REVERSING THE CBS-INEQUALITY FOR SEQUENCES OF VECTORS IN HILBERT SPACES WITH APPLICATIONS (II)

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Abstract. Several new reverses for the Cauchy-Bunyakovsky-Schwarz (CBS) inequality for sequences of vectors in Hilbert spaces which complement the ones obtained in part one are given. Applications in reversing the generalised triangle inequality and for Fourier coefficients are given as well.

1. Introduction

Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$.

One of the key inequalities in inner product spaces with numerous applications is the *Schwarz inequality*

$$|\langle x,y\rangle|^2 \leq \|x\|^2\|y\|^2, \qquad x,y\in H \tag{1.1}$$

which is known as the quadratic version of it, while

$$|\langle x,y\rangle| \leq \|x\|\|y\|, \qquad x,y\in H \tag{1.2}$$

is the simple version. The case of equality holds in either (1.1) or (1.2) if and only if the vectors x and y are linearly dependent.

By a *multiplicative reverse* of the Schwarz inequality in either simple or quadratic version, we understand an inequality of the form

$$(1\leq)\,\frac{\|x\|\|y\|}{|\langle x,y\rangle|}\leq k_1 \qquad \text{or} \qquad (1\leq)\,\frac{\|x\|^2\|y\|^2}{|\langle x,y\rangle|^2}\leq k_2 \tag{1.3}$$

with appropriate k_1 and k_2 and under various assumptions for the vectors x and y, while by the *additive reverse* we understand an inequality of the form

$$(0\leq)\,\|x\|\|y\|-|\langle x,y\rangle|\leq h_1 \qquad \text{or} \qquad (0\leq)\,\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\leq h_2. \tag{1.4}$$

Similar definitions apply when $|\langle x,y\rangle|$ is replaced by $\operatorname{Re}\langle x,y\rangle$ or $|\operatorname{Re}\langle x,y\rangle|$.

The following reverses for the Schwarz inequality hold (see [4], or the monograph on line [5, p. 27]).

Theorem 1. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$. If $x,a\in H$ and $r>0$ are such that*

$$x\in B(x,r):=\{z\in H|\ \|z-a\|\leq r\}, \tag{1.5}$$

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then we have the inequalities

$$(0 \leq) \|x\| \|a\| - |\langle x, a\rangle| \leq \|x\| \|a\| - |\mathrm{Re}\langle x, a\rangle| \leq \|x\| \|a\| - \mathrm{Re}\langle x, a\rangle \leq \frac{1}{2} r^2. \tag{1.6}$$

The constant $\frac{1}{2}$ is best possible in (1.5) in the sense that it cannot be replaced by a smaller quantity.

An additive version for the Schwarz inequality that may be more useful in applications is incorporated in [4] (see also [5, p. 28]).

Theorem 2. *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$ and $x, y \in H$ and $\gamma, \Gamma \in \mathbb{K}$ with $\Gamma \neq -\gamma$ and either*

$$\mathrm{Re}\langle \Gamma y - x, x - \gamma y\rangle \geq 0, \tag{1.7}$$

or, equivalently,

$$\left\| x - \frac{\gamma + \Gamma}{2} y \right\| \leq \frac{1}{2} |\Gamma - \gamma| \|y\| \tag{1.8}$$

holds. Then we have the inequalities

$$\begin{aligned} 0 &\leq \|x\| \|y\| - |\langle x, y\rangle| \\ &\leq \|x\| \|y\| - \left| \mathrm{Re}\left[\frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} \cdot \langle x, y\rangle \right] \right| \\ &\leq \|x\| \|y\| - \mathrm{Re}\left[\frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} \cdot \langle x, y\rangle \right] \\ &\leq \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \|y\|^2 . \end{aligned} \tag{1.9}$$

The constant $\frac{1}{4}$ in the last inequality is best possible.

We remark that a simpler version of the above result may be stated if one assumed that the scalars are real:

Corollary 1. *If $M \geq m > 0$, and either*

$$\mathrm{Re}\langle My - x, x - my\rangle \geq 0, \tag{1.10}$$

or, equivalently,

$$\left\| x - \frac{m + M}{2} y \right\| \leq \frac{1}{2} (M - m) \|y\| \tag{1.11}$$

holds, then

$$\begin{aligned} 0 &\leq \|x\| \|y\| - |\langle x, y\rangle| \\ &\leq \|x\| \|y\| - |\mathrm{Re}\langle x, y\rangle| \\ &\leq \|x\| \|y\| - \mathrm{Re}\langle x, y\rangle \\ &\leq \frac{1}{4} \cdot \frac{(M - m)^2}{M + m} \|y\|^2 . \end{aligned} \tag{1.12}$$

The constant $\frac{1}{4}$ is sharp.

Now, let $(K,\langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbb{K}$, $p_i \geq 0$, $i \in \mathbb{N}$ with $\sum_{i=1}^{\infty} p_i = 1$. Consider $\ell_{\mathbf{p}}^2(K)$ as the space

$$\ell_{\mathbf{p}}^2(K) := \left\{ x = (x_i)\,|x_i \in K,\ i \in \mathbb{N} \ \text{ and } \ \sum_{i=1}^{\infty} p_i \|x_i\|^2 < \infty \right\}.$$

It is well known that $\ell_{\mathbf{p}}^2(K)$ endowed with the inner product

$$\langle x, y\rangle_{\mathbf{p}} := \sum_{i=1}^{\infty} p_i \langle x_i, y_i\rangle$$

is a Hilbert space over $\mathbb{K}$. The norm $\|\cdot\|_{\mathbf{p}}$ of $\ell_{\mathbf{p}}^2(K)$ is given by

$$\|x\|_{\mathbf{p}} := \left(\sum_{i=1}^{\infty} p_i \|x_i\|^2\right)^{\frac{1}{2}}.$$

If $x, y \in \ell_{\mathbf{p}}^2(K)$, then the following Cauchy-Bunyakovsky-Schwarz (CBS) inequality holds true:

$$\sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2 \geq \left|\sum_{i=1}^{\infty} p_i \langle x_i, y_i\rangle\right|^2 \tag{1.13}$$

with equality iff there exists a $\lambda \in \mathbb{K}$ such that $x_i = \lambda y_i$ for each $i \in \mathbb{N}$.

If

$$\alpha \in \ell_{\mathbf{p}}^2(K) := \left\{ \alpha = (\alpha_i)_{i\in\mathbb{N}} \big|\, \alpha_i \in \mathbb{K},\ i \in \mathbb{N} \ \text{ and } \ \sum_{i=1}^{\infty} p_i |\alpha_i|^2 < \infty \right\}$$

and $x \in \ell_{\mathbf{p}}^2(K)$, then the following (CBS)-type inequality is also valid:

$$\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \geq \left\|\sum_{i=1}^{\infty} p_i \alpha_i x_i\right\|^2 \tag{1.14}$$

with equality if and only if there exists a vector $v \in K$ such that $x_i = \overline{\alpha_i} v$ for each $i \in \mathbb{N}$.

Note that the inequality (1.14) follows by the obvious identity

$$\sum_{i=1}^{n} p_i |\alpha_i|^2 \sum_{i=1}^{n} p_i \|x_i\|^2 - \left\|\sum_{i=1}^{n} p_i \alpha_i x_i\right\|^2 = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} p_i p_j \left\|\overline{\alpha_i} x_j - \overline{\alpha_j} x_i\right\|^2 \tag{1.15}$$

for each $n \in \mathbb{N}$, $n \geq 1$.

In [6], by the use of some preliminary results obtained in [3], various reverses for the (CBS)-type inequalities (1.13) and (1.14) for sequences of vectors in Hilbert spaces were obtained. Applications for bounding the distance to a finite-dimensional subspace and in reversing the generalised triangle inequality have also been provided.

The aim of the present paper is to provide different results by employing some inequalities discovered in [4]. Similar applications are pointed out.

2. Reverses of the (CBS)-Inequality for Two Sequences in $\ell_{\mathbf{p}}^2(K)$

The following proposition may be stated.

Proposition 1. *Let $x, y \in \ell_{\mathbf{p}}^2(K)$ and $r > 0$. If*

$$\|x_i - y_i\| \le r \quad \text{for each} \quad i \in \mathbb{N}, \tag{2.1}$$

then

$$\begin{aligned}(0 \le) & \left(\sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{\infty} p_i \langle x_i, y_i\rangle\right| \\ & \le \left(\sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2\right)^{\frac{1}{2}} - \left|\sum_{i=1}^{\infty} p_i \operatorname{Re}\langle x_i, y_i\rangle\right| \\ & \le \left(\sum_{i=1}^{\infty} p_i \|x_i\|^2 \sum_{i=1}^{\infty} p_i \|y_i\|^2\right)^{\frac{1}{2}} - \sum_{i=1}^{\infty} p_i \operatorname{Re}\langle x_i, y_i\rangle \\ & \le \frac{1}{2} r^2. \end{aligned} \tag{2.2}$$

The constant $\frac{1}{2}$ in front of r^2 is best possible in the sense that it cannot be replaced by a smaller quantity.

Proof. If (2.1) holds true, then

$$\|x - y\|_{\mathbf{p}}^2 = \sum_{i=1}^{\infty} p_i \|x_i - y_i\|^2 \le r^2 \sum_{i=1}^{\infty} p_i = r^2$$

and thus $\|x - y\|_{\mathbf{p}} \le r$.

Applying the inequality (1.6) for the inner product $\left(\ell_{\mathbf{p}}^2(K), \langle\cdot,\cdot\rangle_{\mathbf{p}}\right)$, we deduce the desired result (2.2).

The sharpness of the constant follows by Theorem 1 and we omit the details. ∎

The following result may be stated as well.

Proposition 2. *Let $x, y \in \ell_{\mathbf{p}}^2(K)$ and $\gamma, \Gamma \in \mathbb{K}$ with $\Gamma \ne -\gamma$. If either*

$$\operatorname{Re}\langle \Gamma y_i - x_i, x_i - \gamma y_i\rangle \ge 0 \quad \text{for each} \quad i \in \mathbb{N} \tag{2.3}$$

or, equivalently,

$$\left\|x_i - \frac{\gamma + \Gamma}{2} y_i\right\| \le \frac{1}{2}|\Gamma - \gamma| \|y_i\| \quad \text{for each} \quad i \in \mathbb{N} \tag{2.4}$$

holds, then:

$$(0\leq)\left(\sum_{i=1}^{\infty}p_i\left\|x_i\right\|^2\sum_{i=1}^{\infty}p_i\left\|y_i\right\|^2\right)^{\frac{1}{2}}-\left|\sum_{i=1}^{\infty}p_i\left\langle x_i,y_i\right\rangle\right| \tag{2.5}$$
$$\leq\left(\sum_{i=1}^{\infty}p_i\left\|x_i\right\|^2\sum_{i=1}^{\infty}p_i\left\|y_i\right\|^2\right)^{\frac{1}{2}}-\left|\operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}\sum_{i=1}^{\infty}p_i\left\langle x_i,y_i\right\rangle\right]\right|$$
$$\leq\left(\sum_{i=1}^{\infty}p_i\left\|x_i\right\|^2\sum_{i=1}^{\infty}p_i\left\|y_i\right\|^2\right)^{\frac{1}{2}}-\operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}\sum_{i=1}^{\infty}p_i\left\langle x_i,y_i\right\rangle\right]$$
$$\leq\frac{1}{4}\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\sum_{i=1}^{\infty}p_i\left\|y_i\right\|^2.$$

The constant $\frac{1}{4}$ is best possible in (2.5).

Proof. Since, by (2.3),

$$\operatorname{Re}\left\langle \Gamma y-x,x-\gamma y\right\rangle_{\mathbf{p}}=\sum_{i=1}^{\infty}p_i\operatorname{Re}\left\langle \Gamma y_i-x_i,x_i-\gamma y_i\right\rangle\geq 0,$$

hence, on applying the inequality (1.9) for the Hilbert space $\left(\ell^2_{\mathbf{p}}(K),\langle\cdot,\cdot\rangle_{\mathbf{p}}\right)$, we deduce the desired inequality (2.5).

The best constant follows by Theorem 2 and we omit the details. ∎

Corollary 2. *If the conditions (2.3) and (2.4) hold for $\Gamma=M$, $\gamma=m$ with $M\geq m>0$, then*

$$(0\leq)\left(\sum_{i=1}^{\infty}p_i\left\|x_i\right\|^2\sum_{i=1}^{\infty}p_i\left\|y_i\right\|^2\right)^{\frac{1}{2}}-\left|\sum_{i=1}^{\infty}p_i\left\langle x_i,y_i\right\rangle\right| \tag{2.6}$$
$$\leq\left(\sum_{i=1}^{\infty}p_i\left\|x_i\right\|^2\sum_{i=1}^{\infty}p_i\left\|y_i\right\|^2\right)^{\frac{1}{2}}-\left|\sum_{i=1}^{\infty}p_i\operatorname{Re}\left\langle x_i,y_i\right\rangle\right|$$
$$\leq\left(\sum_{i=1}^{\infty}p_i\left\|x_i\right\|^2\sum_{i=1}^{\infty}p_i\left\|y_i\right\|^2\right)^{\frac{1}{2}}-\sum_{i=1}^{\infty}p_i\operatorname{Re}\left\langle x_i,y_i\right\rangle$$
$$\leq\frac{1}{4}\cdot\frac{(M-m)^2}{M+m}\sum_{i=1}^{\infty}p_i\left\|y_i\right\|^2.$$

The constant $\frac{1}{4}$ is best possible.

3. Reverses of the (CBS)-Inequality for Mixed Sequences

The following result holds:

Theorem 3. *Let $\alpha\in\ell^2_{\mathbf{p}}(K)$, $x\in\ell^2_{\mathbf{p}}(K)$ and $v\in K\setminus\{0\}$, $r>0$. If*

$$\left\|x_i-\overline{\alpha_i}v\right\|\leq r\left|\alpha_i\right|\quad\textit{for each}\quad i\in\mathbb{N} \tag{3.1}$$

(note that if $\alpha_i \neq 0$ for any $i \in \mathbb{N}$, then the condition (3.1) is equivalent to the simpler one

$$\left\| \frac{x_i}{\alpha_i} - v \right\| \leq r \quad \textit{for each} \quad i \in \mathbb{N}), \tag{3.2}$$

then

$$\begin{aligned}(0 \leq) \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \right)^{\frac{1}{2}} - \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\| \\ \leq \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \right)^{\frac{1}{2}} - \left| \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, \frac{v}{\|v\|} \right\rangle \right| \\ \leq \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \right)^{\frac{1}{2}} - \left| \operatorname{Re} \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, \frac{v}{\|v\|} \right\rangle \right| \\ \leq \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \right)^{\frac{1}{2}} - \operatorname{Re} \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, \frac{v}{\|v\|} \right\rangle \\ \leq \frac{1}{2} \cdot \frac{r^2}{\|r\|} \sum_{i=1}^{\infty} p_i |\alpha_i|^2 .\end{aligned} \tag{3.3}$$

The constant $\frac{1}{2}$ is best possible in (3.3).

Proof. From (3.1) we deduce

$$\|x_i\|^2 - 2 \operatorname{Re} \langle \alpha_i x_i, v \rangle + |\alpha_i|^2 \|v\|^2 \leq r^2 |\alpha_i|^2 ,$$

which is clearly equivalent to

$$\|x_i\|^2 + |\alpha_i|^2 \|v\|^2 \leq 2 \operatorname{Re} \langle \alpha_i x_i, v \rangle + r^2 |\alpha_i|^2 \tag{3.4}$$

for each $i \in \mathbb{N}$.

If we multiply (3.4) by $p_i \geq 0$, $i \in \mathbb{N}$ and sum over $i \in \mathbb{N}$, then we deduce

$$\sum_{i=1}^{\infty} p_i \|x_i\|^2 + \|v\|^2 \sum_{i=1}^{\infty} p_i |\alpha_i|^2 \leq 2 \operatorname{Re} \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, v \right\rangle + r^2 \sum_{i=1}^{\infty} p_i |\alpha_i|^2 . \tag{3.5}$$

Since, obviously

$$2 \|v\| \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \right)^{\frac{1}{2}} \leq \sum_{i=1}^{\infty} p_i \|x_i\|^2 + \|v\|^2 \sum_{i=1}^{\infty} p_i |\alpha_i|^2 , \tag{3.6}$$

hence, by (3.5) and (3.6), we deduce

$$2 \|v\| \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \right)^{\frac{1}{2}} \leq 2 \operatorname{Re} \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, v \right\rangle + r^2 \sum_{i=1}^{\infty} p_i |\alpha_i|^2 ,$$

which is clearly equivalent to the last inequality in (3.3).

The other inequalities are obvious.

The best constant follows by Theorem 1. ∎

The following corollary may be stated.

Corollary 3. *Let* $\alpha \in \ell_{\mathbf{p}}^2(K)$, $x \in \ell_{\mathbf{p}}^2(K)$, $e \in H$, $\|e\| = 1$ *and* $\gamma, \Gamma \in \mathbb{K}$ *with* $\Gamma \neq -\gamma$. *If*

$$\left\| x_i - \overline{\alpha_i} \frac{\gamma + \Gamma}{2} \cdot e \right\| \leq \frac{1}{2} |\Gamma - \gamma| |\alpha_i| \tag{3.7}$$

for each $i \in \mathbb{N}$, *or, equivalently,*

$$\operatorname{Re} \langle \Gamma \overline{\alpha_i} e - x_i, x_i - \gamma \overline{\alpha_i} e \rangle \tag{3.8}$$

for each $i \in \mathbb{N}$ *(note that, if* $\alpha_i \neq 0$ *for any* $i \in \mathbb{N}$, *then (3.7) is equivalent to*

$$\left\| \frac{x_i}{\overline{\alpha_i}} - \frac{\gamma + \Gamma}{2} e \right\| \leq \frac{1}{2} |\Gamma - \gamma| \tag{3.9}$$

for each $i \in \mathbb{N}$ *and (3.8) is equivalent to*

$$\operatorname{Re} \left\langle \Gamma e - \frac{x_i}{\overline{\alpha_i}}, \frac{x_i}{\overline{\alpha_i}} - \gamma e \right\rangle \geq 0 \tag{3.10}$$

for each $i \in \mathbb{N}$*), then the following reverse of the (CBS)-inequality is valid:*

$$\begin{aligned} (0 \leq) & \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \right)^{\frac{1}{2}} - \left\| \sum_{i=1}^{\infty} p_i \alpha_i x_i \right\| \\ & \leq \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \right)^{\frac{1}{2}} - \left| \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, e \right\rangle \right| \\ & \leq \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \right)^{\frac{1}{2}} - \left| \operatorname{Re} \left[\frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, e \right\rangle \right] \right| \\ & \leq \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2 \right)^{\frac{1}{2}} - \operatorname{Re} \left[\frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} \left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, e \right\rangle \right] \\ & \leq \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \sum_{i=1}^{\infty} p_i |\alpha_i|^2 . \end{aligned} \tag{3.11}$$

The constant $\frac{1}{4}$ *is best possible.*

Remark 1. *If* $M \geq m > 0$, $\alpha_i \neq 0$ *and for* e *as above, either*

$$\left\| \frac{x_i}{\overline{\alpha_i}} - \frac{M + m}{2} e \right\| \leq \frac{1}{2} (M - m) \quad \text{for each} \quad i \in \mathbb{N} \tag{3.12}$$

or, equivalently,

$$\operatorname{Re} \left\langle Me - \frac{x_i}{\overline{\alpha_i}}, \frac{x_i}{\overline{\alpha_i}} - me \right\rangle \geq 0 \quad \text{for each} \quad i \in \mathbb{N}$$

holds, then

$$\begin{aligned}(0 \le) & \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2\right)^{\frac{1}{2}} - \left\|\sum_{i=1}^{\infty} p_i \alpha_i x_i\right\| \\ & \le \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2\right)^{\frac{1}{2}} - \left|\left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, e\right\rangle\right| \\ & \le \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2\right)^{\frac{1}{2}} - \left|\operatorname{Re}\left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, e\right\rangle\right| \\ & \le \left(\sum_{i=1}^{\infty} p_i |\alpha_i|^2 \sum_{i=1}^{\infty} p_i \|x_i\|^2\right)^{\frac{1}{2}} - \operatorname{Re}\left\langle \sum_{i=1}^{\infty} p_i \alpha_i x_i, e\right\rangle \\ & \le \frac{1}{4} \cdot \frac{(M-m)^2}{M+m} \sum_{i=1}^{\infty} p_i |\alpha_i|^2 .\end{aligned}$$

The constant $\frac{1}{4}$ *is best possible.*

4. Reverses for the Generalised Triangle Inequality

In 1966, Diaz and Metcalf [2] proved the following interesting reverse of the generalised triangle inequality:

$$r \sum_{i=1}^{\infty} \|x_i\| \le \left\|\sum_{i=1}^{\infty} x_i\right\|, \tag{4.1}$$

provided the vectors $x_1, \dots, x_n \in H \backslash \{0\}$ satisfy the assumption

$$0 \le r \le \frac{\operatorname{Re}\langle x_i, a\rangle}{\|x_i\|}, \qquad i \in \{1, \dots, n\}, \tag{4.2}$$

where $a \in H$, $\|a\| = 1$ and $(H; \langle \cdot, \cdot \rangle)$ is a real or complex inner product space.

In an attempt to provide other sufficient conditions for (4.1) to hold, the author pointed out in [7] that

$$\sqrt{1-\rho^2} \sum_{i=1}^{\infty} \|x_i\| \le \left\|\sum_{i=1}^{\infty} x_i\right\| \tag{4.3}$$

where the vectors x_i, $i \in \{1, \dots, n\}$ satisfy the condition

$$\|x_i - a\| \le \rho, \qquad i \in \{1, \dots, n\}, \tag{4.4}$$

where $r \in H$, $\|a\| = 1$ and $\rho \in (0, 1)$.

Following [7], if $M \ge m > 0$ and the vectors $x_i \in H$, $i \in \{1, \dots, n\}$ verify either

$$\operatorname{Re}\langle Ma - x_i, x_i - ma\rangle \ge 0, \qquad i \in \{1, \dots, n\}, \tag{4.5}$$

or, equivalently,

$$\left\|x_i - \frac{M+m}{2} \cdot a\right\| \le \frac{1}{2}(M-m), \qquad i \in \{1, \dots, n\}, \tag{4.6}$$

where $a \in H$, $\|a\| = 1$, then

$$\frac{2\sqrt{mM}}{M+m}\sum_{i=1}^{n}\|x_i\| \le \left\|\sum_{i=1}^{n} x_i\right\|. \tag{4.7}$$

It is obvious from Theorem 3, that, if

$$\|x_i - v\| \le r, \qquad \text{for} \quad i \in \{1,\dots,n\}, \tag{4.8}$$

where $x_i \in H$, $i \in \{1,\dots,n\}$, $v \in H\backslash\{0\}$ and $r > 0$, then we can state the inequality

$$\begin{aligned}(0 \le)\left(\frac{1}{n}\sum_{i=1}^{n}\|x_i\|^2\right)^{\frac{1}{2}} - \left\|\frac{1}{n}\sum_{i=1}^{n}x_i\right\| & \\ \le \left(\frac{1}{n}\sum_{i=1}^{n}\|x_i\|^2\right)^{\frac{1}{2}} - \left|\left\langle\frac{1}{n}\sum_{i=1}^{n}x_i, \frac{v}{\|v\|}\right\rangle\right| & \\ \le \left(\frac{1}{n}\sum_{i=1}^{n}\|x_i\|^2\right)^{\frac{1}{2}} - \left|\mathrm{Re}\left\langle\frac{1}{n}\sum_{i=1}^{n}x_i, \frac{v}{\|v\|}\right\rangle\right| & \\ \le \left(\frac{1}{n}\sum_{i=1}^{n}\|x_i\|^2\right)^{\frac{1}{2}} - \mathrm{Re}\left\langle\frac{1}{n}\sum_{i=1}^{n}x_i, \frac{v}{\|v\|}\right\rangle & \\ \le \frac{1}{2}\cdot\frac{r^2}{\|v\|}. &\end{aligned} \tag{4.9}$$

Since, by the (CBS)-inequality we have

$$\frac{1}{n}\sum_{i=1}^{n}\|x_i\| \le \left(\frac{1}{n}\sum_{i=1}^{n}\|x_i\|^2\right)^{\frac{1}{2}}, \tag{4.10}$$

hence, by (4.9) and (4.5) we have:

$$(0 \le)\sum_{i=1}^{n}\|x_i\| - \left\|\sum_{i=1}^{n}x_i\right\| \le \frac{1}{2}n\cdot\frac{r^2}{\|v\|} \tag{4.11}$$

provided that (4.8) holds true.

Utilising Corollary 3, we may state that, if

$$\left\|x_i - \frac{\gamma+\Gamma}{2}\cdot e\right\| \le \frac{1}{2}|\Gamma-\gamma|, \qquad i \in \{1,\dots,n\}, \tag{4.12}$$

or, equivalently,

$$\mathrm{Re}\langle \Gamma e - x_i, x_i - \gamma e\rangle \ge 0, \qquad i \in \{1,\dots,n\}, \tag{4.13}$$

where $e \in H$, $\|e\| = 1, \gamma, \Gamma \in \mathbb{K}$, $\Gamma \neq -\gamma$ and $x_i \in H$, $i \in \{1, \ldots, n\}$, then

$$(0 \leq) \left(\frac{1}{n} \sum_{i=1}^{n} \|x_i\|^2 \right)^{\frac{1}{2}} - \left\| \frac{1}{n} \sum_{i=1}^{n} x_i \right\| \tag{4.14}$$
$$\leq \left(\frac{1}{n} \sum_{i=1}^{n} \|x_i\|^2 \right)^{\frac{1}{2}} - \left| \left\langle \frac{1}{n} \sum_{i=1}^{n} x_i, e \right\rangle \right|$$
$$\leq \left(\frac{1}{n} \sum_{i=1}^{n} \|x_i\|^2 \right)^{\frac{1}{2}} - \left| \operatorname{Re} \left[\frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} \left\langle \frac{1}{n} \sum_{i=1}^{n} x_i, e \right\rangle \right] \right|$$
$$\leq \left(\frac{1}{n} \sum_{i=1}^{n} \|x_i\|^2 \right)^{\frac{1}{2}} - \operatorname{Re} \left[\frac{\bar{\Gamma} + \bar{\gamma}}{|\Gamma + \gamma|} \left\langle \frac{1}{n} \sum_{i=1}^{n} x_i, e \right\rangle \right]$$
$$\leq \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|}.$$

Now, making use of (4.10) and (4.14) we can establish the following additive reverse of the generalised triangle inequality

$$(0 \leq) \sum_{i=1}^{n} \|x_i\| - \left\| \sum_{i=1}^{n} x_i \right\| \leq \frac{1}{4} n \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|}, \tag{4.15}$$

provided either (4.12) or (4.13) hold true.

5. Applications for Fourier Coefficients

Let $(H; \langle \cdot, \cdot \rangle)$ be a Hilbert space over the real or complex number field $\mathbb{K}$ and $\{e_i\}_{i \in I}$ an *orthonormal basis* for H. Then (see for instance [1, p. 54 – 61]):

(i) Every element $x \in H$ can be expanded in a *Fourier series*, i.e.,

$$x = \sum_{i \in I} \langle x, e_i \rangle e_i,$$

where $\langle x, e_i \rangle$, $i \in I$ are the *Fourier coefficients* of x;

(ii) (Parseval identity)

$$\|x\|^2 = \sum_{i \in I} \langle x, e_i \rangle e_i, \quad x \in H;$$

(iii) (Extended Parseval's identity)

$$\langle x, y \rangle = \sum_{i \in I} \langle x, e_i \rangle \langle e_i, y \rangle, \qquad x, y \in H;$$

(iv) (Elements are uniquely determined by their Fourier coefficients)

$$\langle x, e_i \rangle = \langle y, e_i \rangle \quad \text{for every } i \in I \text{ implies that } x = y.$$

We must remark that all the results from the second and third sections may be stated for $K = \mathbb{K}$ where $\mathbb{K}$ is the Hilbert space of complex (real) numbers endowed with the usual norm and inner product.

Therefore we can state the following reverses of the Schwarz inequality:

Proposition 3. *Let $(H;\langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbb{K}$ and $\{e_i\}_{i\in I}$ an orthonormal base for H. If $x,y\in H$, $y\neq 0$, $a\in\mathbb{K}$ $(\mathbb{C},\mathbb{R})$ with $r>0$ such that*

$$\left|\frac{\langle x,e_i\rangle}{\langle y,e_i\rangle}-a\right|\leq r \qquad \text{for each } \ i\in I, \tag{5.1}$$

then we have the following reverse of the Schwarz inequality:

$$\begin{aligned}(0\leq)\ \|x\|\,\|y\|-|\langle x,y\rangle| &\leq \|x\|\,\|y\|-\left|\operatorname{Re}\left[\langle x,y\rangle\cdot\frac{\bar{a}}{|a|}\right]\right| \\ &\leq \|x\|\,\|y\|-\operatorname{Re}\left[\langle x,y\rangle\cdot\frac{\bar{a}}{|a|}\right] \\ &\leq \frac{1}{2}\cdot\frac{r^2}{|a|}\,\|y\|^2 .\end{aligned} \tag{5.2}$$

The constant $\frac{1}{2}$ is best possible in (5.2).

The proof is similar to the one in Theorem 3, where instead of x_i we take $\langle x,e_i\rangle$, instead of α_i we take $\langle e_i,y\rangle$, $\|\cdot\|=|\cdot|$, $p_i=1$ and use the Parseval identities mentioned above in (ii) and (iii). We omit the details.

The following result may be stated as well.

Proposition 4. *Let $(H;\langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbb{K}$ and $\{e_i\}_{i\in I}$ an orthonormal base for H. If $x,y\in H$, $y\neq 0$, $e,\gamma,\Gamma\in\mathbb{K}$ with $|e|=1$, $\Gamma\neq-\gamma$ and*

$$\left|\frac{\langle x,e_i\rangle}{\langle y,e_i\rangle}-\frac{\gamma+\Gamma}{2}\cdot e\right|\leq\frac{1}{2}\,|\Gamma-\gamma| \tag{5.3}$$

or equivalently,

$$\operatorname{Re}\left[\left(\Gamma e-\frac{\langle x,e_i\rangle}{\langle y,e_i\rangle}\right)\left(\frac{\langle e_i,x\rangle}{\langle e_i,y\rangle}-\bar{\gamma}\bar{e}\right)\right]\geq 0 \tag{5.4}$$

for each $i\in I$, then

$$\begin{aligned}(0\leq)\ \|x\|\,\|y\|-|\langle x,y\rangle| &\leq \|x\|\,\|y\|-\left|\operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}\,\langle x,y\rangle\cdot\bar{e}\right]\right| \\ &\leq \|x\|\,\|y\|-\operatorname{Re}\left[\frac{\bar{\Gamma}+\bar{\gamma}}{|\Gamma+\gamma|}\,\langle x,y\rangle\cdot\bar{e}\right] \\ &\leq \frac{1}{4}\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\,\|y\|^2 .\end{aligned} \tag{5.5}$$

The constant $\frac{1}{4}$ is best possible.

Remark 2. *If $\Gamma=M\geq m=\gamma>0$, then one may state simpler inequalities from (5.5). We omit the details.*

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