rangle\right] \le |\lambda|\left|\left\langle T^2x, x\right\rangle\right|$$

and

$$\left(\left|\left\langle T^2x, x\right\rangle\right|^{\frac{1}{2}} - |\lambda|\,\|Tx\|\right)^2 \ge 0,$$

then

$$\begin{aligned} I &\le r^2 - 2|\lambda|\left|\left\langle T^2x, x\right\rangle\right|^{\frac{1}{2}}\left(\|Tx\| - \left|\left\langle T^2x, x\right\rangle\right|^{\frac{1}{2}}\right) \\ &\le r^2 - 2|\lambda|\,\delta(T)\left|\left\langle T^2x, x\right\rangle\right|^{\frac{1}{2}}. \end{aligned}$$

Utilising (2.9) we get

$$\|Tx\|^2 \le \left|\left\langle T^2x, x\right\rangle\right| - 2|\lambda|\,\delta(T)\left|\left\langle T^2x, x\right\rangle\right|^{\frac{1}{2}} + r^2$$

for any $x \in H$, $\|x\| = 1$, which implies the desired result. ∎

3. Inequalities Under More Restrictions

Now, observe that, for a normal operator $T : H \to H$ and for $\lambda \in \mathbb{C}\setminus\{0\}$, $r > 0$, the following two conditions are equivalent

(c) $$\|Tx - \lambda T^*x\| \le r \le |\lambda|\,\|T^*x\| \quad \text{for any} \quad x \in H, \|x\| = 1$$

and

(cc) $$\|T - \lambda T^*\| \le r \quad \text{and} \quad \xi(T) := \inf_{\|x\|=1}\|Tx\| \ge \frac{r}{|\lambda|}.$$

We can state the following result.

Theorem 3. *Assume that the normal operator $T : H \to H$ satisfies either (c) or, equivalently, (cc) for a given $\lambda \in \mathbb{C}\setminus\{0\}$ and $r > 0$. Then:*

$$(0 \le)\,\|Tx\|^4 - \left|\left\langle T^2x, x\right\rangle\right|^2 \le r^2\|Tx\|^2 \tag{3.1}$$

and

$$\|Tx\|\left(\xi^2(T) - \frac{r^2}{|\lambda|^2}\right)^{\frac{1}{2}} \le \left|\left\langle T^2x, x\right\rangle\right|, \tag{3.2}$$

for any $x \in H$, $\|x\| = 1$.

Proof. We use the following elementary reverse of Schwarz's inequality for vectors in inner product spaces (see [3] or [2]):

$$\|y\|^2\|a\|^2 - [\operatorname{Re}\langle y, a\rangle]^2 \le r^2\|y\|^2 \tag{3.3}$$

provided $\|y - a\| \le r \le \|a\|$.

If in (3.3) we choose $x \in H$, $\|x\| = 1$ and $y = Tx$, $a = \lambda T^*x$, then we have:

$$\|Tx\|^2\|\lambda T^*x\|^2 - |\langle Tx, \lambda T^*x\rangle|^2 \le r^2\|\lambda T^*x\|^2$$

giving

$$\|Tx\|^4 \le \left|\left\langle T^2x, x\right\rangle\right|^2 + r^2\|T^*x\|^2, \tag{3.4}$$

from where we deduce (3.1).

We also know that, if $\|y-a\| \leq r \leq \|a\|$, then (see [3] or [2])

$$\|y\| \left(\|a\|^2 - r^2\right)^{\frac{1}{2}} \leq \operatorname{Re}\langle y, a\rangle,$$

which gives:

$$\|Tx\| \left(|\lambda|^2 \|Tx\|^2 - r^2\right)^{\frac{1}{2}} \leq \operatorname{Re}\langle Tx, \lambda T^* x\rangle \leq |\lambda| \left|\left\langle T^2 x, x\right\rangle\right|$$

i.e.,

$$\|Tx\| \left(\|Tx\|^2 - \frac{r^2}{|\lambda|^2}\right)^{\frac{1}{2}} \leq \left|\left\langle T^2 x, x\right\rangle\right| \tag{3.5}$$

for any $x \in H$, $\|x\| = 1$. Since, obviously

$$\left(\|Tx\|^2 - \frac{r^2}{|\lambda|^2}\right)^{\frac{1}{2}} \geq \left(\xi^2(T) - \frac{r^2}{|\lambda|^2}\right)^{\frac{1}{2}},$$

hence, by (3.5) we get (3.2). ∎

Theorem 4. *Assume that the normal operator $T : H \to H$ satisfies either (c) or, equivalently, (cc) for a given $\lambda \in \mathbb{C}\setminus\{0\}$ and $r > 0$. Then:*

$$\begin{aligned} (0 \leq)\ & \|Tx\|^4 - \left|\left\langle T^2 x, x\right\rangle\right|^2 \\ & \leq 2\left|\left\langle T^2 x, x\right\rangle\right| \|Tx\| \left[|\lambda| \|T\| - \left(|\lambda|^2 \xi^2(T) - r^2\right)^{\frac{1}{2}}\right] \\ & \left(\leq 2|\lambda| \left|\left\langle T^2 x, x\right\rangle\right| \|T\|^2\right), \end{aligned} \tag{3.6}$$

for any $x \in H$, $\|x\| = 1$.

Proof. We use the following reverse of the Schwarz inequality obtained in [2]:

$$0 \leq \|y\|^2 \|a\|^2 - |\langle y, a\rangle|^2 \leq 2|\langle y, a\rangle| \|a\| \left(\|a\| - \sqrt{\|a\|^2 - r^2}\right)$$

provided $\|y-a\| \leq r \leq \|a\|$.

Now, let $x \in H$, $\|x\| = 1$ and choose $y = Tx$, $a = \lambda T^* x$ to get from (3.6) that:

$$\begin{aligned} & \|Tx\|^2 |\lambda|^2 \|T^* x\|^2 - |\lambda|^2 \left|\left\langle T^2 x, x\right\rangle\right|^2 \\ & \qquad \leq 2|\lambda|^2 \left|\left\langle T^2 x, x\right\rangle\right| \|T^* x\| \left[|\lambda| \|T^* x\| - \left(|\lambda|^2 \|T^* x\|^2 - r^2\right)^{\frac{1}{2}}\right] \end{aligned}$$

giving

$$\begin{aligned} & \|Tx\|^4 - \left|\left\langle T^2 x, x\right\rangle\right|^2 \leq \\ & \qquad \leq 2\left|\left\langle T^2 x, x\right\rangle\right| \|Tx\| \left[|\lambda| \|Tx\| - \left(|\lambda|^2 \|Tx\|^2 - r^2\right)^{\frac{1}{2}}\right], \end{aligned}$$

which, by employing a similar argument to that used in the previous theorem, gives the desired inequality (3.6). ∎

4. Other Results for Accreative Operators

For a bounded linear operator $T : H \to H$ the following two statements are equivalent

$$\text{(d)} \qquad \operatorname{Re}\langle \Gamma T^{*}x - Tx, Tx - \gamma T^{*}x\rangle \geq 0 \quad \text{for any } \; x \in H, \|x\| = 1;$$

and

$$\text{(dd)} \qquad \left\| Tx - \frac{\gamma+\Gamma}{2} T^{*}x \right\| \leq \frac{1}{2}\left|\Gamma - \gamma\right| \left\|T^{*}x\right\| \quad \text{for any } \; x \in H, \|x\| = 1.$$

This follows by the elementary fact that in any inner product space $(H; \langle\cdot,\cdot\rangle)$ we have, for $x, z, Z \in H$, that

$$\operatorname{Re}\langle Z - x, x - z\rangle \geq 0 \tag{4.1}$$

if and only if

$$\left\| x - \frac{z+Z}{2} \right\| \leq \frac{1}{2}\left\|Z - z\right\|. \tag{4.2}$$

An operator $B : H \to H$ is called *accreative* [5, p. 26] if $\operatorname{Re}\langle Bx, x\rangle \geq 0$ for any $x \in H$. We observe that, the condition (d) is in fact equivalent with the condition that

(ddd) *the operator* $(T^{*} - \bar{\gamma}T)(\Gamma T^{*} - T)$ *is accreative.*

Now, if $T : H \to H$ is a normal operator, then the following statements are equivalent

$$\text{(e)} \qquad (T^{*} - \bar{\gamma}T)(\Gamma T^{*} - T) \geq 0$$

and

$$\text{(ee)} \qquad \Gamma\left[T^{*}\right]^{2} - (\bar{\gamma}\Gamma + 1)\, T^{*}T + \bar{\gamma}T^{2} \geq 0.$$

This is obvious since for T a normal operator we have $T^{*}T = TT^{*}$.

We also must remark that (e) implies that

$$0 \leq \langle \Gamma T^{*}x - Tx, Tx - \bar{\gamma}T^{*}x\rangle \quad \text{for any } \; x \in H, \|x\| = 1.$$

Therefore, (e) (or equivalently (ee)) is a sufficient condition for (d) (or equivalently (dd)[or (ddd)]) to hold true.

The following result may be stated.

Theorem 5. *Let $\gamma, \Gamma \in \mathbb{C}$ with $\Gamma \neq -\gamma$. For a normal operator $T : H \to H$ assume that (ddd) holds true. Then:*

$$(0 \leq)\, \|Tx\|^{2} - \left|\left\langle T^{2}x, x\right\rangle\right| \leq \frac{1}{4}\cdot\frac{|\Gamma-\gamma|^{2}}{|\Gamma+\gamma|}\|Tx\|^{2} \tag{4.3}$$

for any $x \in H$, $\|x\| = 1$.

Proof. We use the following reverse of the Schwarz inequality established in [4] (see also [2]):

$$\|z\|\,\|y\| - \frac{\operatorname{Re}(\Gamma+\gamma)\operatorname{Re}\langle z, y\rangle + \operatorname{Im}(\Gamma+\gamma)\operatorname{Im}\langle z, y\rangle}{|\Gamma+\gamma|} \leq \frac{1}{4}\cdot\frac{|\Gamma-\gamma|^{2}}{|\Gamma+\gamma|}\|y\|^{2}, \tag{4.4}$$

provided $\gamma, \Gamma \in \mathbb{C}$, $\Gamma \neq -\gamma$ and $z, y \in H$ satisfy either the condition

$$(\ell) \qquad \operatorname{Re}\langle \Gamma y - z, z - \gamma y\rangle \geq 0,$$

or, equivalently the condition

$$(\ell\ell) \qquad \left\| z - \frac{\gamma + \Gamma}{2} y \right\| \leq \frac{1}{2} |\Gamma - \gamma| \, \|y\| .$$

Now, if in (4.4) we choose $z = Tx$, $y = T^*x$ for $x \in H$, $\|x\| = 1$, then we obtain

$$\|Tx\| \, \|T^*x\| - |\langle Tx, T^*x\rangle| \leq \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \|T^*x\|^2 ,$$

which is equivalent with (4.3). ∎

Remark 2. *The second inequality in (4.3) is equivalent with*

$$\|Tx\|^2 \left(1 - \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \right) \leq \left| \langle T^2 x, x\rangle \right|$$

for any $x \in H$, $\|x\| = 1$. *This inequality is of interest if* $4|\Gamma + \gamma| \geq |\Gamma - \gamma|^2$.

The following result may be stated as well.

Theorem 6. *Let* $\gamma, \Gamma \in \mathbb{C}$ *with* $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$. *If* $T : H \to H$ *is a normal operator such that (ddd) holds true, then:*

$$\|Tx\|^2 \leq \frac{|\Gamma + \gamma|}{2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}} \left| \langle T^2 x, x\rangle \right| \tag{4.5}$$

for any $x \in H$, $\|x\| = 1$.

Proof. We can use the following reverse of the Schwarz inequality:

$$\|z\| \, \|y\| \leq \frac{|\Gamma + \gamma|}{2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}} |\langle z, y\rangle| , \tag{4.6}$$

provided $\gamma, \Gamma \in \mathbb{C}$ with $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$ and $z, y \in H$ are satisfying either the condition (ℓ) or, equivalently the condition $(\ell\ell)$.

Now, if in (4.6) we choose $z = Tx$, $y = T^*x$ for $x \in H$, $\|x\| = 1$, then we get

$$\|Tx\| \, \|T^*x\| \leq \frac{|\Gamma + \gamma|}{2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}} |\langle Tx, T^*x\rangle|$$

which is equivalent with (4.5). ∎

Also, we have:

Theorem 7. *If* γ, Γ, T *satisfy the hypothesis of Theorem 6, then we have the inequality:*

$$(0 \leq) \|Tx\|^4 - \left| \langle T^2 x, x\rangle \right|^2 \leq \left[|\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})} \right] \left| \langle T^2 x, x\rangle \right| \|Tx\|^2 , \tag{4.7}$$

for any $x \in H$, $\|x\| = 1$.

Proof. We make use of the following inequality [2]:

$$(0 \leq) \|z\|^2 \|y\|^2 - |\langle z, y\rangle|^2 \leq \left[|\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}\right] |\langle z, y\rangle| \|y\|^2 \tag{4.8}$$

that holds for $\gamma, \Gamma \in \mathbb{C}$ with $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$ and provided the vectors $z, y \in H$ satisfy either the condition (ℓ) or, equivalently the condition $(\ell\ell)$.

Now, if in (4.8) we choose $z = Tx$, $y = T^*x$ with $x \in H$, $\|x\| = 1$, then we get the desired result (4.7). ∎

Remark 3. *If we choose* $\Gamma = M \geq m = \gamma > 0$, *then, obviously*

$$\operatorname{Re}\langle MT^*x - Tx, Tx - mT^*x\rangle \geq 0 \quad \text{for any} \quad x \in H, \|x\| = 1 \tag{4.9}$$

is equivalent with

$$\left\|Tx - \frac{m+M}{2}T^*x\right\| \leq \frac{1}{2}(M - m) \quad \text{for any} \quad x \in H, \|x\| = 1, \tag{4.10}$$

or with the fact that

$$\text{the operator } (T^* - mT)(MT^* - T) \text{ is accreative.} \tag{4.11}$$

If T is normal, then the above are implied by the following two conditions that are equivalent between them:

$$(T^* - mT)(MT^* - T) \geq 0 \tag{4.12}$$

and

$$M[T^*]^2 - (mM + 1)T^*T + mT^2 \geq 0. \tag{4.13}$$

Now, if (4.11) holds, then

$$(0 \leq) \|Tx\|^2 - \left|\langle T^2x, x\rangle\right| \leq \frac{1}{4} \cdot \frac{(M - m)^2}{M + m} \|Tx\|^2, \tag{4.14}$$

$$\|Tx\|^2 \leq \frac{M + m}{2\sqrt{mM}} \left|\langle T^2x, x\rangle\right| \tag{4.15}$$

or, equivalently

$$(0 \leq) \|Tx\|^2 - \left|\langle T^2x, x\rangle\right| \leq \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} \left|\langle T^2x, x\rangle\right| \tag{4.16}$$

and

$$(0 \leq) \|Tx\|^4 - \left|\langle T^2x, x\rangle\right|^2 \leq \left(\sqrt{M} - \sqrt{m}\right)^2 \left|\langle T^2x, x\rangle\right| \|Tx\|^2, \tag{4.17}$$

for any $x \in H$, $\|x\| = 1$.

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