On Lehman's inequality and electrical networks

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Abstract. We obtain generalizations of Lehman's inequality, arising in electrical networks [6]. Connections with subadditivity and convexity are pointed out.

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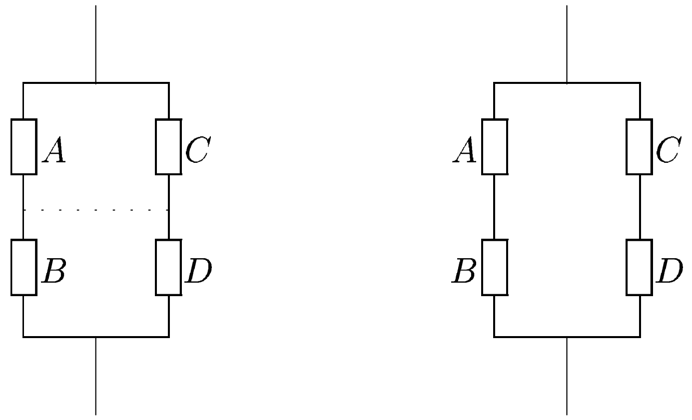
1 Introduction

Lehman's inequality (see [6], [2]) states that if A, B, C, D are positive numbers, then

$$\frac{(A+B)(C+D)}{A+B+C+D} \geq \frac{AC}{A+C} + \frac{BD}{B+D}. \tag{1}$$

This was discovered as follows: interpret A, B, C, D as resistances of an electrical network. It is well-known that if two resistances R_1 and R_2

are serially connected, then their compound resistance is $R = R_1 + R_2$, while in parallel connecting one has $1/R = 1/R_1 + 1/R_2$. Now consider two networks, as given in the following two figures:



$$R = \frac{(A+B)(C+D)}{A+B+C+D} \qquad\qquad R' = \frac{AC}{A+C} + \frac{BD}{B+D}$$

By Maxwell's principle, the current chooses a distribution such as to minimize the energy (or power), so clearly $R' \le R$, i.e. Lehman's inequality (1).

In fact, the above construction may be repeated with $2n$ resistances, in order to obtain:

Theorem 1. *If a_i, b_i $(i = \overline{1,n})$ are positive numbers, then*

$$\frac{(a_1 + \cdots + a_n)(b_1 + \cdots + b_n)}{a_1 + \cdots + a_n + b_1 + \cdots + b_n} \ge \frac{a_1 b_1}{a_1 + b_1} + \cdots + \frac{a_n b_n}{a_n + b_n} \qquad (2)$$

for any $n \ge 2$.

Remark. Since $\frac{2ab}{a+b} = H(a,b)$ is in fact the harmonic mean of two positive numbers, Lehman's inequality (2) can be written also as

$$H(a_1 + \cdots + a_n, b_1 + \cdots + b_n) \ge H(a_1, b_1) + \cdots + H(a_n, b_n) \qquad (3)$$

2 Two variable generalizations

In what follows, by using convexity methods, we shall extend (3) in various ways. First we introduce certain definitions. Let $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ be a function with two arguments, where A is a cone (e.g. $A = \mathbb{R}^2_+$). Let $k \in \mathbb{R}$ be a real number. Then we say that f is k**-homogeneous**, if

$$f(rx, ry) = r^k f(x, y) \tag{4}$$

for any $r > 0$ and $x, y \in A$. When $k = 1$, we simply say that f is **homogeneous**.

Let $F : I \subset \mathbb{R} \to \mathbb{R}$ be a function of an argument defined on an interval I. We say that F is k**-convex** (k-concave), if

$$F(\lambda a + \mu b) \underset{(\geq)}{\leq} \lambda^k F(a) + \mu^k f(b), \tag{5}$$

for any $a, b \in I$, and any $\lambda, \mu > 0$, $\lambda + \mu = 1$. We note, that if $k = 1$, then F will be called simply convex. For example, $F(t) = |t|^k$, $t \in \mathbb{R}$ is k-convex, for $k \geq 1$, since $|\lambda a+\mu b|^k \leq \lambda^k |a|^k + \mu^k |b|^k$ by $(u+v)^k \leq u^k + v^k$ $(u, v > 0)$, $k \geq 1$, which is well-known. On the other hand, the function $F(t) = |t|$, though is convex, is not 2-convex on $\mathbb{R}$.

The k-convex functions have been introduced for the first time by W. W. Breckner [4]. See also [5] for other examples and results. A similar convexity notion, when in (5) one replaces $\lambda + \mu = 1$ by $\lambda^k + \mu^k = 1$, was introduced by W. Orlicz [12] (see also [8] for these convexities).

Now, let $A = (0, +\infty) \times (0, +\infty) = \mathbb{R}^2_+$ and $I = (0, +\infty)$. Define $F(t) = f(1, t)$ for $t \in I$.

Theorem 2. *If f is k-homogeneous, and F is k-convex (k-concave) then*

$$f(a_1 + \cdots + a_n, b_1 + \cdots + b_n) \underset{(\geq)}{\leq} f(a_1, b_1) + \cdots + f(a_n, b_n) \tag{6}$$

for any $a_i, b_i \in A$ $(i = 1, 2, \dots, n)$.

Proof. First remark, that by (4) and the definition of F, one has

$$a^k F\left(\frac{b}{a}\right) = a^k f\left(1, \frac{b}{a}\right) = f(a, b) \tag{7}$$

On the other hand, by induction it can be proved the following Jensen-type inequality:

$$F(\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n) \underset{(\geq)}{\leq} \lambda_1^k F(x_1) + \lambda_2^k F(x_2) + \dots + \lambda_n^k F(x_n), \tag{8}$$

for any $x_i \in I$, $\lambda_i > 0$ $(i = \overline{1, n})$, $\lambda_1 + \dots + \lambda_n = 1$.

E.g. for $n = 3$, relation (8) can be proved as follows:

Put $a = \dfrac{\lambda_1}{\lambda_1 + \lambda_2} x_1 + \dfrac{\lambda_2}{\lambda_1 + \lambda_2} x_2$, $b = x_3$, $\lambda = \lambda_1 + \lambda_2$, $\mu = \lambda_3$ in (5). Then, as $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = \lambda a + \mu b$, we have

$$F(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3) \leq \lambda^k F(a) + \mu^k F(b) \leq$$

$$\leq (\lambda_1 + \lambda_2)^k \left[\frac{\lambda_1^k}{(\lambda_1 + \lambda_2)^k} F(x_1) + \frac{\lambda_2^k}{(\lambda_1 + \lambda_2)^k} F(x_2)\right] + \lambda_3^k F(x_3) =$$

$$= \lambda_1^k F(x_1) + \lambda_2^k F(x_2) + \lambda_3^k F(x_3).$$

Put now in (8)

$$x_1 = \frac{b_1}{a_1},\ x_2 = \frac{b_2}{a_2}, \dots, x_n = \frac{b_n}{a_n},$$

$$\lambda_1 = \frac{a_1}{a_1 + \dots + a_n},\ \lambda_2 = \frac{a_2}{a_1 + \dots + a_n}, \dots, \lambda_n = \frac{a_n}{a_1 + \dots + a_n}$$

in order to obtain

$$F\left(\frac{b_1 + \dots + b_n}{a_1 + \dots + a_n}\right) \underset{(\geq)}{\leq} \frac{a_1^k F\left(\dfrac{b_1}{a_1}\right) + a_2^k F\left(\dfrac{b_2}{a_2}\right) + \dots + a_n^k F\left(\dfrac{b_n}{a_n}\right)}{(a_1 + \dots + a_n)^k} \tag{9}$$

Now, by (7) this gives $f(a_1 + \dots + a_n, b_1 + \dots + b_n) \leq f(a_1, b_1) + \dots + f(a_n, b_n)$, i.e. relation (6).

Remark. Let $f(a,b) = \dfrac{a+b}{ab}$. Then f is homogeneous (i.e. $k=1$), and $F(t) = f(1,t) = \dfrac{t+1}{t}$ is 1-convex (i.e., convex), since $F''(t) = 2/t^3 > 0$. Then relation (6) gives the following inequality:

$$\frac{1}{H(a_1+\cdots+a_n, b_1+\cdots+b_n)} \le \frac{1}{H(a_1,b_1)} + \cdots + \frac{1}{H(a_n,b_n)}. \qquad (10)$$

Let now $f(a,b) = \dfrac{ab}{a+b}$. Then f is homogeneous, with $F(t) = \dfrac{t}{t+1}$, which is concave. From (6) (with $\geq$ inequality), we recapture Lehman's inequality (3).

The following theorem has a similar proof:

Theorem 3. *Let f be k-homogeneous, and suppose that F is l-convex (l-concave) ($k, l \in \mathbb{R}$). Then*

$$(a_1+\cdots+a_n)^{l-k} f(a_1+\cdots+a_n, b_1+\cdots+b_n) \le (\geq)$$
$$a_1^{l-k} f(a_1,b_1) + \cdots + a_n^{l-k} f(a_n,b_n). \qquad (11)$$

Remarks. For $k = l$, (11) gives (9).

For example, let $f(a,b) = \dfrac{a}{b}$, where $a, b \in (0,\infty)\times(0,\infty)$. Then $k = 0$ (i.e. f is homogeneous of order 0), and $F(t) = \dfrac{1}{t}$, which is 1-convex, since $F''(t) = \dfrac{2}{t^3} > 0$. Thus $l = 1$, and relation (11) gives the inequality

$$\frac{(a_1+\cdots+a_n)^2}{b_1+\cdots+b_n} \le \frac{a_1^2}{b_1} + \cdots + \frac{a_n^2}{b_n} \qquad (12)$$

Finally, we given another example of this type. Put $f(a,b) = \dfrac{a^2+b^2}{a+b}$. Then $k = 1$. Since $F(t) = \dfrac{t^2+1}{t+1}$, after elementary computations, $F''(t) = 4/(t+1)^3 > 0$, so $l = 1$, and (11) (or (9)) gives the relation

$$\frac{(a_1+\cdots+a_n)^2 + (b_1+\cdots+b_n)^2}{a_1+\cdots+a_n+b_1+\cdots+b_n} \le \frac{a_1^2+b_1^2}{a_1+b_1} + \cdots + \frac{a_n^2+b_n^2}{a_n+b_n} \qquad (13)$$

Since $L_1(a,b) = \dfrac{a^2+b^2}{a+b}$ (more generally, $L_p(a,b) = \dfrac{a^{p+1}+b^{p+1}}{a^p+b^p}$) are the so-called "Lehmer means" [9], [7], [1] of $a, b > 0$, (13) can be written also as

$$L_1(a_1+\cdots+a_n, b_1+\cdots+b_n) \le L_1(a_1,b_1)+\cdots+L_1(a_n,b_1). \qquad (14)$$

Clearly, one can obtain more general forms for L_p. For inequalities on more general means (e.g. Gini means), see [10], [11].

3 Hölder's inequality

As we have seen, there are many applications to Theorems 2 and 3. Here we wish to give an important application; namely a new proof of Hölder's inequality (one of the most important inequalities in Mathematics).

Let $f(a,b) = a^{1/p}b^{1/q}$, where $1/p+1/q = 1$ ($p > 1$). Then clearly f is homogeneous ($k = 1$), with $F(t) = t^{1/q}$. Since $F'(t) = \dfrac{1}{q}t^{-1/p}$, $F''(t) = -\dfrac{1}{pq}t^{-(1/p)-1} < 0$, so by Theorem 2 one gets

$$(a_1+\cdots+a_n)^{1/p}(b_1+\cdots+b_n)^{1/q} \ge a_1^{1/p}b_1^{1/q}+\cdots+a_n^{1/p}b_n^{1/q} \qquad (15)$$

Replace now $a_i = A_i^p$, $b_i = B_i^q$ ($i = \overline{1,n}$) in order to get

$$\sum_{i=1}^n A_iB_i \le \left(\sum_{i=1}^n A_i^p\right)^{1/p}\left(\sum_{i=1}^n B_i^q\right)^{1/q}, \qquad (16)$$

which is the classical Hölder inequality.

4 Many variables generalization

Let $f : A \subset \mathbb{R}_+^n \to \mathbb{R}$ be of n arguments ($n \ge 2$). For simplicity, put $p = (x_1,\ldots,x_n)$, $p' = (x_1',\ldots,x_n')$, when $p+p' = (x_1+x_1',\ldots,x_n+x_n')$

and $rp = (rx_1, \dots, rx_n)$ for $r \in \mathbb{R}$. Then the definitions of k-homogeneity and k-convexity can be extended to this case, similarly to paragraph 2. If A is a cone, then f is k-homogeneous, if $f(rp) = r^k f(p)$ $(r > 0)$ and if A is convex set then f is **k-convex**, if $f(\lambda p + \mu p') \leq \lambda^k f(p) + \mu^k f(p')$ for any $p, p' \in A$, $\lambda, \mu > 0$, $\lambda + \mu = 1$. We say that f is **k-Jensen convex**, if $f\left(\frac{p+p'}{2}\right) \leq \frac{f(p)+f(p')}{2^k}$. We say that f is **r-subhomogeneous of order** k, if $f(rp) \leq r^k f(p)$. Particularly, if $k = 1$ (i.e. $f(rp) \leq rf(p)$), we say that f is **r-subhomogeneous** (see e.g. [14], [15]). If f is r-subhomogeneous of order k for any $r > 1$, we say that f is **subhomogeneous of order** k. For $k = 1$, see [13]. We say that f is **subadditive** on A, if

$$f(p + p') \leq f(p) + f(p') \qquad (17)$$

We note that in the particular case of $n = 2$, inequality (6) with "$\leq$" says exactly that $f(a, b)$ of two arguments is subadditive.

Theorem 4. *If f is homogeneous of order k, then f is subadditive if and only if it is k-Jensen convex.*

Proof. If f is subadditive, i.e. $f(p+p') \leq f(p)+f(p')$ for any $p, p' \in A$, then

$$f\left(\frac{p+p'}{2}\right) = \frac{1}{2^k} f(p+p') \leq \frac{f(p)+f(p')}{2^k},$$

so f is k-Jensen convex. Reciprocally, if f is k-Jensen convex, then $f\left(\frac{p+p'}{2}\right) \leq \frac{f(p)+f(p')}{2^k}$, so

$$f(p+p') = f\left[2\left(\frac{p+p'}{2}\right)\right] = 2^k f\left(\frac{p+p'}{2}\right) \leq f(p) + f(p'),$$

i.e. (17) follows.

Remark. Particularly, a homogeneous subadditive function is convex, a simple, but very useful result in the theory of convex bodies (e.g. "distance function", "supporting function", see e.g. [3], [16]).

Theorem 5. *If f is 2-subhomogeneous of order k, and is k-Jensen convex, then it is subadditive.*

Proof. Since $f(p+p') = f\left(2\left(\frac{p+p'}{2}\right)\right) \le 2^k f\left(\frac{p+p'}{2}\right)$, and $f\left(\frac{p+p'}{2}\right) \le \frac{f(p)+f(p')}{2^k}$, we get $f(p+p') \le f(p)+f(p')$, so (17) follows.

Remark. Particularly, if f is 2-subhomogeneous, and Jensen convex, then it is subadditive. (18)

It is well-known that a continuous Jensen convex function (defined on an open convex set $A \subset \mathbb{R}^n$) is convex. Similarly, for continuous k-Jensen convex functions, see [4].

To give an interesting example, connected with Lehman's inequality, let us consider $A = \mathbb{R}_+^n$, $f(p) = H(p) = n/\left(\frac{1}{x_1}+\cdots+\frac{1}{x_n}\right)$.
Let $\frac{1}{g(p)} = \frac{1}{x_1}+\cdots+\frac{1}{x_n}$. Then

$$\frac{dg}{g^2} = \sum_{i=1}^n \frac{dx_i}{x_i^2}, \quad \frac{d^2g}{g^2} - 2\frac{dg^2}{g^3} = -2\sum_{i=1}^n \frac{dx_i^2}{x_i^3},$$

so

$$\frac{1}{2}\frac{d^2g}{g^3} = \left(\sum_{i=1}^n \frac{dx_i}{x_i^2}\right)^2 - \left(\sum_{i=1}^n \frac{1}{x_i}\right)\left(\sum_{i=1}^n \frac{dx_i^2}{x_i^3}\right).$$

(Here d denotes a differential.) Now apply Hölder's inequality (16) for $p = q = 2$ (i.e. Cauchy-Bunjakovski inequality), $A_i = 1/\sqrt{x_i}$, $B_i = (1/x_i\sqrt{x_i})dx_i$. Then one obtains $\frac{d^2g}{g^3} \le 0$, and since $g > 0$, we get $d^2g \le 0$. It is well-known ([16]) that this implies the concavity of function $g(p) = H(p)/n$, so $-H(p)$ will be a convex function. By consequence (17) of Theorem 5, $H(p)$ is subadditive, i.e.

$$H(x_1+x_1', x_2+x_2', \ldots, x_n+x_n') \ge H(x_1, x_2, \ldots, x_n) + \\ + H(x_1', x_2', \ldots, x_n'), \quad (x_i, x_i' > 0). \tag{19}$$

For $n = 2$ this coincides with (3), i.e. Lehman's inequality (1).

Finally, we prove a result, which is a sort of reciprocal to Theorem 5:

Theorem 6. *Let us suppose that f is subadditive, and k-convex, where $k \geq 1$. Then f is subhomogeneous of order k.*

Proof. For any $r > 1$ one can find a positive integer n such that $r \in [n, n+1]$. Then r can be written as a convex combination of n and $n+1$: $r = n\lambda + (n+1)\mu$. By the k-convexity of f one has

$$f(rp) = f(n\lambda p + (n+1)\mu p) \leq \lambda^k f(np) + \mu^k f[(n+1)p].$$

Since f is subadditive, from (17) it follows by induction that $f(np) \leq nf(p)$, so we get

$$f(rp) \leq n\lambda^k f(p) + (n+1)\mu^k f(p) = [n\lambda^k + (n+1)\mu^k]f(p).$$

Now, since $k \geq 1$, it is well-known that

$$[\lambda n + (n+1)\mu]^k \geq (\lambda n)^k + ((n+1)\mu)^k.$$

But $(\lambda n)^k \geq n\lambda^k$ and $((n+1)\mu)^k \geq (n+1)\mu^k$, so finally we can write

$$f(rp) \leq [\lambda n + (n+1)\mu]^k f(p) = r^k f(p),$$

which means that f is subhomogeneous of order k.

Remark. For $k = 1$ Theorem 6 contains a result by R. A. Rosenbaum [13].

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NOTE ADDED IN PROOF. Recently (23th February, 2005) we have discovered that Lehman's inequality (2) (or (3)) appears also as Theorem 67 in G. H. Hardy, J. E. Littlewood and G. Polya [Inequalities, Cambridge Univ. Press, 1964; see p.61], and is due to E. A. Milne [Note on Rosseland's integral for the stellar absorption coefficient, Monthly Notices, R.A.S. 85(1925), 979-984]. Though we are unable to read Milne's paper, perhaps we should call Lehman's inequality as the "Milne-Lehman inequality". We note also that the Milne-Lehman inequality is published as a Proposed Problem 2113 (by M.E. Kuczma), as well as Problem 2392 (by G. Tsintsifas) in the Canadian journal Crux Mathematicorum.

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