COMPARISON THEOREM FOR GENERALIZATION OF STOLARSKY MEANS

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Abstract. For real u, v, r, s and positive x, y we define

$$R(u, v; r, s; x, y) = \left(\frac{E(u, v; x^s, y^s)}{E(u, v; x^r, y^r)}\right)^{1/(s-r)}$$

where E is the Stolarsky mean. R contains Gini, Heronian and many other known means.
In this paper we give sufficient and necessary conditions for

$$R(u, v; a, b; x, y) \le R(u, v; c, d; x, y)$$

to hold.
We also investigate convexity properties of $R(\alpha, \alpha+1; r, s; x, y)$ and obtain new inequalities between Gini, Heronian and Stolarsky means.

1. Introduction

The comparison theorem for Stolarsky means

$$(1)\quad E(r, s) = E(r, s; x, y) = \begin{cases} \left(\frac{r}{s}\frac{y^s - x^s}{y^r - x^r}\right)^{1/(s-r)} & sr(s-r)(x-y) \ne 0, \\ \left(\frac{1}{r}\frac{y^r - x^r}{\log y - \log x}\right)^{1/r} & r(x-y) \ne 0,\ s = 0, \\ e^{-1/r}\left(y^{y^r}/x^{x^r}\right)^{1/(y^r - x^r)} & r = s,\ r(x-y) \ne 0, \\ \sqrt{xy} & r = s = 0,\ x - y \ne 0, \\ x & x = y. \end{cases}$$

proved first by Leach and Scholander in [2] states that $E(a, b; x, y) \le E(c, d; x, y)$ holds for all $x, y > 0$ if and only if

$$(2)\qquad a + b \le c + d$$

and

$$(3)\qquad e(a, b) \le e(c, d)$$

where

$$(4)\qquad e(r, s) = \begin{cases} \frac{|r| - |s|}{r - s} & \min(a, b, c, d) < 0 < \max(a, b, c, d) \\ \begin{cases} \frac{r-s}{\log(r/s)} & rs > 0, \\ 0 & rs = 0 \end{cases} & \text{otherwise,} \end{cases}$$

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In [4] Páles gave a proof of the comparison theorem for Gini means

$$G(r,s)=G(r,s;x,y)=\begin{cases}\left(\frac{x^s+y^s}{x^r+y^r}\right)^{1/(s-r)} & s-r\neq 0,\\ \exp\left(\frac{x^r\log x+y^r\log y}{x^r+y^r}\right) & s=r\neq 0,\\ \sqrt{xy} & s=r=0\end{cases} \tag{5}$$

which states that $G(a,b,x,y)\le G(c,d,x,y)$ holds for all $x,y>0$ if and only if

$$a+b\le c+d \tag{6}$$

and

$$m(a,b)\le m(c,d) \tag{7}$$

where

$$m(r,s)=\begin{cases}\min(r,s) & \text{if} \quad \min(a,b,c,d)\ge 0,\\ \max(r,s) & \text{if} \quad \max(a,b,c,d)\le 0,\\ \dfrac{|r|-|s|}{r-s} & \text{if} \quad \min(a,b,c,d)<0<\max(a,b,c,d).\end{cases} \tag{8}$$

It is natural to ask if similar results can be obtained for other two-parameter families of means like Heronian

$$N(r,s)=N(r,s;x,y)=\left(\frac{x^s+(\sqrt{xy})^s+y^s}{x^r+(\sqrt{xy})^r+y^r}\right)^{1/(s-r)} \tag{9}$$

or centroidal

$$T(r,s)=T(r,s;x,y)==\left(\frac{x^{2s}+(xy)^s+y^{2s}}{x^s+y^s}\Big/\frac{x^{2r}+(xy)^r+y^{2r}}{x^r+y^r}\right)^{1/(s-r)} \tag{10}$$

means.

In this paper we introduce S-means - a one parameter family of means that contains all the means mentioned so far - and prove the comparison theorem for them. The paper is organised as follows: in the next section we introduce S-means and investigate their monotonicity and convexity properties. Section 3 is devoted to the proof of the comparison theorem. In the last section we apply our results to some other families of means.

2. S-means

For real α we define S-means as

$$S(\alpha;r,s)=S(\alpha;r,s;x,y)=\frac{E(r,s;x^{\alpha+1},y^{\alpha+1})}{E(r,s;x^{\alpha},y^{\alpha})} \tag{11}$$

It is easy to verify that

$$S(0;r,s,x,y)=E(r,s;x,y) \tag{12}$$

$$S(\tfrac{1}{2};r,s,x,y)=N(r,s;x,y) \tag{13}$$

$$S(1;r,s,x,y)=G(r,s;x,y) \tag{14}$$

$$S(2;r,s,x,y)=T(r,s;x,y). \tag{15}$$

Observe that S-means are homogeneous of order 1 with respect to x and y and symmetric in r,s and x,y.

Investigating monotonicity properties of S-means we will use the weighted extended mean values introduced by the author in [6] and defined as

$$F(r,s;a,b;x,y)=\frac{E(r,s;ax,by)}{E(r,s;a,b)}. \tag{16}$$

so the S-means can be redefined as $F(r,s;x^\alpha,y^\alpha;x,y)$.

Theorem 1. *S-means are means (i.e.* $\min(x,y)\le S(\alpha;r,s;x,y)\le \max(x,y)$*).*

Proof. By Theorem 4 in [6] $\min(x,y)\le F\le\max(x,y)$. □

Theorem 2. $S(\alpha,r,s;x,y)$ *is increasing in variables* r *and* s *if* $\alpha>-\frac{1}{2}$ *and decreasing if* $\alpha<-\frac{1}{2}$.

Proof. Theorem 2 in [6] states that $F(r,s;a,b;x,y)$ increases (resp. decreases) in r and s if $(x-y)(a^2x-b^2y)>(<$, resp.) 0. For S-means this condition reads $(x-y)(x^{2\alpha+1}-y^{2\alpha+1})>(<$, resp.) 0 and this is equivalent to $2\alpha+1>(<$, resp.) 0. □

Theorem 3. $S(\alpha,r,s;x,y)$ *is increasing in* α *if* $r+s>0$ *and decreasing if* $r+s<0$.

Proof. Since S-means are homogeneous we can assume that $y=1$. By [6] [Theorem 3] $F(r,s;a,b;x,y)$ increases (resp. decreases) in a if $(x-y)(r+s)>(<$, resp.) 0, so

$$\begin{aligned}\operatorname{sgn}(S(\alpha,r,s;x,1)-S(\beta,r,s;x,1))&\\ &=\operatorname{sgn}(F(r,s;x^\alpha,1;x,1)-F(r,s;x^\beta,1;x,1))\\ &=\operatorname{sgn}((x^\alpha-x^\beta)(x-1)(r+s))\\ &=\operatorname{sgn}(x^\beta(x^{\alpha-\beta}-1)(x-1)(r+s))\\ &=\operatorname{sgn}(\alpha-\beta)(r+s).\end{aligned}$$

because $\operatorname{sgn}((x^{\alpha-\beta}-1)(x-1))=\operatorname{sgn}(\alpha-\beta)$ □

Corollary 1.

$$E(r,s)\le N(r,s)\le G(r,s)\le T(r,s)$$

if $r+s>0$. *For* $r+s<0$ *the inequalities reverse.*

The above inequality between Stolarsky and Gini means was obtained by Neuman and Páles [3].
It is worth noting that in general S-means are not monotone in x and y.

Before investigating convexity of S-means let us recall some elementary properties of convex functions:

Property 1. *If* f *is convex (resp. concave) then for* $h>0$ $f(x+h)-f(x)$ *increases (resp. decreases) in* x. *For negative* h *the monotonicity of* $f(x+h)-f(x)$ *reverses.*

Property 2. *If* f *is convex (resp. concave) then for* $t>0$ *and fixed* x_0 $f(x_0-t)+f(x_0+t)$ *increases (resp. decreases) in* t.

Property 3. f *is convex (resp. concave) if and only if the function* $\frac{f(s)-f(t)}{s-t}$ *increases (resp. decreases) in* s *and* t.

The following lemmas will be useful

Lemma 1. *If $f(x)$ is odd and convex for $x>0$ then the function $h(t)=f(x_0+t)+f(x_0-t)$ is increasing in t for $t>0$ if $x_0>0$ and decreasing otherwise.*
If $f(x)$ is concave for $x>0$ then the monotonicity of h reverses.

Proof. Let $x_0>0$ and f be convex on $(0,\infty)$. If $0<t<x_0$ the h increases by property 2. If $t>x_0$ then

$$h(t)=f(x_0+t)-f(t-x_0)=f((t-x_0)+2x_0)-f(t-x_0)$$

increases by property 1.
The proof is the same in case $x_0<0$. For a concave f it suffices to consider $-f$. □

For real t and positive $A,B\neq 1$ let

$$g(t,A,B)=\frac{A^t\log^2 A}{(A^t-1)^2}-\frac{B^t\log^2 B}{(B^t-1)^2}.$$

Lemma 2.
(a) *$g(t,A,B)=g(\pm t,A^{\pm 1},B^{\pm 1})$ for arbitrary choice of signs,,*
(b) *g is increasing in t on $(0,\infty)$ if $\log^2 A-\log^2 B>0$ and decreasing otherwise.*

Proof. (a) becomes obvious when we write

$$g(t,A,B)=\frac{\log^2 A}{A^t-2+A^{-t}}-\frac{\log^2 B}{B^t-2+B^{-t}}.$$

From (a) it follows that replacing A,B with A^{-1},B^{-1} if necessary we may assume that $A,B>1$. In this case $\operatorname{sgn}(\log^2 A-\log^2 B)=\operatorname{sgn}(A^t-B^t)$.

$$\begin{aligned}\frac{\partial g}{\partial t}&=-\frac{A^t(A^t+1)\log^3 A}{(A^t-1)^3}+\frac{B^t(B^t+1)\log^3 B}{(B^t-1)^3}\\&=-\frac{1}{t^3}(\phi(A^t)-\phi(B^t))=-\frac{1}{t^3}(A^t-B^t)\phi'(\xi),\end{aligned}$$

where

$$\phi(u)=\frac{u(u+1)\log^3 u}{(u-1)^3}$$

and $\xi>1$ lies between A^t and B^t. In order to complete the proof it is enough to show that $\phi'(u)<0$ for $u>1$.

$$\phi'(u)=\frac{(u^2+4u+1)\log^2 u}{(u-1)^4}\left[\frac{3(u^2-1)}{u^2+4u+1}-\log u\right],$$

so the sign of ϕ' is the same as the sign of $\psi(u)=\frac{3(u^2-1)}{u^2+4u+1}-\log u$. But $\psi(1)=0$ and $\psi'(u)=-(u-1)^4/(u^2+4u+1)^2$¡0, so $\phi(u)<0$. □

As $E(r,s;x,y)$ is positive and continuous in all variables S-means are also continuous which allows us to consider only the general case ($r\neq s,\ rs\neq 0$) and have the other cases follow by continuity.
Our next step will be investigation of convexity of $S(\alpha)$. Once again due to homogeneity of S as a function of x and y we can consider only case $y=1$.
Given that

$$\frac{\partial^2}{\partial\alpha^2}\log|b^\alpha-1|=-\frac{b^\alpha\log^2 b}{(b^\alpha-1)^2}$$

and

$$S(\alpha; r, s; x, 1) = \left(\frac{x^{(\alpha+1)s} - 1}{x^{\alpha s} - 1} \Big/ \frac{x^{(\alpha+1)r} - 1}{x^{\alpha r} - 1} \right)^{1/s-r}$$

we see, that

$$\frac{\partial^2}{\partial \alpha^2} \log S = \frac{g(\alpha+1, x^r, x^s) - g(\alpha, x^r, x^s)}{s - r}. \tag{17}$$

This leads to the following

Theorem 4. *If $r + s > 0$ then $S(\alpha)$ is log-concave for $\alpha > -\frac{1}{2}$ and log-convex for $\alpha < -\frac{1}{2}$.*
If $r + s < 0$ then $S(\alpha)$ is log-convex for $\alpha > -\frac{1}{2}$ and log-concave for $\alpha < -\frac{1}{2}$.

Proof. By Lemma 2(a) we can replace α and $\alpha + 1$ in the right hand side of the formula (17) with their absolute values. Since $\alpha > -\frac{1}{2}$ is equivalent to $|\alpha+1| > |\alpha|$ applying Lemma 2(b) we get

$$\begin{aligned}
\operatorname{sgn} \frac{\partial^2}{\partial \alpha^2} \log S &= \operatorname{sgn} \frac{g(|\alpha+1|, x^r, x^s) - g(|\alpha|, x^r, x^s)}{s - r} \\
&= \operatorname{sgn} \left(\left(\alpha + \tfrac{1}{2}\right) \frac{\log^2(x^r) - \log^2(x^s)}{s - r} \right) \\
&= -\operatorname{sgn} \left(\left(\alpha + \tfrac{1}{2}\right) (r + s) \right)
\end{aligned}$$

which completes the proof. □

An easy calculation reveals the following properties of $S(\alpha)$:

$$S(-\tfrac{1}{2}; r, s; x, y) = \sqrt{xy} \tag{18}$$

$$S(-\tfrac{1}{2} - \alpha) S(-\tfrac{1}{2} + \alpha) = S^2(-\tfrac{1}{2}) = xy, \tag{19}$$

which enables us to prove the

Theorem 5. *If $r + s > 0$ then*

- *if $\alpha_0 > -\frac{1}{2}$ then $S(\alpha_0 - t) S(\alpha_0 + t)$ is decreasing for positive t,*
- *if $\alpha_0 < -\frac{1}{2}$ then $S(\alpha_0 - t) S(\alpha_0 + t)$ is increasing for positive t.*

For $r + s < 0$ the monotonicities reverse.

Proof. From (18) and (19) we see, that the function $\log S(\alpha - \frac{1}{2}) - \log S(-\frac{1}{2})$ is odd, so the theorem follows from Lemma 1 and Theorem 4. □

The following table shows some inequalities between the means that can be obtained using Theorems 4 and 5. In all the cases we assume $r + s > 0$, otherwise the inequalities reverse

Inequality	How obtained
$N^2(r,s) \geq E(r,s) G(r,s)$	$\frac{1}{2} = \frac{1}{2} 0 + \frac{1}{2} 1$
$E^2(r,s) \geq \sqrt{xy} N(r,s)$	$0 = \frac{1}{2}(-\frac{1}{2}) + \frac{1}{2}\frac{1}{2}$
$E^3(r,s) \geq xy G(r,s)$	$0 = \frac{2}{3}(-\frac{1}{2}) + \frac{1}{3} 1$
$N^3(r,s) \geq \sqrt{xy} G^2(r,s)$	$\frac{1}{2} = \frac{1}{3}(-\frac{1}{2}) + \frac{2}{3} 1$
$N(r,s) G(r,s) \geq \sqrt{xy} T(r,s)$	$\alpha_0 = \frac{3}{4}, t_1 = \frac{1}{4}, t_2 = \frac{5}{4}$ in th. 5

The last theorem of this section refines the inequalities between Stolarsky, Gini and Heronian means.

Theorem 6. *If $0 < \alpha$ and $r + s > 0$ then*

$$E(r, s) \le S(\alpha; r/(2\alpha + 1), s/(2\alpha + 1)).$$

For $r + s < 0$ the inequality reverses.

Setting $\alpha = \frac{1}{2}$ and $\alpha = 1$ we get

Corollary 2. *If $r + s > 0$ then*

$$E(r, s) \le G(r/3, s/3) \quad \textit{and} \quad E(r, s) \le N(r/2, s/2).$$

The inequalities reverse if $r + s < 0$.

(the first inequality was obtained by Czinder and Páles in [1]).

Proof of Theorem 6. For $-\frac{1}{2} < \beta < 0$ let $\mu = 2\beta + 1$ and $\nu = -\beta/(2\beta + 1)$. Then $-\beta = \mu\nu$ and $\beta + 1 = \mu(\nu + 1)$.

$$\begin{aligned}
S(\beta; r, s) &= \left(\frac{x^{s(\beta+1)} - y^{s(\beta+1)}}{x^{s\beta} - y^{s\beta}} \frac{x^{r\beta} - y^{r\beta}}{x^{r(\beta+1)} - y^{r(\beta+1)}}\right)^{\frac{1}{s-r}} \\
&= \left(\frac{(xy)^{r\beta}}{(xy)^{s\beta}} \frac{x^{s(\beta+1)} - y^{s(\beta+1)}}{x^{-s\beta} - y^{-s\beta}} \frac{x^{-r\beta} - y^{-r\beta}}{x^{r(\beta+1)} - y^{r(\beta+1)}}\right)^{\frac{1}{s-r}} \\
&= (xy)^{-\beta} \left(\frac{x^{\mu s(\nu+1)} - y^{\mu s(\nu+1)}}{x^{\mu s\nu} - y^{\mu s\nu}} \frac{x^{\mu r\nu} - y^{\mu r\nu}}{x^{\mu r(\nu+1)} - y^{\mu r(\nu+1)}}\right)^{\frac{\mu}{\mu s-\mu r}} \\
&= (xy)^{-\beta} S^{\mu}(\nu; \mu r, \mu s) \\
&= (xy)^{-\beta} S^{2\beta+1}(\tfrac{-\beta}{2\beta+1}; (2\beta + 1)r, (2\beta + 1)s).
\end{aligned} \tag{20}$$

If $r + s > 0$ then $S(\beta)$ is log-concave in $(-\frac{1}{2}, \infty)$. As $\beta = -\frac{1}{2}\gamma + 0(1 - \gamma)$, where $\gamma = -2\beta$, we get

$$\begin{aligned}
(xy)^{-\beta} S^{2\beta+1}(\tfrac{-\beta}{2\beta+1}; (2\beta + 1)r, (2\beta + 1)s) &= S(\beta; r, s) \\
&\ge S^{-2\beta}(-\tfrac{1}{2}; r, s) S^{1+2\beta}(0; r, s) \\
&= (\sqrt{xy})^{-2\beta} E^{2\beta+1}(r, s).
\end{aligned} \tag{21}$$

In order to complete the proof we set $\alpha = \frac{-\beta}{2\beta+1}$ and observe that if β varies from 0 to $-\frac{1}{2}$ then α increases from 0 to ∞.
In case $r + s < 0$ S is log-convex, which makes the inequality in (21) reverse. □

3. Comparison theorem for S-means

The theorem that follows is the main result of this paper

Theorem 7. *For $\alpha > 0$*

$$S(\alpha; a, b; x, y) \le S(\alpha; c, d; x, y) \tag{22}$$

holds for all $x, y > 0$ if and only if the following conditions are satisfied

$$a + b \le c + d \tag{23}$$

$$m(a, b) \le m(c, d) \tag{24}$$

where m is defined by (8).

In other words for positive α S-means are comparable if and only if corresponding Gini means are.
To start with the proof we need some prerequisites. For $x \in (0,1)$, $a < b$, $c < d$, $abcd \neq 0$ define

$$h(x) = h(x;a,b;c,d) = \frac{\dfrac{bx^b}{x^b-1} - \dfrac{ax^a}{x^a-1}}{b-a} - \frac{\dfrac{dx^d}{x^d-1} - \dfrac{cx^{c-1}}{x^c-1}}{d-c} \tag{25}$$

$$= \left(\frac{\dfrac{b}{x^b-1} - \dfrac{a}{x^a-1}}{b-a} - \frac{\dfrac{d}{x^d-1} - \dfrac{c}{x^c-1}}{d-c} \right) \tag{26}$$

$$= \left(\frac{\dfrac{b}{2}\dfrac{x^b+1}{x^b-1} - \dfrac{a}{2}\dfrac{x^a+1}{x^a-1}}{b-a} - \frac{\dfrac{d}{2}\dfrac{x^d+1}{x^d-1} - \dfrac{c}{2}\dfrac{x^c+1}{x^c-1}}{d-c} \right) \tag{27}$$

$$= \frac{1}{\ln x}\left(\frac{k(\frac{b\ln x}{2}) - k(\frac{a\ln x}{2})}{b-a} - \frac{k(\frac{d\ln x}{2}) - k(\frac{c\ln x}{2})}{d-c} \right) \tag{28}$$

where $k(t) = \dfrac{t}{\tanh t}$.
The expressions in numerators of (25), (26) and (27) differ by a linear term that gets annihilated by the difference of difference quotients. The following lemmas will be used in the proof

Lemma 3. *If $a+b < (>$, resp. $)$ $c+d$ then $h(x)$ is positive (resp. negative) for x close to 1.*

Proof. Writing $(1+t)^a - 1 = at + \frac{a(a-1)}{2}t^2 + \frac{a(a-1)(a-2)}{6}t^3 + o(t^3)$ etc. in (26) we obtain after elementary calculations that

$$h(1+t) = \frac{((a+b)-(c+d))t + o(t)}{12 + O(t)}$$

□

Lemma 4. *If $m(a,b) < (>$, resp. $)$ $m(c,d)$ then $h(x)$ is positive (resp. negative) for x close to 0.*
If $m(a,b) = m(c,d)$ then $h(x)$ is positive (resp. negative) for x close to 0 if $a+b < (>$, resp. $)$ $c+d$.
If $m(a,b) = m(c,d)$ and $a+b = c+d$ then $h \equiv 0$.

Proof. Consider three cases:
If $\min(a,b,c,d) > 0$ then applying l'Hôpital's rule we see that

$$\lim_{x\to 0} \frac{h(x)}{x^{\min(a,c)}} = \begin{cases} 1/(b-a) > 0 & a < c, \\ -1/(d-c) < 0 & a > c, \\ 1/(b-a) - 1/(d-c) & a = c \end{cases}$$

Note that in case $a = c$ the sign od h near 0 depends on the condition (23)
Case $\max(a,b,c,d) < 0$ is similar.
If $\min(a,b,c,d) < 0 < \max(a,b,c,d)$ then we see from (27) that

$$\lim_{x\to 0} h(x) = \frac{1}{2}\frac{|a|-|b|}{b-a} - \frac{1}{2}\frac{|c|-|d|}{d-c} = \frac{m(c,d) - m(a,b)}{2}$$

which completes the proof in case $m(a,b) \neq m(c,d)$.
Consider now the case $m(a,b) = m(c,d)$. Geometrically this means that the segments joining corresponding points on the graph of $-|x|$ are parallel. But this is possible only if $a, c < 0 < b, d$. If the segments are parallel to the x-axis then $a = -b$ and $c = -d$. In this case $h \equiv 0$.
If they are not horizontal then their midpoints lie on a line passing through the origin and we see that

$$a+b < c+d \quad \text{if and only if} \quad \min(|a|,|b|,|c|,|d|) \in \{|b|,|c|\} \tag{29}$$

$$a+b > c+d \quad \text{if and only if} \quad \min(|a|,|b|,|c|,|d|) \in \{|a|,|d|\}. \tag{30}$$

On the other hand

$$\lim_{x\to 0} \frac{h(x)}{x^{\min(|a|,|b|,|c|,|b|)}} = \begin{cases} -a/(b-a) > 0 & |a| = \min(|a|,|b|,|c|,|b|), \\ -b/(b-a) < 0 & |b| = \min(|a|,|b|,|c|,|b|), \\ c/(d-c) < 0 & |c| = \min(|a|,|b|,|c|,|b|), \\ d/(d-c) > 0 & |d| = \min(|a|,|b|,|c|,|b|) \end{cases} \tag{31}$$

Comparing (31, (29) and (30) we obtain required result. □

Lemma 5 ([5] Theorem 2)**.** *If f is an even function satisfying $f''(x) > 0, f'''(x) < 0$ and $(xf'''(x)/f''(x))' < 0$ for $x > 0$ and*

$$F_{a,b}(t) = \frac{f(at) - f(bt)}{a-b}$$

then the inequality $F_{a,b}(t) \le F_{c,d}(t)$ holds for all $t \in [-s,s]$ if and only if $a+b \le c+d$ and $F_{a,b}(s) \le F_{c.d}(s)$.

Now, equipped with the these tools we can prove the theorem.

Proof of Theorem 7. Due to homogeneity we can assume that $y = 1$, and because of symmetry $a \le b$ and $c \le d$. Assume also that $a \neq b$, $c \neq d$ and $abcd \neq 0$. By (11) the inequality $S(\alpha; a, b; x, 1) \le S(\alpha; c, d; x, 1)$ is eqivalent to

$$\frac{E(a,b;x^{\alpha+1},1)}{E(c,d;x^{\alpha+1},1)} \le \frac{E(a,b;x^{\alpha},1)}{E(c,d;x^{\alpha},1)}$$

Let

$$H(a,b;c,d;x) = \log \frac{E(a,b;x,1)}{E(c,d;x,1)}.$$

The comparison theorem reduces to the question when

$$H(a,b;c,d;x^{\alpha+1}) \le H(a,b;c,d;x^{\alpha}) \tag{32}$$

holds for all x.
Note that

$$H_x(a,b;c,d;x) = x^{-1} h(x;a,b;c,d) \tag{33}$$

where h is defined by (25).
We will show first that the the conditions (23) and (24) are necessary.
If (32) holds then by the mean value theorem $H(a,b;c,d;x^{\alpha+1}) - H(a,b;c,d;x^{\alpha}) = H_x(a,b;c,d;\xi)(x^{\alpha+1} - x^{\alpha})$. But this means that H_x takes positive values arbitrarily close to 0 and 1. By (33) and by Lemmas 3 and 4 this is impossible if (23) and (24) are not satisfied.

To proof sufficiency assume that (23) and (24) hold. Suppose we know already that the function k in (28) satisfies assumptions of Lemma 5. Then by Lemma 4 or $h \equiv 0$ in which case H is constant in x, or h is positive for small x, which means (using (28)) that

$$\frac{k(\frac{b\ln x}{2}) - k(\frac{a\ln x}{2})}{b-a} \leq \frac{k(\frac{d\ln x}{2}) - k(\frac{c\ln x}{2})}{d-c}$$

for small x. By Lemma 5 the above inequality holds for all $x \in (0,1)$. But this means that h is nonnegative, and by (33) $H(a,b;c,d;x)$ increases for $x < 1$. As $H(a,b;c,d;x) = H(a,b;c,d;x^{-1})$ it decreases for $x > 1$ hence (32) holds for all x. The case $a = b$, $c = d$ and $abcd = 0$ can be easily proved using continuity of H.

To complete the proof we have to show that $k(t) = t/\tanh t$ satisfies $k''(t) > 0, k'''(t) < 0$ and $(kf'''(t)/k''(t))' < 0$ for $t > 0$.

$$k''(t) = 2\frac{t\cosh t - \sinh t}{\sinh^3 t} > 0 \tag{34}$$

$$\begin{aligned} k'''(t) &= 2\frac{3\sinh t\cosh t - t(1+2\cosh^2 t)}{\sinh^4 t} \\ &= \frac{2}{\sinh^4 t}(3\sinh(2t) - 2t(2+\cosh(2t))) < 0 \end{aligned} \tag{35}$$

because $3\sinh(t) - t(2+\cosh(t)) = \sum_{k=2}^{\infty}(\frac{3}{2k+1} - 1)\frac{t^{2k+1}}{(2k)!} < 0$.
Now we have to show that $tk'''(t)/k''$ is decreasing

$$tk'''(t)/k''(t) = t\frac{3\sinh t\cosh t - t(1+2\cosh^2 t)}{t\cosh t\sinh t - \sinh^2 t} \tag{36}$$

$$= -2t\frac{4t + 2t\cosh(2t) - 3\sinh(2t)}{2t\sinh(2t) + 2 - 2\cosh(2t)} \equiv -2tu(2t). \tag{37}$$

Since $u(t)$ is positive by (34) and (35) it is enough to show that u is increasing.

$$u'(t) = \frac{\cosh^2 t - 8\cosh t + 7 + 6t\sinh t + t^2(1-2\cosh t)}{(t\sinh t + 2 - 2\cosh t)^2} \tag{38}$$

$$= \frac{1}{(t\sinh t + 2 - 2\cosh t)^2}\sum_{\substack{k=6 \\ k \text{ even}}}^{\infty}\frac{2(2^{k-2} - (k-2)^2)}{k!}t^k > 0 \tag{39}$$

□

Theorem 8. *For* $-\frac{1}{2} < \alpha < 0$ $S(\alpha;a,b) \leq S(\alpha;c,d)$ *for all* $x, y > 0$ *if and only if (23) and (24) hold.*

Proof. Follows immediately from (20) and the fact that (23) and (24) are invariant under multiplication by a positive constant. □

Finally from (19) we obtain

Corollary 3. *For* $\alpha < -\frac{1}{2}$ *,* $\alpha \neq 1$ $S(\alpha;a,b) \leq S(\alpha;c,d)$ *if and only if* $a+b \geq c+d$ *and* $m(a,b) \geq m(c,d)$ *hold.*

The following table summarises the results of this section

$S(\alpha;a,b) \le S(\alpha;c,d)$

$\alpha \in (-\frac{1}{2},0) \cup (0,\infty)$	$a+b \le c+d$ and $m(a,b) \le m(c,d)$
$\alpha = 0$	$a+b \le c+d$ and $e(a,b) \le e(c,d)$
$\alpha = -\frac{1}{2}$	$S(\alpha;a,b) = S(\alpha;c,d) = \sqrt{xy}$
$\alpha = -1$	$a+b \ge c+d$ and $e(a,b) \ge e(c,d)$
$\alpha \in (-\infty,-1) \cup (-1,-\frac{1}{2})$	$a+b \ge c+d$ and $m(a,b) \ge m(c,d)$

4. Other means

Observe that for $r \ne s$ S-means can be written as

$$S(\alpha;r,s;x,y) = \left(\frac{E(\alpha,\alpha+1;x^s,y^s)}{E(\alpha,\alpha+1;x^r,y^r)}\right)^{\frac{1}{s-r}} \tag{40}$$

and this encourages us to consider more general means defined by

$$R(u,v;r,s;x,y) = \left(\frac{E(u,v;x^s,y^s)}{E(u,v;x^r,y^r)}\right)^{\frac{1}{s-r}} \tag{41}$$

As E is continuously differentiable the definition of R can be extended for $r = s$. R contains some well known means like

$$R(1,n+1;0,1;x,y) = \left(\frac{x^n + x^{n-1}y + \cdots + xy^{n-1} + y^n}{n+1}\right)^{1/n}$$

and its two-parameter generalization

$$R(1,n+1;r,s;x,y) = \left(\frac{x^{ns} + x^{(n-1)s}y^s + \cdots + x^s y^{(n-1)s} + y^{ns}}{x^{nr} + x^{(n-1)r}y^r + \cdots + x^r y^{(n-1)r} + y^{nr}}\right)^{1/n(s-r)}.$$

Note that in general case

$$\begin{aligned} R(u,v;r,s;x,y) &= \left(\frac{E(u,v;x^s,y^s)}{E(u,v;x^r,y^r)}\right)^{\frac{1}{s-r}} \\ &= \left(\frac{x^{sv}-y^{sv}}{v}\frac{u}{x^{su}-y^{su}}\frac{v}{x^{rv}-y^{rv}}\frac{x^{ru}-y^{ru}}{u}\right)^{\frac{1}{v-u}\frac{1}{s-r}} \\ &= \left(\frac{x^{sv}-y^{sv}}{s}\frac{r}{x^{rv}-y^{rv}}\frac{x^{ru}-y^{ru}}{r}\frac{s}{x^{su}-y^{su}}\right)^{\frac{1}{s-r}\frac{1}{v-u}} \\ &= \left(\frac{E(r,s;x^v,y^v)}{E(r,s;x^u,y^u)}\right)^{\frac{1}{v-u}} = R(r,s;u,v,x,y) \end{aligned} \tag{42}$$

Using this identity we can express R in terms of S for $u \ne v$. In this case $u = (v-u)\frac{u}{v-u}$ and $v = (v-u)(\frac{u}{v-u}+1)$, so that from (42)

$$R(u,v;r,s;x,y) = \left(\frac{E(r,s;x^v,y^v)}{E(r,s;x^u,y^u)}\right)^{\frac{1}{v-u}} = S(\tfrac{u}{v-u};r,s;x^{v-u},y^{v-u})^{\frac{1}{v-u}} \tag{43}$$

This shows that R is indeed a mean and allows us to prove the

Theorem 9. *If $u+v>0$ then $R(u,v;a,b;x,y) \le R(u,v;c,d;x,y)$ if and only if*

$$a+b \le c+d$$

and

$$w(a,b) \le w(c,d),$$

where

$$w(r,s) == \begin{cases} e(r,s) & uv = 0, \\ m(r,s) & uv \neq 0. \end{cases} \tag{44}$$

If $u+v<0$ *then* $R(u,v;a,b;x,y) \leq R(u,v;c,d;x,y)$ *if and only if*

$$a+b \geq c+d$$

and

$$w(a,b) \geq w(c,d).$$

Proof. Let $u \neq v$. By (43)

$$R(u,v;a,b;x,y) \leq R(u,v;c,d;x,y)$$

is equivalent to

$$S(\tfrac{u}{v-u};a,b;x^{v-u},y^{v-u})^{\frac{1}{v-u}} \leq S(\tfrac{u}{v-u};c,d;x^{v-u},y^{v-u})^{\frac{1}{v-u}}$$

The assertion follows from the comparison theorem for S-means and Stolarsky means by noting that the sign of $\frac{u}{v-u} + \frac{1}{2} = \frac{u+v}{2(v-u)}$ is responsible for signs in the comparison criteria, and additional sign change may be caused by removal of exponents $\frac{1}{v-u}$. Note that $uv = 0$ if and only if $\frac{u}{v-u} = 0$ or -1, so in this case the e function must be used instead of m.
The case $u = v$ follows by continuity. □

Corollary 4. *R increases in r and s if $u+v>0$ and decreases otherwise.*

Proof. Follows from the previous theorem and the fact that the comparison criteria are preserved when parameters increase. □

The following corollary is an immediate consequence of (42)

Corollary 5. *R increases in u and v if $r+s>0$ and decreases otherwise.*

Next corollary is a simple consequence of Theorem 4 and Property 3.

Corollary 6. *The function $l(r) = \log E(u,v;x^r,y^r)$ is convex (resp. concave) if $u+v > (<$, resp. $)\ 0$.*

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