INTEGRAL INEQUALITIES ON INFINITE INTERVALS

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Abstract. Inequalities concerning the distance between a function and some integrals on infinite intervals are given.

1. Introduction

Let $-\infty \leq a < b \leq +\infty$ and $w \in L(a,b)$ a Lebesgue integrable function on (a,b) with $\int_a^b w(s)\,ds \neq 0$.

The following identity holding for locally absolutely continuous functions $f : (a,b) \to \mathbb{R}$, where (a,b) is finite or infinite, is known in the literature as the *weighted Montgomery* identity:

$$f(x) - \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t) f(t)\,dt = \frac{1}{\int_a^b w(s)\,ds}\int_a^x \left(\int_a^t w(s)\,ds\right) f'(t)\,dt - \frac{1}{\int_a^b w(s)\,ds}\int_x^b \left(\int_t^b w(s)\,ds\right) f'(s)\,ds \tag{1.1}$$

for any $x \in (a,b)$.

For a simple proof of this fact we refer to the monograph [2, p. 376] where further similar results are provided.

For generalisations to the case of $n-$time differentiable functions we refer to [3].

In [1] a different representation for the left hand side of (1.1) has been provided

$$f(x) - \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t) f(t)\,dt = \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)(x-t)\left(\int_0^1 f'[(1-\lambda)x + \lambda t]\,d\lambda\right) dt \tag{1.2}$$

for any $x \in (a,b)$.

If $a = 0$, $b = \infty$, $w(t) = e^{-t}$, then from (1.1) we obtain the identity:

$$f(x) - \int_0^\infty e^{-t} f(t)\,dt = \int_0^x \left(1 - e^{-t}\right) f'(t)\,dt - \int_x^\infty e^{-t} f'(t)\,dt \tag{1.3}$$

for any $x \in [0,\infty)$, provided the involved integrals exist for f locally absolutely continuous on $[0,\infty)$.

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Taking the modulus in (1.3), we get

$$\left|f(x)-\int_0^\infty e^{-t}f(t)\,dt\right| \leq \int_0^x \left(1-e^{-t}\right)|f'(t)|\,dt+\int_x^\infty e^{-t}|f'(t)|\,dt := I(x), \tag{1.4}$$

for $x\in[0,\infty)$.

One can obtain various upper bounds for I. For instance,

$$\begin{aligned} I(x) &\leq ess\sup_{t\in[0,x]}|f'(t)|\int_0^x\left(1-e^{-t}\right)dt+ess\sup_{t\in[x,\infty)}|f'(t)|\int_x^\infty e^{-t}dt \\ &= \left(e^{-x}+x-1\right)\|f'\|_{[0,\infty),\infty}+e^{-x}\|f'\|_{[x,\infty)} \\ &\leq \left(2e^{-x}+x-1\right)\|f'\|_{[0,\infty),\infty}, \qquad x\in[0,\infty), \end{aligned} \tag{1.5}$$

provided $f'\in L_\infty[0,\infty)$.

The inequalities between the first and last term in (1.5) have been pointed out in [2, p. 377].

Also,

$$\begin{aligned} I(x) &\leq \sup_{t\in[0,x]}\left(1-e^{-t}\right)\|f'\|_{[0,x],1}+\sup_{t\in[x,\infty)}e^{-t}\|f'\|_{[x,\infty),1} \\ &= \left(1-e^{-x}\right)\|f'\|_{[0,x],1}+e^{-x}\|f'\|_{[x,\infty),1} \\ &\leq \max\left\{1-e^{-x},e^{-x}\right\}\|f'\|_{[0,\infty),1} \\ &= \frac{1+|1-2e^{-x}|}{2}\|f'\|_{[0,\infty),1}, \qquad x\in[0,\infty), \end{aligned} \tag{1.6}$$

provided $f'\in L_1[0,\infty)$.

If one uses Hölder type inequalities, then one may deduce other bounds for $I(x)$ in terms of the $p-$norms of f', $p>1$.

Now, if we use (1.2) for $a=0$, $b=\infty$ and $w(t)=e^{-t}$, then we may state

$$f(x)-\int_0^\infty e^{-t}f(t)\,dt=\int_0^\infty e^{-t}(x-t)\left(\int_0^1 f'[(1-\lambda)x+\lambda t]\,d\lambda\right)dt \tag{1.7}$$

for any $\alpha\in(0,\infty)$, provided that the involved integrals exist.

Taking the modulus on (1.7) we have

$$\left|f(x)-\int_0^\infty e^{-t}f(t)\,dt\right| \leq \int_0^\infty e^{-t}|x-t|\left(\int_0^1 |f'[(1-\lambda)x+\lambda t]|\,d\lambda\right)dt =: J(x) \tag{1.8}$$

for $x\in[0,\infty)$.

On making use of similar arguments outlined above, we may produce various bounds for $J(x)$ in terms of the $p-$norms $\|f'\|_p$. If $|f'|$ is convex on $(0,\infty)$, then

$$|f'[(1-\lambda)x+\lambda t]|\leq(1-\lambda)|f'(x)|+\lambda|f'(t)|$$

for any $x, y \in [0, \infty)$ and $\lambda \in [0, 1]$. Then

$$\begin{aligned} J(x) &\leq \int_0^\infty e^{-t} |x-t| \left[|f'(x)| \int_0^1 (1-\lambda)\, dx + |f'(t)| \int_0^1 \lambda d\lambda \right] \\ &= \frac{1}{2} \int_0^\infty e^{-t} |x-t| |f'(t)| \, dt + \frac{1}{2} |f'(x)| \int_0^\infty e^{-t} |x-t| \, dt \\ &\leq \frac{1}{2} \left[\|f'\|_{[0,\infty),\infty} + |f'(x)| \right] \int_0^\infty e^{-t} |x-t| \, dt \\ &= \frac{1}{2} \left[\|f'\|_{[0,\infty),\infty} + |f'(x)| \right] \left(2e^{-x} + x - 1 \right), \end{aligned}$$

for any $x \in [0, \infty)$, which is an improvement on the result

$$\left| f(x) - \int_0^\infty e^{-t} f(t)\, dt \right| \leq \left(2e^{-x} + x - 1 \right) \|f'\|_{[0,\infty),\infty}, \qquad x \geq 0 \tag{1.9}$$

that has been obtained in [2, p. 377].

We note that for $x \to \infty$ the bound (1.9) is tending to ∞ as well, showing that for large $x \in (0, \infty)$, $\int_0^\infty e^{-t} f(t)\, dt$ is far from $f(x)$ even if $f' \in L_\infty(0, \infty)$.

It is natural to enquire how we can modify the expression under the integral such that its absolute distance from $f(x)$ will remain finite for any $x \in [0, \infty)$.

The aim of this paper is to provide some inequalities for which the absolute value of the difference between a function and an integral transform of it remain finite for any x in an infinite interval.

2. The Results

The following result holds.

Theorem 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a locally absolutely continuous function on $\mathbb{R}$. Then for any $x \in \mathbb{R}$ we have the inequalities*

$$\begin{aligned} &\left| f(x) - \int_0^\infty \left[\frac{f(x-v) + f(x+v)}{2} \right] e^{-v} dv \right| \\ &\leq \frac{1}{2} \left[\int_{-\infty}^x e^{t-x} |f'(t)| \, dt + \int_x^\infty |f'(t)| \, dt \right] \\ &\leq \begin{cases} \frac{1}{2} \left[\|f'\|_{(-\infty,x],\infty} + \|f'\|_{[x,\infty),\infty} \right] & \text{if } f' \in L_\infty(\mathbb{R}); \\ \frac{1}{2 \cdot q^{\frac{1}{q}}} \left[\|f'\|_{(-\infty,x],p} + \|f'\|_{[x,\infty),p} \right] & \text{if } f' \in L_p(\mathbb{R}), \\ & p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2} \|f'\|_{\mathbb{R},1} & \text{if } f' \in L(\mathbb{R}), \end{cases} \\ &\leq \begin{cases} \|f'\|_{\mathbb{R},\infty} & \text{if } f' \in L_\infty(\mathbb{R}); \\ \frac{1}{2^{\frac{1}{p}} \cdot q^{\frac{1}{q}}} \|f'\|_{\mathbb{R},p} & \text{if } f' \in L_p(\mathbb{R}),\ p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2} \|f'\|_{\mathbb{R},1} & \text{if } f' \in L(\mathbb{R}). \end{cases} \end{aligned} \tag{2.1}$$

Proof. Define the function $p:\mathbb{R}^2\to\mathbb{R}$,

$$p(t,x):=\begin{cases}\exp(t-x) & \text{if } -\infty<t\le x<\infty,\\ \\ -\exp(x-t) & \text{if } -\infty<x<t<\infty,\end{cases} \tag{2.2}$$

then we have

$$\begin{aligned}\int_{-\infty}^{\infty}p(x,t)f'(t)\,dt &= \int_{-\infty}^{x}e^{t-x}f'(t)\,dt+\int_{x}^{\infty}e^{x-t}f'(t)\,dt \\ &= e^{t-x}f(t)\Big|_{-\infty}^{x}-\int_{-\infty}^{x}e^{t-x}f(t)\,dt \\ &\quad -\left[e^{x-t}f(t)\Big|_{x}^{\infty}+\int_{x}^{\infty}e^{x-t}f(t)\,dt\right] \\ &= f(x)-\int_{-\infty}^{x}e^{t-x}f(t)\,dt+f(x)-\int_{x}^{\infty}e^{x-t}f(t)\,dt \\ &= 2f(x)-\left[\int_{-\infty}^{x}e^{t-x}f(t)\,dt+\int_{x}^{\infty}e^{x-t}f(t)\,dt\right].\end{aligned} \tag{2.3}$$

On the other hand, by changing the variable $t-x=u$, we have

$$\int_{-\infty}^{x}e^{t-x}f(t)\,dt=\int_{-\infty}^{0}e^{u}f(x+u)\,du$$

and by $v=-u$, we deduce

$$\begin{aligned}\int_{-\infty}^{0}e^{u}f(x+u)\,du &= \int_{\infty}^{0}e^{-v}f(x-v)\,d(-v) \\ &= \int_{0}^{\infty}e^{-v}f(x-v)\,dv.\end{aligned}$$

Also, if we choose $v=t-x$ in the second integral, we have

$$\int_{x}^{\infty}e^{x-t}f(t)\,dt=\int_{0}^{\infty}e^{-v}f(x+v)\,dv$$

and thus, by (2.3), we get the following identity that is of interest in itself as well

$$f(x)-\int_{0}^{\infty}\left[\frac{f(x-v)+f(x+v)}{2}\right]e^{-v}dv=\frac{1}{2}\int_{-\infty}^{\infty}p(x,t)f'(t)\,dt \tag{2.4}$$

for any $x\in\mathbb{R}$.

Now, if we take the modulus in (2.4), we deduce

$$\begin{aligned}&\left|f(x)-\int_{0}^{\infty}\left[\frac{f(x-v)+f(x+v)}{2}\right]e^{-v}dv\right| \\ &\qquad\le\frac{1}{2}\left[\int_{-\infty}^{x}e^{t-x}\left|f'(t)\right|dt+\int_{x}^{\infty}e^{x-t}\left|f'(t)\right|dt\right],\end{aligned}$$

and the first inequality in (2.1) is proven.

Now, if $f'\in L_{\infty}(\mathbb{R})$, then obviously

$$\begin{aligned}\int_{-\infty}^{x}e^{t-x}\left|f'(t)\right|dt &\le \left\|f'\right\|_{(-\infty,x],\infty}\int_{-\infty}^{x}e^{t-x}dt \\ &= \left\|f'\right\|_{(-\infty,x],\infty}\end{aligned}$$

and

$$\int_x^{\infty} e^{x-t} |f'(t)| \, dt \le \|f'\|_{(-\infty,x],\infty} \int_x^{\infty} e^{x-t} dt = \|f'\|_{(-\infty,x],\infty} .$$

If $f' \in L_p(\mathbb{R})$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then, by Hölder's inequality, we have

$$\begin{aligned} \int_{-\infty}^{x} e^{t-x} |f'(t)| \, dt &\le \left(\int_{-\infty}^{x} e^{q(t-x)} dt \right)^{\frac{1}{q}} \left(\int_{-\infty}^{x} |f'(t)|^p \, dt \right)^{\frac{1}{p}} \\ &= \frac{1}{q^{\frac{1}{q}}} \|f'\|_{(-\infty,x],p} \end{aligned}$$

and, similarly,

$$\int_x^{\infty} e^{x-t} |f'(t)| \, dt \le \frac{1}{q^{\frac{1}{q}}} \|f'\|_{[x,\infty),p} ,$$

getting the second part of the second inequality in (2.1).

Also, since

$$\int_{-\infty}^{x} e^{t-x} |f'(t)| \, dt \le \sup_{-\infty < t \le x} e^{t-x} \int_{-\infty}^{x} |f'(t)| \, dt = \|f'\|_{(-\infty,x],1} ,$$

$$\int_x^{\infty} e^{x-t} |f'(t)| \, dt \le \|f'\|_{[x,\infty),1}$$

and

$$\|f'\|_{(-\infty,x],1} + \|f'\|_{[x,\infty),1} = \|f'\|_{\mathbb{R},1} ,$$

then the last part of the second inequality in (2.1) is also proven.

Now, since

$$\begin{aligned} \frac{1}{2} \left[\|f'\|_{(-\infty,x],\infty} + \|f'\|_{[x,\infty),\infty} \right] &\le \max \left\{ \|f'\|_{(-\infty,x],\infty} , \|f'\|_{[x,\infty),\infty} \right\} \\ &= \|f'\|_{\mathbb{R},\infty} \end{aligned}$$

the first part of the third inequality in (2.1) is proved.

Using the elementary inequality

$$\alpha + \beta \le 2^{\frac{1}{q}} \left(\alpha^p + \beta^p \right)^{\frac{1}{p}} , \quad p > 1, \; \frac{1}{p} + \frac{1}{q} = 1, \; \alpha, \beta \ge 0,$$

we deduce that

$$\begin{aligned} \|f'\|_{(-\infty,x],p} + \|f'\|_{[x,\infty),p} &\le 2^{\frac{1}{q}} \left(\|f'\|^p_{(-\infty,x],p} + \|f'\|^p_{[x,\infty),p} \right)^{\frac{1}{p}} \\ &= 2^{\frac{1}{q}} \|f'\|_{\mathbb{R},p} \end{aligned}$$

and the second part of the third inequality is also proven.

The proof is completed. ∎

The following result may be stated as well.

Theorem 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a locally absolutely continuous function on $\mathbb{R}$ such that there exist the constants $\gamma, \Gamma \in \mathbb{R}$ such that*

$$\gamma \le f'(t) \le \Gamma \quad \text{for a.e.} \quad t \in \mathbb{R}, \tag{2.5}$$

then

$$\left| f(x) - \int_0^{\infty} \left[\frac{f(x-v) + f(x+v)}{2} \right] e^{-v} dv \right| \le \frac{1}{2} (\Gamma - \gamma) , \tag{2.6}$$

for any $x \in \mathbb{R}$.

Proof. From the proof of Theorem 1, we know that

$$f(x)-\int_0^\infty\left[\frac{f(x-v)+f(x+v)}{2}\right]e^{-v}dv = \frac{1}{2}\left[\int_{-\infty}^x e^{t-x}f'(t)\,dt-\int_x^\infty e^{x-t}f'(t)\,dt\right]. \tag{2.7}$$

Utilising (2.5) we have, for a fixed $x\in\mathbb{R}$, that

$$\gamma e^{t-x}\le e^{t-x}f'(t)\le\Gamma e^{t-x}\quad\text{for a.e.}\quad t\in(-\infty,x]$$

and

$$-\Gamma e^{x-t}\le -f'(t)\,e^{x-t}\le-\gamma e^{x-t}\quad\text{for a.e.}\quad t\in[x,\infty),$$

which gives, by integration,

$$\gamma e^{-x}\int_{-\infty}^x e^t dt\le\int_{-\infty}^x e^{t-x}f'(t)\,dt\le\Gamma e^{-x}\int_{-\infty}^x e^t dt$$

and

$$-\Gamma e^x\int_x^\infty e^{-t}dt\le-\int_x^\infty e^{x-t}f'(t)\,dt\le-\gamma e^x\int_x^\infty e^{-t}dt$$

i.e.,

$$\gamma\le\int_{-\infty}^x e^{t-x}f'(t)\,dt\le\Gamma$$

and

$$-\Gamma\le-\int_x^\infty e^{x-t}f'(t)\,dt\le-\gamma,$$

which, by addition, provide the desired inequality (2.6). ∎

Remark 1. *The inequality (2.6) is better than the inequality*

$$\left|f(x)-\int_0^\infty\left[\frac{f(x-v)+f(x+v)}{2}\right]e^{-v}dv\right|\le\|f'\|_{\mathbb{R},\infty},$$

which has been obtained in (2.1). This follows by the fact that, if (2.5) holds true, then $-\|f'\|_{\mathbb{R},\infty}\le\gamma$ *and* $\Gamma\le\|f'\|_{\mathbb{R},\infty}$, *where* $\|f'\|_{\mathbb{R},\infty}:=ess\sup_{t\in\mathbb{R}}|f'(t)|$.

The case of convex functions is incorporated in the following theorem.

Theorem 3. *Let* $f:\mathbb{R}\to\mathbb{R}$ *be a convex function on* $\mathbb{R}$ *and* $f'_+(x)$, $f'_-(x)$ *the lateral derivatives in* x, $x\in\mathbb{R}$, *then*

$$f(x)-\int_0^\infty\left[\frac{f(x-v)+f(x+v)}{2}\right]e^{-v}dv\le\frac{1}{2}\left[f'_-(x)-f'_+(x)\right]\le 0 \tag{2.8}$$

for any $x\in\mathbb{R}$.

Proof. Since f is convex, hence $f'(t)\le f'_-(x)$ for a.e. $t\in(-\infty,x]$ and $f'(t)\ge f'_+(x)$ for a.e. $t\in[x,\infty)$. This implies that,

$$\int_{-\infty}^x e^{t-x}f'(t)\,dt\le\int_{-\infty}^x e^{t-x}f'_-(x)\,dt=f'_-(x) \tag{2.9}$$

and

$$-\int_x^\infty e^{x-t}f'(t)\,dt\le-\int_x^\infty e^{x-t}f'_+(x)\,dt=-f'_+(x) \tag{2.10}$$

for any $x\in\mathbb{R}$.

Adding (2.9) to (2.10) and utilising the representation (2.7), we deduce the desired inequality (2.8). ∎

Remark 2. *If f is convex on $\mathbb{R}$, then we have the inequality:*

$$\int_0^{\infty}\left[\frac{f(x-v)+f(x+v)}{2}\right]e^{-v}dv \geq f(x) \tag{2.11}$$

for each $x \in \mathbb{R}$. This inequality may be proved on using the definition of convexity as well. Namely, since

$$\frac{f(x-v)+f(x+v)}{2} \geq f(x),$$

then

$$\int_0^{\infty}\left[\frac{f(x-v)+f(x+v)}{2}\right]e^{-v}dv \geq f(x)\int_0^{\infty}e^{-v}dv = f(x),$$

which is exactly (2.11).

Note that in general (2.8) is a better result than (2.11) since, for instance, if one considers the convex function $f(t) := |t-x|$, $t \in \mathbb{R}$, then $\frac{1}{2}\left[f'_{-}(x) - f'_{+}(x)\right] = -1$.

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