REVERSING THE CBS-INEQUALITY FOR SEQUENCES OF VECTORS IN HILBERT SPACES WITH APPLICATIONS (I)

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Abstract. Several reverses for the Cauchy-Bunyakovsky-Schwarz (CBS) inequality for sequences of vectors in Hilbert spaces are obtained. Applications for bounding the distance to a finite-dimensional subspace, in reversing the generalised triangle inequality and for Fourier coefficients are also given.

1. Introduction

Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$. One of the most important inequalities in inner product spaces with numerous applications, is the *Schwarz inequality*

$$|\langle x,y\rangle|^2 \leq \|x\|^2 \|y\|^2, \quad x,y\in H \tag{1.1}$$

or, equivalently,

$$|\langle x,y\rangle| \leq \|x\| \|y\|, \quad x,y\in H. \tag{1.2}$$

The case of equality holds iff there exists a scalar $\alpha\in\mathbb{K}$ such that $x=\alpha y$.

By a *multiplicative reverse* of the Schwarz inequality we understand an inequality of the form

$$(1\leq)\frac{\|x\|\|y\|}{|\langle x,y\rangle|}\leq k_1 \quad\text{or}\quad (1\leq)\frac{\|x\|^2\|y\|^2}{|\langle x,y\rangle|^2}\leq k_2 \tag{1.3}$$

with appropriate k_1 and k_2 and under various assumptions for the vectors x and y, while by an *additive reverse* we understand an inequality of the form

$$(0\leq)\|x\|\|y\|-|\langle x,y\rangle|\leq h_1 \quad\text{or}\quad (0\leq)\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\leq h_2. \tag{1.4}$$

Similar definition apply when $|\langle x,y\rangle|$ is replaced by $\operatorname{Re}\langle x,y\rangle$ or $|\operatorname{Re}\langle x,y\rangle|$.

The following recent reverses for the Schwarz inequality hold (see for instance the monograph on line [3, p. 20]):

Theorem 1. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$. If $x,y\in H$ and $r>0$ are such that*

$$\|x-y\|\leq r<\|y\|, \tag{1.5}$$

then we have the following multiplicative reverse of the Schwarz inequality

$$(1\leq)\frac{\|x\|\|y\|}{|\langle x,y\rangle|}\leq\frac{\|x\|\|y\|}{\operatorname{Re}\langle x,y\rangle}\leq\frac{\|y\|}{\sqrt{\|y\|^2-r^2}} \tag{1.6}$$

Date: 07 October, 2004.

2000 *Mathematics Subject Classification.* 46C05, 26D15.

Key words and phrases. CSB-inequality, Gram determinant, Gram inequality, Fourier coefficients, Triangle inequality.

and the subsequent additive reverses

$$(0\leq)\,\|x\|\,\|y\|-|\langle x,y\rangle|\leq\|x\|\,\|y\|-\operatorname{Re}\langle x,y\rangle \leq\frac{r^{2}}{\sqrt{\|y\|^{2}-r^{2}}\left(\|y\|+\sqrt{\|y\|^{2}-r^{2}}\right)}\operatorname{Re}\langle x,y\rangle \tag{1.7}$$

and

$$(0\leq)\,\|x\|^{2}\,\|y\|^{2}-|\langle x,y\rangle|^{2}\leq\|x\|^{2}\,\|y\|^{2}-[\operatorname{Re}\langle x,y\rangle]^{2} \leq r^{2}\,\|x\|^{2}. \tag{1.8}$$

All the above inequalities are sharp.

Other additive reverses of the quadratic Schwarz's inequality are incorporated in the following result [3, p. 18-19]:

Theorem 2. *Let $x,y\in H$ and $a,A\in\mathbb{K}$. If*

$$\operatorname{Re}\langle Ay-x,x-ay\rangle\geq0 \tag{1.9}$$

or, equivalently,

$$\left\|x-\frac{a+A}{2}\cdot y\right\|\leq\frac{1}{2}\,|A-a|\,\|y\|, \tag{1.10}$$

then

$$(0\leq)\,\|x\|^{2}\,\|y\|^{2}-|\langle x,y\rangle|^{2} \leq\frac{1}{4}\,|A-a|^{2}\,\|y\|^{4}-\begin{cases}\left|\frac{A+a}{2}\,\|y\|^{2}-\langle x,y\rangle\right|^{2}\\ \\ \|y\|^{2}\operatorname{Re}\langle Ay-x,x-ay\rangle\end{cases} \leq\frac{1}{4}\,|A-a|^{2}\,\|y\|^{4}. \tag{1.11}$$

The constant $\frac{1}{4}$ is best possible in all inequalities.

If one were to assume more about the complex numbers A and a, then one may state the following result as well [3, p. 21-23].

Theorem 3. *With the assumptions of Theorem 2 and, if in addition,* $\operatorname{Re}(A\bar{a})>0$, *then*

$$\|x\|\,\|y\|\leq\frac{1}{2}\cdot\frac{\operatorname{Re}\left[\left(\bar{A}+\bar{a}\right)\langle x,y\rangle\right]}{\sqrt{\operatorname{Re}(A\bar{a})}}\leq\frac{1}{2}\cdot\frac{|A+a|}{\sqrt{\operatorname{Re}(A\bar{a})}}\,|\langle x,y\rangle|, \tag{1.12}$$

$$(0\leq)\,\|x\|\,\|y\|-\operatorname{Re}\langle x,y\rangle\leq\frac{1}{2}\cdot\frac{\operatorname{Re}\left[\left(\bar{A}+\bar{a}-2\sqrt{\operatorname{Re}(A\bar{a})}\right)\langle x,y\rangle\right]}{\sqrt{\operatorname{Re}(A\bar{a})}} \tag{1.13}$$

and

$$(0\leq)\,\|x\|^{2}\,\|y\|^{2}-|\langle x,y\rangle|^{2}\leq\frac{1}{4}\cdot\frac{|A-a|^{2}}{\operatorname{Re}(A\bar{a})}\,|\langle x,y\rangle|^{2}. \tag{1.14}$$

The constants $\frac{1}{2}$ and $\frac{1}{4}$ are best possible.

Remark 1. *If $A = M$, $a = m$ and $M \geq m > 0$, then (1.12) and (1.13) may be written in a more convenient form as*

$$\|x\| \|y\| \leq \frac{M+m}{2\sqrt{mM}} \operatorname{Re}\langle x, y\rangle \tag{1.15}$$

and

$$(0 \leq) \|x\| \|y\| - \operatorname{Re}\langle x, y\rangle \leq \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{mM}} \operatorname{Re}\langle x, y\rangle . \tag{1.16}$$

Here the constant $\frac{1}{2}$ is sharp in both inequalities.

In this paper several reverses for the Cauchy-Bunyakovsky-Schwarz (CBS) inequality for sequences of vectors in Hilbert spaces are obtained. Applications for bounding the distance to a finite-dimensional subspace and in reversing the generalised triangle inequality are also given.

A continuation of this work for different classes of reverse inequalities is planed to be considered in the subsequent paper [5].

2. Reverses of the (CBS) −Inequality for Two Sequences in $\ell^2_{\mathbf{p}}(K)$

Let $(K, \langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbb{K}$, $p_i \geq 0$, $i \in \mathbb{N}$ with $\sum_{i=1}^{\infty} p_i = 1$. Consider $\ell^2_{\mathbf{p}}(K)$ as the space

$$\ell^2_{\mathbf{p}}(K) := \left\{ x = (x_i)_{i\in\mathbb{N}} \middle| x_i \in K,\ i \in \mathbb{N} \text{ and } \sum_{i=1}^{\infty} p_i \|x_i\|^2 < \infty \right\}.$$

It is well known that $\ell^2_{\mathbf{p}}(K)$ endowed with the inner product

$$\langle x, y\rangle_{\mathbf{p}} := \sum_{i=1}^{\infty} p_i \langle x_i, y_i\rangle$$

is a Hilbert space over $\mathbb{K}$. The norm $\|\cdot\|_{\mathbf{p}}$ of $\ell^2_{\mathbf{p}}(K)$ is given by

$$\|x\|_{\mathbf{p}} := \left(\sum_{i=1}^{\infty} p_i \|x_i\|^2 \right)^{\frac{1}{2}}.$$

If $x, y \in \ell^2_{\mathbf{p}}(K)$, then the following Cauchy-Bunyakovsky-Schwarz (CBS) inequality holds true

$$\sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2 \geq \left| \sum_{i=1}^{\infty} p_i \langle x_i, y_i\rangle \right|^2 \tag{2.1}$$

with equality iff there exists a $\lambda \in \mathbb{K}$ such that $x_i = \lambda y_i$ for each $i \in \mathbb{N}$.

This is an obvious consequence of the Schwarz inequality (1.1) written for the inner product $\langle\cdot,\cdot\rangle_{\mathbf{p}}$ defined on $\ell^2_{\mathbf{p}}(K)$.

The following proposition may be stated.

Proposition 1. *Let $x, y \in \ell^2_{\mathbf{p}}(K)$ and $r > 0$. Assume that*

$$\|x_i - y_i\| \leq r < \|y_i\| \quad \textit{for each } i \in \mathbb{N}. \tag{2.2}$$

Then we have the inequality

$$(1\leq)\frac{\left(\sum_{i=1}^{\infty}p_i\left\|x_i\right\|^2\sum_{i=1}^{\infty}p_i\left\|y_i\right\|^2\right)^{\frac{1}{2}}}{\left|\sum_{i=1}^{\infty}p_i\left\langle x_i,y_i\right\rangle\right|}\leq\frac{\left(\sum_{i=1}^{\infty}p_i\left\|x_i\right\|^2\sum_{i=1}^{\infty}p_i\left\|y_i\right\|^2\right)^{\frac{1}{2}}}{\sum_{i=1}^{\infty}p_i\operatorname{Re}\left\langle x_i,y_i\right\rangle}\leq\frac{\left(\sum_{i=1}^{\infty}p_i\left\|y_i\right\|^2\right)^{\frac{1}{2}}}{\sqrt{\sum_{i=1}^{\infty}p_i\left\|y_i\right\|^2-r^2}},\tag{2.3}$$

$$\begin{aligned}(0\leq)&\left(\sum_{i=1}^{\infty}p_i\left\|x_i\right\|^2\sum_{i=1}^{\infty}p_i\left\|y_i\right\|^2\right)^{\frac{1}{2}}-\left|\sum_{i=1}^{\infty}p_i\left\langle x_i,y_i\right\rangle\right|\\
&\leq\left(\sum_{i=1}^{\infty}p_i\left\|x_i\right\|^2\sum_{i=1}^{\infty}p_i\left\|y_i\right\|^2\right)^{\frac{1}{2}}-\sum_{i=1}^{\infty}p_i\operatorname{Re}\left\langle x_i,y_i\right\rangle\\
&\leq\frac{r^2\cdot\sum_{i=1}^{\infty}p_i\operatorname{Re}\left\langle x_i,y_i\right\rangle}{\sqrt{\sum_{i=1}^{\infty}p_i\left\|y_i\right\|^2-r^2}\left[\left(\sum_{i=1}^{\infty}p_i\left\|y_i\right\|^2\right)^{\frac{1}{2}}+\sqrt{\sum_{i=1}^{\infty}p_i\left\|y_i\right\|^2-r^2}\right]}\end{aligned}\tag{2.4}$$

and

$$\begin{aligned}(0\leq)&\sum_{i=1}^{\infty}p_i\left\|x_i\right\|^2\sum_{i=1}^{\infty}p_i\left\|y_i\right\|^2-\left|\sum_{i=1}^{\infty}p_i\left\langle x_i,y_i\right\rangle\right|^2\\
&\leq\sum_{i=1}^{\infty}p_i\left\|x_i\right\|^2\sum_{i=1}^{\infty}p_i\left\|y_i\right\|^2-\left[\sum_{i=1}^{\infty}p_i\operatorname{Re}\left\langle x_i,y_i\right\rangle\right]^2\\
&\leq r^2\sum_{i=1}^{\infty}p_i\left\|x_i\right\|^2.\end{aligned}\tag{2.5}$$

Proof. From (2.2), we have

$$\left\|x-y\right\|_{\mathbf{p}}^2=\sum_{i=1}^{\infty}p_i\left\|x_i-y_i\right\|^2\leq r^2\sum_{i=1}^{\infty}p_i\leq\sum_{i=1}^{\infty}p_i\left\|y_i\right\|^2=\left\|y\right\|_{\mathbf{p}}^2,$$

giving $\left\|x-y\right\|_{\mathbf{p}}\leq r\leq\left\|y\right\|_{\mathbf{p}}$. Applying Theorem 1 for $\ell_{\mathbf{p}}^2(K)$ and $\langle\cdot,\cdot\rangle_{\mathbf{p}}$, we deduce the desired inequality. ∎

The following proposition holds.

Proposition 2. *Let* $x,y\in\ell_{\mathbf{p}}^2(K)$ *and* $a,A\in\mathbb{K}$. *If*

$$\operatorname{Re}\left\langle Ay_i-x_i,x_i-ay_i\right\rangle\geq0\quad\text{for each }i\in\mathbb{N}\tag{2.6}$$

or, equivalently,

$$\left\|x_i-\frac{a+A}{2}y_i\right\|\leq\frac{1}{2}\left|A-a\right|\left\|y_i\right\|\quad\text{for each }i\in\mathbb{N}\tag{2.7}$$

then

$$(2.8)\qquad (0 \leq) \sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2 - \left| \sum_{i=1}^{\infty} p_i \langle x_i, y_i \rangle \right|^2$$

$$\leq \frac{1}{4} |A-a|^2 \left(\sum_{i=1}^{\infty} p_i \|y_i\|^2 \right)^2$$

$$- \left\{ \begin{array}{c} \left| \frac{A+a}{2} \sum_{i=1}^{\infty} p_i \|y_i\|^2 - \sum_{i=1}^{\infty} p_i \langle x_i, y_i \rangle \right|^2 \\ \\ \sum_{i=1}^{\infty} p_i \|y_i\|^2 \sum_{i=1}^{\infty} p_i \operatorname{Re} \langle A y_i - x_i, x_i - a y_i \rangle \end{array} \right.$$

$$\leq \frac{1}{4} |A-a|^2 \left(\sum_{i=1}^{\infty} p_i \|y_i\|^2 \right)^2.$$

The proof follows by Theorem 2, we omit the details.

Finally, on using Theorem 3, we may state:

Proposition 3. *Assume that* x, y, a *and* A *are as in Proposition 2. Moreover, if* $\operatorname{Re}(A\bar{a}) > 0$, *then we have the inequality:*

$$(2.9)\qquad \left(\sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2 \right)^{\frac{1}{2}} \leq \frac{1}{2} \cdot \frac{\operatorname{Re}\left[\left(\bar{A} + \bar{a} \right) \sum_{i=1}^{\infty} p_i \langle x_i, y_i \rangle \right]}{\sqrt{\operatorname{Re}(A\bar{a})}}$$

$$\leq \frac{1}{2} \cdot \frac{|A-a|}{\sqrt{\operatorname{Re}(A\bar{a})}} \left| \sum_{i=1}^{\infty} p_i \langle x_i, y_i \rangle \right|,$$

$$(2.10)\qquad (0 \leq) \left(\sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2 \right)^{\frac{1}{2}} - \sum_{i=1}^{\infty} p_i \operatorname{Re} \langle x_i, y_i \rangle$$

$$\leq \frac{1}{2} \cdot \frac{\operatorname{Re}\left[\left(\bar{A} + \bar{a} - 2\sqrt{\operatorname{Re}(A\bar{a})} \right) \sum_{i=1}^{\infty} p_i \langle x_i, y_i \rangle \right]}{\sqrt{\operatorname{Re}(A\bar{a})}}$$

and

$$(2.11)\qquad (0 \leq) \sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2 - \left| \sum_{i=1}^{\infty} p_i \langle x_i, y_i \rangle \right|^2$$

$$\leq \frac{1}{4} \cdot \frac{|A-a|^2}{\operatorname{Re}(A\bar{a})} \left| \sum_{i=1}^{\infty} p_i \langle x_i, y_i \rangle \right|^2.$$

3. Reverses of the (CBS) −Inequality for Mixed Sequences

Let $(K, \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbb{K}$ and for $p_i \geq 0$, $i \in \mathbb{N}$ with $\sum_{i=1}^{\infty} p_i = 1$, and $\ell_{\mathbf{p}}^2(K)$ the Hilbert space defined in the previous section.

If

$$\alpha \in \ell_{\mathbf{p}}^2(\mathbb{K}) := \left\{ \alpha = (\alpha_i)_{i \in \mathbb{N}} \middle| \alpha_i \in \mathbb{K},\ i \in \mathbb{N} \ \text{ and } \ \sum_{i=1}^{\infty} p_i |\alpha_i|^2 < \infty \right\}$$

and $x \in \ell_{\mathbf{p}}^{2}(K)$, then the following Cauchy-Bunyakovsky-Schwarz (CBS) inequality holds true:

$$\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \geq \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\|^2, \tag{3.1}$$

with equality if and only if there exists a vector $v \in K$ such that $x_i = \overline{\alpha_i} v$ for any $i \in \mathbb{N}$.

The inequality (3.1) follows by the obvious identity

$$\begin{aligned} &\sum_{i=1}^{n} p_i |\alpha_i|^2 \sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{n} p_i \alpha_i x_i \right\|^2 \\ = &\ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j \|\overline{\alpha_i} x_j - \overline{\alpha_j} x_i\|^2, \end{aligned}$$

for any $n \in \mathbb{N}$, $n \geq 1$.

In the following we establish some reverses of the (CBS) $-$inequality in some of its various equivalent forms that will be specified where they occur.

Theorem 4. *Let $\alpha \in \ell_{\mathbf{p}}^{2}(\mathbb{K})$, $x \in \ell_{\mathbf{p}}^{2}(K)$ and $a \in K$, $r > 0$ such that $\|a\| > r$. If the following condition holds*

$$\|x_i - \overline{\alpha_i} a\| \leq r |\alpha_i| \quad \text{for each } i \in \mathbb{N}, \tag{3.2}$$

(note that if $\alpha_i \neq 0$ for any $i \in \mathbb{N}$, then the condition (3.2) is equivalent to

$$\left\| \frac{x_i}{\overline{\alpha_i}} - a \right\| \leq r \quad \text{for each } i \in \mathbb{N}), \tag{3.3}$$

then we have the following inequalities

$$\begin{aligned} \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \right)^{\frac{1}{2}} &\leq \frac{1}{\sqrt{\|a\|^2 - r^2}} \operatorname{Re} \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, a \right\rangle \\ &\leq \frac{\|a\|}{\sqrt{\|a\|^2 - r^2}} \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\|; \end{aligned} \tag{3.4}$$

$$\begin{aligned} 0 &\leq \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \right)^{\frac{1}{2}} - \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\| \\ &\leq \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \right)^{\frac{1}{2}} - \operatorname{Re} \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, \frac{a}{\|a\|} \right\rangle \\ &\leq \frac{r^2}{\sqrt{\|a\|^2 - r^2} \left(\|a\| + \sqrt{\|a\|^2 - r^2} \right)} \operatorname{Re} \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, \frac{a}{\|a\|} \right\rangle \\ &\leq \frac{r^2}{\sqrt{\|a\|^2 - r^2} \left(\|a\| + \sqrt{\|a\|^2 - r^2} \right)} \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\|; \end{aligned} \tag{3.5}$$

$$\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \le \frac{1}{\|a\|^2 - r^2} \left[\operatorname{Re} \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, a \right\rangle \right]^2 \le \frac{\|a\|^2}{\|a\|^2 - r^2} \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\|^2 \tag{3.6}$$

and

$$\begin{aligned} 0 &\le \sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\|^2 \\ &\le \sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 - \left[\operatorname{Re} \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, \frac{a}{\|a\|} \right\rangle \right]^2 \\ &\le \frac{r^2}{\|a\|^2 \left(\|a\|^2 - r^2 \right)} \left[\operatorname{Re} \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, a \right\rangle \right]^2 \\ &\le \frac{r^2}{\|a\|^2 - r^2} \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\|^2 . \end{aligned} \tag{3.7}$$

All the inequalities in (3.4) – (3.7) are sharp.

Proof. From (3.2) we deduce

$$\|x_i\|^2 - 2 \operatorname{Re} \langle x_i, \overline{\alpha_i} a \rangle + |\alpha_i|^2 \|a\|^2 \le |\alpha_i|^2 r^2$$

for any $i \in \mathbb{N}$, which is clearly equivalent to

$$\|x_i\|^2 + \left(\|a\|^2 - r^2 \right) |\alpha_i|^2 \le 2 \operatorname{Re} \langle \alpha_i x_i, a \rangle \tag{3.8}$$

for each $i \in \mathbb{N}$.

If we multiply (3.8) by $p_i \ge 0$ and sum over $i \in \mathbb{N}$, then we deduce

$$\sum_{i=1}^{\infty} p_i \|x_i\|^2 + \left(\|a\|^2 - r^2 \right) \sum_{i=1}^{\infty} p_i |\alpha_i|^2 \le 2 \operatorname{Re} \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, a \right\rangle . \tag{3.9}$$

Now, dividing (3.9) by $\sqrt{\|a\|^2 - r^2} > 0$ we get

$$\frac{1}{\sqrt{\|a\|^2 - r^2}} \sum_{i=1}^{\infty} p_i \|x_i\|^2 + \sqrt{\|a\|^2 - r^2} \sum_{i=1}^{\infty} p_i |\alpha_i|^2 \le \frac{2}{\sqrt{\|a\|^2 - r^2}} \operatorname{Re} \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, a \right\rangle . \tag{3.10}$$

On the other hand, by the elementary inequality

$$\frac{1}{\alpha} p + \alpha q \ge 2\sqrt{pq}, \qquad \alpha > 0, \ p, q \ge 0,$$

we can state that:

$$2\left[\sum_{i=1}^{\infty}p_i\left|\alpha_i\right|^2\sum_{i=1}^{\infty}p_i\left\|x_i\right\|^2\right]^{\frac{1}{2}} \leq \frac{1}{\sqrt{\left\|a\right\|^2-r^2}}\sum_{i=1}^{\infty}p_i\left\|x_i\right\|^2+\sqrt{\left\|a\right\|^2-r^2}\sum_{i=1}^{\infty}p_i\left|\alpha_i\right|^2. \tag{3.11}$$

Making use of (3.10) and (3.11), we deduce the first part of (3.4).

The second part is obvious by Schwarz's inequality

$$\operatorname{Re}\left\langle\sum_{i=1}^{\infty}p_i\alpha_ix_i,a\right\rangle\leq\left\|\sum_{i=1}^{\infty}p_i\alpha_ix_i\right\|\left\|a\right\|.$$

If $p_1=1$, $x_1=x$, $\alpha_1=1$ and $p_i=0$, $\alpha_i=0$, $x_i=0$ for $i\geq 2$, then from (3.4) we deduce the inequality

$$\left\|x\right\|\leq\frac{1}{\sqrt{\left\|a\right\|^2-r^2}}\operatorname{Re}\left\langle x,a\right\rangle\leq\frac{\left\|x\right\|\left\|a\right\|}{\sqrt{\left\|a\right\|^2-r^2}}$$

provided $\left\|x-a\right\|\leq r<\left\|a\right\|$, $x,a\in K$. The sharpness of this inequality has been shown in [3, p. 20], and we omit the details.

The other inequalities are obvious consequences of (3.4) and we omit the details. ∎

The following corollary may be stated.

Corollary 1. *Let* $\alpha\in\ell_{\mathbf{p}}^2(\mathbb{K})$, $x\in\ell_{\mathbf{p}}^2(K)$, $e\in H$, $\left\|e\right\|=1$ *and* $\varphi,\phi\in\mathbb{K}$ *with* $\operatorname{Re}(\phi\bar{\varphi})>0$. *If*

$$\left\|x_i-\overline{\alpha_i}\cdot\frac{\varphi+\phi}{2}\cdot e\right\|\leq\frac{1}{2}\left|\phi-\varphi\right|\left|\alpha_i\right| \tag{3.12}$$

for each $i\in\mathbb{N}$, *or, equivalently*

$$\operatorname{Re}\left\langle\phi\overline{\alpha_i}e-x_i,x_i-\varphi\overline{\alpha_i}e\right\rangle\geq 0 \tag{3.13}$$

for each $i\in\mathbb{N}$, *(note that, if* $\alpha_i\neq 0$ *for any* $i\in\mathbb{N}$, *then (3.12) is equivalent to*

$$\left\|\frac{x_i}{\overline{\alpha_i}}-\frac{\varphi+\phi}{2}\cdot e\right\|\leq\frac{1}{2}\left|\phi-\varphi\right| \tag{3.14}$$

for each $i\in\mathbb{N}$ *and (3.13) is equivalent to*

$$\operatorname{Re}\left\langle\phi e-\frac{x_i}{\overline{\alpha_i}},\frac{x_i}{\overline{\alpha_i}}-\varphi e\right\rangle\geq 0$$

for each $i\in\mathbb{N}$*), then the following reverses of the* (CBS) *−inequality are valid:*

$$\left(\sum_{i=1}^{\infty}p_i\left|\alpha_i\right|^2\sum_{i=1}^{\infty}p_i\left\|x_i\right\|^2\right)^{\frac{1}{2}}\leq\frac{\operatorname{Re}\left[\left(\bar{\phi}+\bar{\varphi}\right)\left\langle\sum_{i=1}^{\infty}p_i\alpha_ix_i,e\right\rangle\right]}{2\left[\operatorname{Re}(\phi\overline{\varphi})\right]^{\frac{1}{2}}} \leq\frac{1}{2}\cdot\frac{\left|\varphi+\phi\right|}{\left[\operatorname{Re}(\phi\overline{\varphi})\right]^{\frac{1}{2}}}\left\|\sum_{i=1}^{\infty}p_i\alpha_ix_i\right\|; \tag{3.15}$$

$$(3.16)\qquad 0 \leq \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2\right)^{\frac{1}{2}} - \left\|\sum_{i=1}^{\infty} p_i \alpha_i x_i\right\|$$
$$\leq \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2\right)^{\frac{1}{2}} - \operatorname{Re}\left[\frac{\bar{\phi}+\bar{\varphi}}{|\varphi+\phi|}\left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, e\right\rangle\right]$$
$$\leq \frac{|\phi-\varphi|^2}{2\sqrt{\operatorname{Re}(\phi\varphi)}\left(|\varphi+\phi| + 2\sqrt{\operatorname{Re}(\phi\overline{\varphi})}\right)} \operatorname{Re}\left[\frac{\bar{\phi}+\bar{\varphi}}{|\varphi+\phi|}\left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, e\right\rangle\right]$$
$$\leq \frac{|\phi-\varphi|^2}{2\sqrt{\operatorname{Re}(\phi\varphi)}\left(|\varphi+\phi| + 2\sqrt{\operatorname{Re}(\phi\overline{\varphi})}\right)} \left\|\sum_{i=1}^{\infty} p_i \alpha_i x_i\right\|;$$

$$(3.17)\qquad \sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \leq \frac{1}{4\operatorname{Re}(\phi\bar{\varphi})}\left[\operatorname{Re}\left\{\left(\bar{\phi}+\bar{\varphi}\right)\left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, e\right\rangle\right\}\right]^2$$
$$\leq \frac{1}{4}\cdot\frac{|\varphi+\phi|^2}{\operatorname{Re}(\phi\bar{\varphi})}\left\|\sum_{i=1}^{\infty} p_i \alpha_i x_i\right\|^2$$

and

$$(3.18)\qquad 0 \leq \sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 - \left\|\sum_{i=1}^{\infty} p_i \alpha_i x_i\right\|^2$$
$$\leq \sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 - \left[\operatorname{Re}\left\{\frac{\bar{\phi}+\bar{\varphi}}{|\varphi+\phi|}\left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, e\right\rangle\right\}\right]^2$$
$$\leq \frac{|\phi-\varphi|^2}{4|\phi+\varphi|^2\operatorname{Re}(\phi\bar{\varphi})}\left\{\operatorname{Re}\left[\left(\bar{\phi}+\bar{\varphi}\right)\left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, e\right\rangle\right]\right\}^2$$
$$\leq \frac{|\phi-\varphi|^2}{4\operatorname{Re}(\phi\bar{\varphi})}\left\|\sum_{i=1}^{\infty} p_i \alpha_i x_i\right\|^2.$$

All the inequalities in (3.15) – (3.18) are sharp.

Remark 2. *We remark that if $M \geq m > 0$ and for $\alpha \in \ell^2_{\mathbf{p}}(\mathbb{K})$, $x \in \ell^2_{\mathbf{p}}(K)$, $e \in H$ with $\|e\| = 1$, one would assume that either*

$$(3.19)\qquad \left\|\frac{x_i}{\alpha_i} - \frac{M+m}{2}\cdot e\right\| \leq \frac{1}{2}(M-m)$$

for each $i \in \mathbb{N}$, or, equivalently

$$(3.20)\qquad \operatorname{Re}\left\langle Me - \frac{x_i}{\alpha_i}, \frac{x_i}{\alpha_i} - me\right\rangle \geq 0$$

for each $i \in \mathbb{N}$, then the following, much simpler reverses of the (CBS) −inequality may be stated:

$$\left(\sum_{i=1}^{\infty} p_i \left|\alpha_i\right|^2 \sum_{i=1}^{\infty} p_i \left\|x_i\right\|^2\right)^{\frac{1}{2}} \leq \frac{M+m}{2\sqrt{mM}} \operatorname{Re}\left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, e\right\rangle \leq \frac{M+m}{2\sqrt{mM}} \left\|\sum_{i=1}^{\infty} p_i \alpha_i x_i\right\|; \tag{3.21}$$

$$\begin{aligned} 0 &\leq \left(\sum_{i=1}^{\infty} p_i \left|\alpha_i\right|^2 \sum_{i=1}^{\infty} p_i \left\|x_i\right\|^2\right)^{\frac{1}{2}} - \left\|\sum_{i=1}^{\infty} p_i \alpha_i x_i\right\| \\ &\leq \left(\sum_{i=1}^{\infty} p_i \left|\alpha_i\right|^2 \sum_{i=1}^{\infty} p_i \left\|x_i\right\|^2\right)^{\frac{1}{2}} - \operatorname{Re}\left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, e\right\rangle \\ &\leq \frac{(M-m)^2}{2\left(\sqrt{M}+\sqrt{m}\right)^2 \sqrt{mM}} \operatorname{Re}\left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, e\right\rangle \\ &\leq \frac{(M-m)^2}{2\left(\sqrt{M}+\sqrt{m}\right)^2 \sqrt{mM}} \left\|\sum_{i=1}^{\infty} p_i \alpha_i x_i\right\|; \end{aligned} \tag{3.22}$$

$$\begin{aligned} &\sum_{i=1}^{\infty} p_i \left|\alpha_i\right|^2 \sum_{i=1}^{\infty} p_i \left\|x_i\right\|^2 - \left\|\sum_{i=1}^{\infty} p_i \alpha_i x_i\right\|^2 \\ &\leq \frac{(M+m)^2}{4mM} \left[\operatorname{Re}\left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, e\right\rangle\right]^2 \\ &\leq \frac{(M+m)^2}{4mM} \left\|\sum_{i=1}^{\infty} p_i \alpha_i x_i\right\|^2 \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} 0 &\leq \sum_{i=1}^{\infty} p_i \left|\alpha_i\right|^2 \sum_{i=1}^{\infty} p_i \left\|x_i\right\|^2 - \left\|\sum_{i=1}^{\infty} p_i \alpha_i x_i\right\|^2 \\ &\leq \sum_{i=1}^{\infty} p_i \left|\alpha_i\right|^2 \sum_{i=1}^{\infty} p_i \left\|x_i\right\|^2 - \left[\operatorname{Re}\left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, e\right\rangle\right]^2 \\ &\leq \frac{(M-m)^2}{4mM} \left[\operatorname{Re}\left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, e\right\rangle\right]^2 \\ &\leq \frac{(M-m)^2}{4mM} \left\|\sum_{i=1}^{\infty} p_i \alpha_i x_i\right\|^2. \end{aligned} \tag{3.24}$$

4. Reverses for the Generalised Triangle Inequality

In 1966, J.B. Diaz and F.T. Metcalf [2] proved the following reverse of the generalised triangle inequality holding in an inner product space $(H; \langle \cdot, \cdot \rangle)$ over the real or complex number field $\mathbb{K}$:

$$r\sum_{i=1}^{n}\|x_i\| \leq \left\|\sum_{i=1}^{n} x_i\right\| \tag{4.1}$$

provided the vectors $x_1,\dots,x_n \in H\backslash\{0\}$ satisfy the assumption

$$0 \leq r \leq \frac{\operatorname{Re}\langle x_i, a\rangle}{\|x_i\|}, \tag{4.2}$$

where $a \in H$ and $\|a\| = 1$.

In an attempt to diversify the assumptions for which such reverse results hold, the author pointed out in [4] that

$$\sqrt{1-\rho^2}\sum_{i=1}^{n}\|x_i\| \leq \left\|\sum_{i=1}^{n} x_i\right\|, \tag{4.3}$$

where the vectors $x_i, i \in \{1,\dots,n\}$ satisfy the condition

$$\|x_i - a\| \leq \rho, \qquad i \in \{1,\dots,n\} \tag{4.4}$$

where $a \in H$, $\|a\| = 1$ and $\rho \in (0,1)$.

If, for $M \geq m > 0$, the vectors $x_i \in H$, $i \in \{1,\dots,n\}$ verify either

$$\operatorname{Re}\langle Ma - x_i, x_i - ma\rangle \geq 0, \qquad i \in \{1,\dots,n\}, \tag{4.5}$$

or, equivalently,

$$\left\|x_i - \frac{M+m}{2}\cdot a\right\| \leq \frac{1}{2}(M-m), \qquad i \in \{1,\dots,n\}, \tag{4.6}$$

where $a \in H$, $\|a\| = 1$, then the following reverse of the generalised triangle inequality may be stated as well [4]

$$\frac{2\sqrt{mM}}{M+m}\sum_{i=1}^{n}\|x_i\| \leq \left\|\sum_{i=1}^{n} x_i\right\|. \tag{4.7}$$

Note that the inequalities (4.1), (4.3), and (4.7) are sharp; necessary and sufficient equality conditions were provided (see [2] and [4]).

It is obvious, from Theorem 4, that, if

$$\|x_i - a\| \leq r, \quad \text{for} \quad i \in \{1,\dots,n\}, \tag{4.8}$$

where $\|a\| > r$, $a \in H$ and $x_i \in H$, $i \in \{1,\dots,n\}$, then one can state the inequalities

$$\begin{aligned} \sum_{i=1}^{n}\|x_i\| &\leq \sqrt{n}\left(\sum_{i=1}^{n}\|x_i\|^2\right)^{\frac{1}{2}} \\ &\leq \frac{1}{\sqrt{\|a\|^2 - r^2}}\operatorname{Re}\left\langle\sum_{i=1}^{n} x_i, a\right\rangle \\ &\leq \frac{\|a\|}{\sqrt{\|a\|^2 - r^2}}\left\|\sum_{i=1}^{n} x_i\right\|; \end{aligned} \tag{4.9}$$

and

$$0 \le \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| \tag{4.10}$$
$$\le \sqrt{n} \left(\sum_{i=1}^{n} \|x_i\|^2 \right)^{\frac{1}{2}} - \left\| \sum_{i=1}^{n} x_i \right\|$$
$$\le \sqrt{n} \left(\sum_{i=1}^{n} \|x_i\|^2 \right)^{\frac{1}{2}} - \operatorname{Re} \left\langle \sum_{i=1}^{n} x_i, \frac{a}{\|a\|} \right\rangle$$
$$\le \frac{r^2}{\sqrt{\|a\|^2 - r^2} \left(\|a\| + \sqrt{\|a\|^2 - r^2} \right)} \operatorname{Re} \left\langle \sum_{i=1}^{n} x_i, \frac{a}{\|a\|} \right\rangle$$
$$\le \frac{r^2}{\sqrt{\|a\|^2 - r^2} \left(\|a\| + \sqrt{\|a\|^2 - r^2} \right)} \left\| \sum_{i=1}^{n} x_i \right\|.$$

We note that for $\|a\| = 1$ and $r \in (0,1)$, the inequality (3.9) becomes

$$\sqrt{1 - r^2} \sum_{i=1}^{n} \|x_i\| \le \sqrt{(1 - r^2)\, n} \left(\sum_{i=1}^{n} \|x_i\|^2 \right)^{\frac{1}{2}} \tag{4.11}$$
$$\le \operatorname{Re} \left\langle \sum_{i=1}^{n} x_i, a \right\rangle \le \left\| \sum_{i=1}^{n} x_i \right\|,$$

which is a refinement of (4.3).

With the same assumptions for a and r, we have from (4.10) the following additive reverse of the generalised triangle inequality:

$$0 \le \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| \tag{4.12}$$
$$\le \frac{r^2}{\sqrt{1 - r^2} \left(1 + \sqrt{1 - r^2} \right)} \operatorname{Re} \left\langle \sum_{i=1}^{n} x_i, a \right\rangle$$
$$\le \frac{r^2}{\sqrt{1 - r^2} \left(1 + \sqrt{1 - r^2} \right)} \left\| \sum_{i=1}^{n} x_i \right\|.$$

We can obtain the following reverses of the generalised triangle inequality from Corollary 1 when the assumptions are in terms of complex numbers ϕ and φ:

If $\varphi, \phi \in \mathbb{K}$ with $\operatorname{Re}(\phi\bar{\varphi}) > 0$ and $x_i \in H$, $i \in \{1, \dots, n\}$, $e \in H$, $\|e\| = 1$ are such that

$$\left\| x_i - \frac{\varphi + \phi}{2} e \right\| \le \frac{1}{2} |\phi - \varphi| \quad \text{for each } \ i \in \{1, \dots, n\}, \tag{4.13}$$

or, equivalently,

$$\operatorname{Re} \langle \phi e - x_i, x_i - \varphi e \rangle \ge 0 \ \text{ for each } \ i \in \{1, \dots, n\},$$

then we have the following reverses of the generalised triangle inequality:

$$\sum_{i=1}^{n}\|x_i\| \le \sqrt{n}\left(\sum_{i=1}^{n}\|x_i\|^2\right)^{\frac{1}{2}} \le \frac{\operatorname{Re}\left[\left(\bar{\phi}+\bar{\varphi}\right)\left\langle\sum_{i=1}^{n}x_i,e\right\rangle\right]}{2\sqrt{\operatorname{Re}\left(\phi\bar{\varphi}\right)}} \le \frac{1}{2}\cdot\frac{\left|\bar{\phi}+\bar{\varphi}\right|}{\sqrt{\operatorname{Re}\left(\phi\bar{\varphi}\right)}}\left\|\sum_{i=1}^{n}x_i\right\| \tag{4.14}$$

and

$$\begin{aligned} 0 &\le \sum_{i=1}^{n}\|x_i\| - \left\|\sum_{i=1}^{n}x_i\right\| \\ &\le \sqrt{n}\left(\sum_{i=1}^{n}\|x_i\|^2\right)^{\frac{1}{2}} - \left\|\sum_{i=1}^{n}x_i\right\| \\ &\le \sqrt{n}\left(\sum_{i=1}^{n}\|x_i\|^2\right)^{\frac{1}{2}} - \operatorname{Re}\left[\frac{\left|\bar{\phi}+\bar{\varphi}\right|}{\sqrt{\operatorname{Re}\left(\bar{\phi}\bar{\varphi}\right)}}\left\langle\sum_{i=1}^{n}x_i,e\right\rangle\right] \\ &\le \frac{|\phi-\varphi|^2}{2\sqrt{\operatorname{Re}\left(\phi\bar{\varphi}\right)}\left(|\phi+\varphi|+2\sqrt{\operatorname{Re}\left(\phi\bar{\varphi}\right)}\right)}\operatorname{Re}\left[\frac{\bar{\phi}+\bar{\varphi}}{\left|\bar{\phi}+\bar{\varphi}\right|}\left\langle\sum_{i=1}^{n}x_i,e\right\rangle\right] \\ &\le \frac{|\phi-\varphi|^2}{2\sqrt{\operatorname{Re}\left(\phi\bar{\varphi}\right)}\left(|\phi+\varphi|+2\sqrt{\operatorname{Re}\left(\phi\bar{\varphi}\right)}\right)}\left\|\sum_{i=1}^{n}x_i\right\|. \end{aligned} \tag{4.15}$$

Obviously (4.14) for $\phi = M$, $\varphi = m$, $M \ge m > 0$ provides a refinement for (4.7).

5. Lower Bounds for the Distance to Finite-Dimensional Subspaces

Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $\{y_1,\dots,y_n\}$ a subset of H and $G(y_1,\dots,y_n)$ the *gram matrix* of $\{y_1,\dots,y_n\}$ where $(i,j)-$entry is $\langle y_i,y_j\rangle$. The determinant of $G(y_1,\dots,y_n)$ is called the *Gram determinant* of $\{y_1,\dots,y_n\}$ and is denoted by $\Gamma(y_1,\dots,y_n)$. Thus,

$$\Gamma(y_1,\dots,y_n) = \left|\begin{array}{cccc} \langle y_1,y_1\rangle & \langle y_1,y_2\rangle & \cdots & \langle y_1,y_n\rangle \\ \langle y_2,y_1\rangle & \langle y_2,y_2\rangle & \cdots & \langle y_2,y_n\rangle \\ \multicolumn{4}{c}{\dots\dots\dots\dots\dots} \\ \langle y_n,y_1\rangle & \langle y_n,y_2\rangle & \cdots & \langle y_n,y_n\rangle \end{array}\right|. \tag{5.1}$$

Following [1, p. 129 − 133], we state here some general results for the Gram determinant that will be used in the sequel.

1. Let $\{x_1,\dots,x_n\}\subset H$. Then $\Gamma(x_1,\dots,x_n)\neq 0$ if and only if $\{x_1,\dots,x_n\}$ is linearly independent;
2. Let $M = \mathit{span}\{x_1,\dots,x_n\}$ be $n-$dimensional in H, i.e., $\{x_1,\dots,x_n\}$ is linearly independent. Then for each $x\in H$, the distance $d(x,M)$ from x to the linear subspace H has the representations

$$d^2(x,M) = \frac{\Gamma(x_1,\dots,x_n,x)}{\Gamma(x_1,\dots,x_n)} \tag{5.2}$$

and

$$d^2(x,M)=\begin{cases}\|x\|^2-\frac{\left(\sum_{i=1}^n|\langle x,x_i\rangle|^2\right)^2}{\left\|\sum_{i=1}^n\langle x,x_i\rangle x_i\right\|^2} & \text{if } x\notin M^{\perp},\\ \|x\|^2 & \text{if } x\in M^{\perp},\end{cases} \tag{5.3}$$

where $M^{\perp}$ denotes the orthogonal complement of M.

(3) If $\{x_1,\dots,x_n\}$ is an orthornormal set in H, i.e., $\langle x_i,x_j\rangle=\delta_{ij}$, $i,j\in\{1,\dots,n\}$, where δ_{ij} is Kronecker's delta, then

$$d^2(x,M)=\|x\|^2-\sum_{i=1}^n|\langle x,x_i\rangle|^2. \tag{5.4}$$

(4) Let $\{x_1,\dots,x_n\}$ be a set of nonzero vectors in H. Then

$$0\le\Gamma(x_1,\dots,x_n)\le\|x_1\|^2\|x_2\|^2\cdots\|x_n\|^2. \tag{5.5}$$

The equality holds on the left (respectively right) side of (5.5) if and only if $\{x_1,\dots,x_n\}$ is linearly dependent (respectively orthogonal). The first inequality in (5.5) is known in the literature as *Gram's inequality* while the second one is known as *Hadamard's inequality.*

The following result may be stated.

Proposition 4. *Let $\{x_1,\dots,x_n\}$ be a system of linearly independent vectors, $M=span\{x_1,\dots,x_n\}$, $x\in H\backslash M^{\perp}$, $a\in H$, $r>0$ and $\|a\|>r$. If*

$$\left\|x_i-\overline{\langle x,x_i\rangle}a\right\|\le|\langle x,x_i\rangle|r \quad \text{for each} \quad i\in\{1,\dots,n\}, \tag{5.6}$$

(note that if $\langle x,x_i\rangle\neq0$ for each $i\in\{1,\dots,n\}$, then (5.6) can be written as

$$\left\|\frac{x_i}{\langle x,x_i\rangle}-a\right\|\le r \quad \text{for each} \quad i\in\{1,\dots,n\}), \tag{5.7}$$

then we have the inequality

$$d^2(x,M)\ge\|x\|^2-\frac{\|a\|^2}{\|a\|^2-r^2}\cdot\frac{\sum_{i=1}^n|\langle x,x_i\rangle|^2}{\sum_{i=1}^n\|x_i\|^2}\ge0. \tag{5.8}$$

Proof. Utilising (5.3) we can state that

$$d^2(x,M)=\|x\|^2-\frac{\sum_{i=1}^n|\langle x,x_i\rangle|^2}{\left\|\sum_{i=1}^n\langle x,x_i\rangle x_i\right\|^2}\cdot\sum_{i=1}^n|\langle x,x_i\rangle|^2. \tag{5.9}$$

Also, by the inequality (3.6) applied for $\alpha_i=\langle x,x_i\rangle$, $p_i=\frac{1}{n}$, $i\in\{1,\dots,n\}$, we can state that

$$\frac{\sum_{i=1}^n|\langle x,x_i\rangle|^2}{\left\|\sum_{i=1}^n\langle x,x_i\rangle x_i\right\|^2}\le\frac{\|a\|^2}{\|a\|^2-r^2}\cdot\frac{1}{\sum_{i=1}^n\|x_i\|^2} \tag{5.10}$$

provided the condition (5.7) holds true.

Combining (5.9) with (5.10) we deduce the first inequality in (5.8).

The last inequality is obvious since, by Schwarz's inequality

$$\|x\|^2\sum_{i=1}^n\|x_i\|^2\ge\sum_{i=1}^n|\langle x,x_i\rangle|^2\ge\frac{\|a\|^2}{\|a\|^2-r^2}\sum_{i=1}^n|\langle x,x_i\rangle|^2.$$

∎

Remark 3. *Utilising (5.2), we can state the following result for Gram determinants*

$$\Gamma\left(x_1,\dots,x_n,x\right)\geq\left[\left\|x\right\|^2-\frac{\left\|a\right\|^2}{\left\|a\right\|^2-r^2}\cdot\frac{\sum_{i=1}^n\left|\left\langle x,x_i\right\rangle\right|^2}{\sum_{i=1}^n\left\|x_i\right\|^2}\right]\Gamma\left(x_1,\dots,x_n\right)\geq 0 \tag{5.11}$$

for $x\notin M^{\perp}$ and x,x_i,a and r are as in Proposition 4.

The following corollary of Proposition 4 may be stated as well.

Corollary 2. *Let $\{x_1,\dots,x_n\}$ be a system of linearly independent vectors, $M=$ span $\{x_1,\dots,x_n\}$, $x\in H\backslash M^{\perp}$ and $\phi,\varphi\in K$ with $\operatorname{Re}(\phi\bar{\varphi})>0$. If $e\in H$, $\|e\|=1$ and*

$$\left\|x_i-\overline{\left\langle x,x_i\right\rangle}\cdot\frac{\varphi+\phi}{2}e\right\|\leq\frac{1}{2}\left|\phi-\varphi\right|\left|\left\langle x,x_i\right\rangle\right| \tag{5.12}$$

or, equivalently,

$$\operatorname{Re}\left\langle\phi\cdot\overline{\left\langle x,x_i\right\rangle}e-x_i,x_i-\varphi\cdot\overline{\left\langle x,x_i\right\rangle}e\right\rangle\geq 0,$$

for each $i\in\{1,\dots,n\}$, then

$$d^2\left(x,M\right)\geq\left\|x\right\|^2-\frac{1}{4}\cdot\frac{\left|\varphi+\phi\right|^2}{\operatorname{Re}\left(\phi\bar{\varphi}\right)}\cdot\frac{\sum_{i=1}^n\left|\left\langle x,x_i\right\rangle\right|^2}{\sum_{i=1}^n\left\|x_i\right\|^2}\geq 0, \tag{5.13}$$

or, equivalently,

$$\Gamma\left(x_1,\dots,x_n,x\right)\geq\left[\left\|x\right\|^2-\frac{1}{4}\cdot\frac{\left|\varphi+\phi\right|^2}{\operatorname{Re}\left(\phi\bar{\varphi}\right)}\cdot\frac{\sum_{i=1}^n\left|\left\langle x,x_i\right\rangle\right|^2}{\sum_{i=1}^n\left\|x_i\right\|^2}\right]\Gamma\left(x_1,\dots,x_n\right)\geq 0. \tag{5.14}$$

6. Applications for Fourier Coefficients

Let $(H;\langle\cdot,\cdot\rangle)$ be a Hilbert space over the real or complex number field $\mathbb{K}$ and $\{e_i\}_{i\in I}$ an *orthornormal basis* for H. Then (see for instance [1, p. 54 – 61]):

(i) Every element $x\in H$ can be expanded in a *Fourier series,* i.e.,

$$x=\sum_{i\in I}\left\langle x,e_i\right\rangle e_i,$$

where $\langle x,e_i\rangle$, $i\in I$ are the *Fourier coefficients* of x;

(ii) (Parseval identity)

$$\left\|x\right\|^2=\sum_{i\in I}\left\langle x,e_i\right\rangle e_i,\qquad x\in H;$$

(iii) (Extended Parseval identity)

$$\left\langle x,y\right\rangle=\sum_{i\in I}\left\langle x,e_i\right\rangle\left\langle e_i,y\right\rangle,\qquad x,y\in H;$$

(iv) (Elements are uniquely determined by their Fourier coefficients)

$$\left\langle x,e_i\right\rangle=\left\langle y,e_i\right\rangle\ \text{ for every } i\in I \text{ implies that } x=y.$$

Now, we must remark that all the results from the second and third sections can be stated for $K = \mathbb{K}$ where $\mathbb{K}$ is the Hilbert space of complex (real) numbers endowed with the usual norm and inner product.

Therefore, we can state the following proposition.

Proposition 5. *Let $(H; \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbb{K}$ and $\{e_i\}_{i \in I}$ an orthornormal base for H. If $x, y \in H$ $(y \neq 0)$, $a \in \mathbb{K}$ $(\mathbb{C}, \mathbb{R})$ and $r > 0$ such that $|a| > r$ and*

$$\left| \frac{\langle x, e_i \rangle}{\langle y, e_i \rangle} - a \right| \leq r \quad \text{for each} \quad i \in I, \tag{6.1}$$

then we have the following reverse of the Schwarz inequality

$$\begin{aligned} \|x\| \|y\| &\leq \frac{1}{\sqrt{|a|^2 - r^2}} \operatorname{Re}[\bar{a} \cdot \langle x, y \rangle] \\ &\leq \frac{|a|}{\sqrt{|a|^2 - r^2}} |\langle x, y \rangle| ; \end{aligned} \tag{6.2}$$

$$\begin{aligned} (0 \leq) & \|x\| \|y\| - |\langle x, y \rangle| \\ &\leq \|x\| \|y\| - \operatorname{Re}\left[\frac{\bar{a}}{|a|} \cdot \langle x, y \rangle\right] \\ &\leq \frac{r^2}{\sqrt{|a|^2 - r^2}\left(|a| + \sqrt{|a|^2 - r^2}\right)} \operatorname{Re}\left[\frac{\bar{a}}{|a|} \cdot \langle x, y \rangle\right] \\ &\leq \frac{r^2}{\sqrt{|a|^2 - r^2}\left(|a| + \sqrt{|a|^2 - r^2}\right)} |\langle x, y \rangle| ; \end{aligned} \tag{6.3}$$

$$\begin{aligned} \|x\|^2 \|y\|^2 &\leq \frac{1}{|a|^2 - r^2} \left(\operatorname{Re}[\bar{a} \cdot \langle x, y \rangle]\right)^2 \\ &\leq \frac{|a|^2}{|a|^2 - r^2} |\langle x, y \rangle|^2 \end{aligned} \tag{6.4}$$

and

$$\begin{aligned} (0 \leq) & \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \\ &\leq \|x\|^2 \|y\|^2 - \left(\operatorname{Re}\left[\frac{\bar{a}}{|a|} \cdot \langle x, y \rangle\right]\right)^2 \\ &\leq \frac{r^2}{|a|^2 \left(|a|^2 - r^2\right)} - \left(\operatorname{Re}\left[\frac{\bar{a}}{|a|} \cdot \langle x, y \rangle\right]\right)^2 \\ &\leq \frac{r^2}{|a|^2 - r^2} |\langle x, y \rangle| . \end{aligned} \tag{6.5}$$

The proof is similar to the one in Theorem 4, where instead of x_i we take $\langle x, e_i \rangle$, instead of α_i we take $\langle e_i, y \rangle$, $\|\cdot\| = |\cdot|$, $p_i = 1$, and we use the Parseval identities mentioned above in (ii) and (iii). We omit the details.

The following result may be stated as well.

Proposition 6. *Let $(H;\langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbb{K}$ and $\{e_i\}_{i\in I}$ an orthornormal base for H. If $x,y\in H$ $(y\neq 0)$, $e,\varphi,\phi\in\mathbb{K}$ with $\operatorname{Re}(\phi\bar{\varphi})>0$, $|e|=1$ and, either*

$$\left|\frac{\langle x,e_i\rangle}{\langle y,e_i\rangle}-\frac{\varphi+\phi}{2}\cdot e\right|\leq\frac{1}{2}|\phi-\varphi| \tag{6.6}$$

or, equivalently,

$$\operatorname{Re}\left[\left(\phi e-\frac{\langle x,e_i\rangle}{\langle y,e_i\rangle}\right)\left(\frac{\langle e_i,x\rangle}{\langle e_i,y\rangle}-\bar{\varphi}\bar{e}\right)\right]\geq 0 \tag{6.7}$$

for each $i\in I$, then the following reverses of the Schwarz inequality hold:

$$\|x\|\,\|y\|\leq\frac{\operatorname{Re}\left[\left(\bar{\phi}+\bar{\varphi}\right)\bar{e}\langle x,y\rangle\right]}{2\sqrt{\operatorname{Re}(\phi\bar{\varphi})}}\leq\frac{1}{2}\cdot\frac{|\varphi+\phi|}{\sqrt{\operatorname{Re}(\phi\bar{\varphi})}}|\langle x,y\rangle|. \tag{6.8}$$

$$\begin{aligned}(0\leq)\ &\|x\|\,\|y\|-|\langle x,y\rangle| \\ &\leq\|x\|\,\|y\|-\operatorname{Re}\left[\frac{\left(\bar{\phi}+\bar{\varphi}\right)\bar{e}}{|\varphi+\phi|}\langle x,y\rangle\right] \\ &\leq\frac{|\phi-\varphi|^2}{2\sqrt{\operatorname{Re}(\phi\bar{\varphi})}\left(|\varphi+\phi|+2\sqrt{\operatorname{Re}(\phi\bar{\varphi})}\right)}\operatorname{Re}\left[\frac{\left(\bar{\phi}+\bar{\varphi}\right)\bar{e}}{|\varphi+\phi|}\langle x,y\rangle\right] \\ &\leq\frac{|\phi-\varphi|^2}{2\sqrt{\operatorname{Re}(\phi\bar{\varphi})}\left(|\varphi+\phi|+2\sqrt{\operatorname{Re}(\phi\bar{\varphi})}\right)}|\langle x,y\rangle|\end{aligned} \tag{6.9}$$

and

$$\begin{aligned}(0\leq)\ &\|x\|^2\|y\|^2-|\langle x,y\rangle|^2 \\ &\leq\|x\|^2\|y\|^2-\left[\operatorname{Re}\left[\frac{\left(\bar{\phi}+\bar{\varphi}\right)\bar{e}}{|\varphi+\phi|}\langle x,y\rangle\right]\right]^2 \\ &\leq\frac{|\phi-\varphi|^2}{4|\phi+\varphi|^2\operatorname{Re}(\phi\bar{\varphi})}\left\{\operatorname{Re}\left[\left(\bar{\phi}+\bar{\varphi}\right)\bar{e}\langle x,y\rangle\right]\right\}^2 \\ &\leq\frac{|\phi-\varphi|^2}{4\operatorname{Re}(\phi\bar{\varphi})}|\langle x,y\rangle|^2.\end{aligned} \tag{6.10}$$

Remark 4. *If $\phi=M\geq m=\varphi>0$, then one may state simpler inequalities from (6.8) – (6.10). We omit the details.*

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