

3-Dimensional L -Summing Method

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Abstract

In this short note, we apply L -Summing Method on some 3-dimensional multiplication tables to yield some new identities involving Riemann zeta function.

Keywords: L -Summing Method, Multiplication Table, Riemann zeta function.

2000 *Mathematics Subject Classification:* 65B10, 65A05, 11S40.

Let a_{ij} be an $n \times n$ array. The 2-dimensional L -Summing Method is the following rearrange:

$$\sum_{1 \leq i, j \leq n} a_{ij} = \sum_{k=1}^n \left(\sum_{i=1}^k a_{ik} + \sum_{j=1}^k a_{kj} - a_{kk} \right).$$

Specially, when $a_{ij} = (ij)^{-s}$, we yield [2]:

$$\sum_{k=1}^n \frac{\zeta_k(s)}{k^s} = \frac{\zeta_n^2(s) + \zeta_n(2s)}{2}, \quad (s \in \mathbb{C}).$$

The base of this array was 2-dimensional multiplication table and since we can generalize multiplication table to high dimensional versions [1], we can generalize L -Summing Method, and we are going to do this generalization in \mathbb{R}^3 . In this case, L -Summing Elements are 3-dimensional as the following figure:

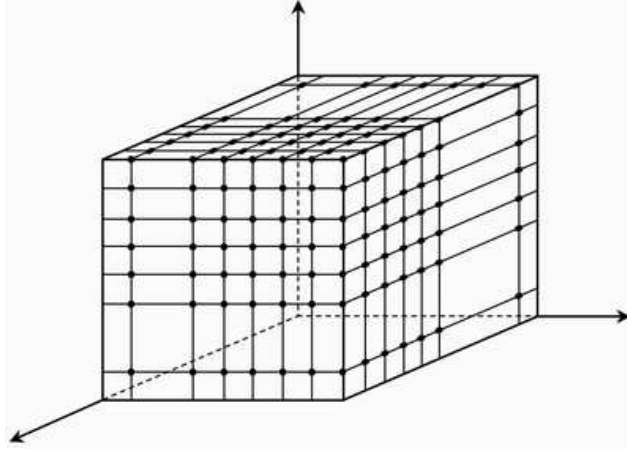


Figure 1: L -Summing Elements in \mathbb{R}^3

For start, let proceed on $MT_{n \times n}^3$ [1]; i.e. let $a_{ijk} = ijk$, in which $1 \leq i, j, k \leq n$. Easily, we have

$$S = \sum_{1 \leq i, j, k \leq n} ijk = \left(\frac{n(n+1)}{2} \right)^3.$$

By L -Summing Method; according to above figure, L -Summing Element in this table is:

$$L_k = 3k \left(\frac{k(k+1)}{2} \right)^2 - 3k^2 \left(\frac{k(k+1)}{2} \right) + k^3 = \frac{3}{4}k^5 + \frac{1}{4}k^3.$$

So, we have the following known identity:

$$\sum_{k=1}^n \frac{3}{4}k^5 + \frac{1}{4}k^3 = \left(\frac{n(n+1)}{2} \right)^3.$$

Now, suppose $s \in \mathbb{C}$ and let $a_{ijk} = (ijk)^{-s}$. It is clear that

$$S = \sum_{1 \leq i, j, k \leq n} (ijk)^{-s} = \left(\sum_{k=1}^n \frac{1}{k^s} \right)^3 = \zeta_n^3(s).$$

By L -Summing Method we have:

$$L_k = 3 \frac{\zeta_k^2(s)}{k^s} - 3 \frac{\zeta_k(s)}{k^{2s}} + \frac{1}{k^{3s}},$$

and we can reform $\sum L_k = S$ as follows:

$$\sum_{k=1}^n \frac{\zeta_k^2(s)}{k^s} - \frac{\zeta_k(s)}{k^{2s}} = \frac{\zeta_n^3(s) - \zeta_n(3s)}{3}.$$

Note that if $\Re(s) > 1$, then $\lim_{n \rightarrow \infty} \zeta_n(s) = \zeta(s)$. So, for $\Re(s) > 1$ we have

$$\sum_{k=1}^{\infty} \frac{\zeta_k^2(s)}{k^s} - \frac{\zeta_k(s)}{k^{2s}} = \frac{\zeta^3(s) - \zeta(3s)}{3}.$$

Also, if $s = 1$, then $\zeta_n(1) = H_n = \sum_{k=1}^n \frac{1}{k}$, and so, we have

$$\sum_{k=1}^n \frac{H_k^2}{k} - \frac{H_k}{k^2} = \frac{H_n^3 - \zeta_n(3)}{3}.$$

References

- [1] M. Hassani, A Generalization of Multiplication Table, *RGMI*A Research Report Collection, **7**(4), Article 13, 2004.
Available at: <http://rgmia.vu.edu.au/v7n4.html>
- [2] M. Hassani and S. Rahimpour, L-Summing Method, *RGMI*A Research Report Collection, **7**(4), Article 10, 2004.
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