

ON INTEGRAL VERSION OF ALZER'S INEQUALITY AND MARTINS' INEQUALITY

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ABSTRACT. Let $c > b > a$ and r be real numbers, and let f be a positive, twice differentiable function and satisfy $f'(t) > 0$ and $(\ln f(t))'' \geq 0$ on $(a, +\infty)$. Then

$$\frac{\sup_{x \in [a, b]} f(x)}{\sup_{x \in [a, c]} f(x)} < \left(\frac{\frac{1}{b-a} \int_a^b f^r(x) dx}{\frac{1}{c-a} \int_a^c f^r(x) dx} \right)^{1/r} < 1 \text{ for all real } r,$$

$$\left(\frac{\frac{1}{b-a} \int_a^b f^r(x) dx}{\frac{1}{c-a} \int_a^c f^r(x) dx} \right)^{1/r} \leq \frac{\exp\left(\frac{1}{b-a} \int_a^b \ln f(x) dx\right)}{\exp\left(\frac{1}{c-a} \int_a^c \ln f(x) dx\right)} \text{ according as } r \geq 0.$$

This solves a recently open problem of B.-N. Guo and F. Qi.

1. INTRODUCTION

It was shown in [1, 2, 8, 13, 17] that let n be a positive integer, then for $r > 0$,

$$\frac{n}{n+1} < \left(\frac{\frac{1}{n} \sum_{i=1}^n i^r}{\frac{1}{n+1} \sum_{i=1}^{n+1} i^r} \right)^{1/r} < \frac{\sqrt[n]{n!}}{n+1} \quad (1)$$

We call the left-hand side of (1) H. Alzer's inequality [1], and the right-hand side of (1) J. S. Martins' inequality [8]. In [3, 14] Alzer's inequality is extended to all real r . In [5] it was proved that Martins' inequality is reversed for $r < 0$.

F. Qi and B.-N. Guo [10, 11] presented an integral version of inequality (1) as follows: Let $b > a > 0$ and $\delta > 0$, then for $r > 0$,

$$\frac{b}{b+\delta} < \left(\frac{\frac{1}{b-a} \int_a^b x^r dx}{\frac{1}{b+\delta-a} \int_a^{b+\delta} x^r dx} \right)^{1/r} < \frac{[b^b/a^a]^{1/(b-a)}}{[(b+\delta)^{b+\delta}/a^a]^{1/(b+\delta-a)}}. \quad (2)$$

We note that the inequality (4) can be written for $r > 0$ as

$$\frac{b}{b+\delta} < \frac{L_r(a, b)}{L_r(a, b+\delta)} < \frac{I(a, b)}{I(a, b+\delta)}, \quad (3)$$

where $L_r(a, b)$ and $I(a, b)$ are respectively the generalized logarithmic mean and the exponential mean of two positive numbers a, b , defined in [6, 15, 16] by, for $a = b$ by $L_r(a, b) = a$ and for $a \neq b$ by

$$L_r(a, b) = \left(\frac{b^{r+1} - a^{r+1}}{(r+1)(b-a)} \right)^{1/r}, \quad r \neq -1, 0;$$

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$$L_{-1}(a, b) = \frac{b-a}{\ln b - \ln a} = L(a, b);$$

$$L_0(a, b) = \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{1/(b-a)} = I(a, b).$$

$L(a, b)$ is the logarithmic mean of two positive numbers a, b . When $a \neq b$, $L_r(a, b)$ is a strictly increasing function of r . In particular,

$$\lim_{r \rightarrow -\infty} L_r(a, b) = \min\{a, b\}, \quad \lim_{r \rightarrow +\infty} L_r(a, b) = \max\{a, b\}.$$

In [4], it was indirectly shown that the function $r \mapsto L_r(a, b)/L_r(a, b + \delta)$ is strictly decreasing with $r \in (-\infty, +\infty)$. This yields that

$$\frac{b}{b + \delta} < \left(\frac{\frac{1}{b-a} \int_a^b x^r dx}{\frac{1}{b+\delta-a} \int_a^{b+\delta} x^r dx} \right)^{1/r} \quad \text{for all real } r, \quad (4)$$

$$\left(\frac{\frac{1}{b-a} \int_a^b x^r dx}{\frac{1}{b+\delta-a} \int_a^{b+\delta} x^r dx} \right)^{1/r} \leq \frac{[b^b/a^a]^{1/(b-a)}}{[(b+\delta)^{b+\delta}/a^a]^{1/(b+\delta-a)}} \quad \text{according as } r \geq 0. \quad (5)$$

In [7], B.-N. Guo and F. Qi ask under which conditions the inequality

$$\frac{\sup_{x \in [a, b]} f(x)}{\sup_{x \in [a, b+\delta]} f(x)} < \left(\frac{\frac{1}{b-a} \int_a^b f^r(x) dx}{\frac{1}{b+\delta-a} \int_a^{b+\delta} f^r(x) dx} \right)^{1/r} < \frac{\exp\left(\frac{1}{b-a} \int_a^b \ln f(x) dx\right)}{\exp\left(\frac{1}{b+\delta-a} \int_a^{b+\delta} \ln f(x) dx\right)} \quad (6)$$

holds for $b > a > 0$, $\delta > 0$ and $r > 0$.

V. Mascioni [9] found the sufficient conditions on the function f , and proved the right-hand inequality of (6) for $r > 0$. Motivated by the paper of Mascioni [9], we establish the following

Theorem. *Let $c > b > a$ and r be real numbers, and let f be a positive, twice differentiable function and satisfy $f'(t) > 0$ and $(\ln f(t))'' \geq 0$ on $(a, +\infty)$. Then*

$$\frac{\sup_{x \in [a, b]} f(x)}{\sup_{x \in [a, c]} f(x)} < \left(\frac{\frac{1}{b-a} \int_a^b f^r(x) dx}{\frac{1}{c-a} \int_a^c f^r(x) dx} \right)^{1/r} < 1 \quad \text{for all real } r, \quad (7)$$

$$\left(\frac{\frac{1}{b-a} \int_a^b f^r(x) dx}{\frac{1}{c-a} \int_a^c f^r(x) dx} \right)^{1/r} \leq \frac{\exp\left(\frac{1}{b-a} \int_a^b \ln f(x) dx\right)}{\exp\left(\frac{1}{c-a} \int_a^c \ln f(x) dx\right)} \quad \text{according as } r \geq 0. \quad (8)$$

Both bounds in (7) are best possible.

2. LEMMAS

Lemma 1. *Let the function f be a positive and twice differentiable on $(a, +\infty)$, where a is a given real number, and let*

$$G(t) = \frac{\frac{1}{t-a} \int_a^t f(x) dx}{f(t)}, \quad t > a.$$

Then we have

(i) *If $f'(t) > 0$ and $(\ln f(t))'' \geq 0$, then the function G is strictly decreasing on $(a, +\infty)$.*

(ii) If $f'(t) < 0$ and $(\ln f(t))'' \leq 0$, then the function G is strictly increasing on $(a, +\infty)$.

Proof. Easy calculation reveals that

$$\begin{aligned} \frac{[(t-a)f(t)]^2 G'(t)}{f(t) + (t-a)f'(t)} &= \frac{(t-a)f^2(t)}{f(t) + (t-a)f'(t)} - \int_a^t f(x) \triangleq H(t), \\ \frac{[f(t) + (t-a)f'(t)]^2 H'(t)}{(t-a)f^3(t)} &= -(t-a) \frac{f''(t)f(t) - [f'(t)]^2}{f^2(t)} - \frac{f'(t)}{f(t)} \\ &= -[(t-a)(\ln(f(t)))'' + (\ln f(t))']. \end{aligned}$$

If $(\ln f(t))' > (<)0$ and $(\ln(f(t)))'' \geq (\leq)0$ for $t > a$, then $H'(t) < (>)0$ for $t > a$, and then, $H(t) < (>)H(a) = 0$ and $G'(t) < (>)0$ for $t > a$. The proof is complete. \square

Lemma 2 ([12]). *If $\mathcal{F}(t)$ is a strictly increasing (decreasing) integrable function on an interval $I \subseteq \mathbb{R}$, then the arithmetic mean $\mathcal{G}(r, s)$ of function $\mathcal{F}(t)$,*

$$\mathcal{G}(r, s) = \begin{cases} \frac{1}{s-r} \int_r^s \mathcal{F}(t) dt, & r \neq s, \\ \mathcal{F}(r), & r = s, \end{cases}$$

is also strictly increasing (decreasing) with both r and s on I .

3. PROOF OF THEOREM

For $r = 0$, (7) can be interpreted as

$$\frac{f(b)}{f(c)} < \frac{\exp\left(\frac{1}{b-a} \int_a^b \ln f(x) dx\right)}{\exp\left(\frac{1}{c-a} \int_a^c \ln f(x) dx\right)} < 1. \quad (9)$$

Define for $t > a$,

$$P(t) = \frac{\exp\left(\frac{1}{t-a} \int_a^t \ln f(x) dx\right)}{f(t)}.$$

A simple computation yields

$$\begin{aligned} (t-a)^2 \frac{P'(t)}{P(t)} &= (t-a) \ln f(t) - \int_a^t \ln f(x) dx - (t-a)(\ln f(t))' \triangleq Q(t), \\ Q'(t) &= -(t-a) [(\ln f(t))' + (t-a)(\ln(f(t)))''] < 0. \end{aligned}$$

Hence, we have $Q(t) < Q(a) = 0$ and $P'(t) < 0$ for $t > a$. This means the left-hand inequality of (9) holds for $c > b > a$. By Lemma 2, the right-hand inequality of (9) holds clearly.

For $r \neq 0$, (7) is equivalent to

$$\frac{f^r(b)}{f^r(c)} \leq \frac{\frac{1}{b-a} \int_a^b f^r(x) dx}{\frac{1}{c-a} \int_a^c f^r(x) dx} \leq 1, \quad \text{according as } r \geq 0. \quad (10)$$

Define for $t > a$,

$$G_r(t) = \frac{\frac{1}{t-a} \int_a^t f^r(x) dx}{f^r(t)}.$$

It is easy to see that

$$(\ln f^r(t))'_t \geq 0 \quad \text{and} \quad (\ln(f^r(t)))''_t \geq 0, \quad \text{according as } r \geq 0. \quad (11)$$

By Lemma 1, the function $t \mapsto G_r(t)$ strictly $\begin{matrix} \text{decreases} \\ \text{increases} \end{matrix}$ with respect to $t \in (a, +\infty)$ according as $r \geq 0$. This produces the left-hand inequality of (10). By Lemma 2, the right-hand inequality of (10) holds clearly.

Both bounds in (7) are best possible because of

$$\begin{aligned} \lim_{r \rightarrow +\infty} \left(\frac{\frac{1}{b-a} \int_a^b f^r(x) dx}{\frac{1}{c-a} \int_a^c f^r(x) dx} \right)^{1/r} &= \frac{\sup_{x \in [a,b]} f(x)}{\sup_{x \in [a,c]} f(x)}, \\ \lim_{r \rightarrow -\infty} \left(\frac{\frac{1}{b-a} \int_a^b f^r(x) dx}{\frac{1}{c-a} \int_a^c f^r(x) dx} \right)^{1/r} &= \frac{\inf_{x \in [a,b]} f(x)}{\inf_{x \in [a,c]} f(x)} = 1. \end{aligned}$$

The inequality (8) is equivalent to

$$\frac{\frac{1}{b-a} \int_a^b f^r(x) dx}{\frac{1}{c-a} \int_a^c f^r(x) dx} < \frac{\exp\left(\frac{1}{b-a} \int_a^b \ln f^r(x) dx\right)}{\exp\left(\frac{1}{c-a} \int_a^c \ln f^r(x) dx\right)} \quad \text{for } r \neq 0. \quad (12)$$

Define for $t > a$,

$$F_r(t) = \frac{\frac{1}{t-a} \int_a^t f^r(x) dx}{\exp\left(\frac{1}{t-a} \int_a^t \ln f^r(x) dx\right)}.$$

It is easy to see from the proof of Theorem 1 of [9] that if $f'(t) > 0$ and $(\ln f(t))'' \geq 0$, then the function F_1 is strictly increasing on $(a, +\infty)$; If $f'(t) < 0$ and $(\ln f(t))'' \leq 0$, then the function F_1 is strictly decreasing on $(a, +\infty)$. Applying this result, together with (11), we obviously imply the function $t \mapsto F_r(t)$ strictly $\begin{matrix} \text{increases} \\ \text{decreases} \end{matrix}$ with respect to $t \in (a, +\infty)$ according as $r \geq 0$. This produces (12). The proof is complete.

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