AN KLAMKIN TYPE INEQUALITY ON THE TRIANGLE

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Abstract. In this short note, we give a new Klamkin type inequality on the triangle: for any point P inside the triangle ABC, then

$$(x+y+z)\left(xK_1^2+yK_2^2+zK_3^2\right)\geqslant a^2yz+b^2zx+c^2xy$$

where x,y,z are three real numbers, a,b,c the sides of triangle ABC and $K_1=XD, K_2=YE, K_3=ZF$ if X,Y,Z and D,E,F are the midpoints of PA,PB,PC and BC,CA,AB respectively.

1. Introduction

Throughout the paper we assume A,B,C the angles of triangle ABC, a,b,c the sides, s the semi-perimeter, R_1,R_2,R_3 the distances from P to A,B,C, and $K_1=XD, K_2=YE, K_3=ZF$ if X,Y,Z and D,E,F are the midpoints of PA,PB,PC and BC,CA,AB respectively. Moreover, we will customarily use the cyclic sum symbol and cyclic product symbol, that is: $\sum f(a)=f(a)+f(b)+f(c)$, $\sum f(a,b)=f(a,b)+f(b,c)+f(c,a)$ and $\prod f(a)=f(a)f(b)f(c)$, similarly, one defines others.

The following inequality (1.1) is well-known and it was obtained by M. S. Klamkin [1] in 1975:

Theorem 1.1. *If $x,y,z\in R$, then*

$$(x+y+z)\left(xR_1^2+yR_2^2+zR_3^2\right)\geqslant a^2yz+b^2zx+c^2xy. \tag{1.1}$$

The equality in (1.1) holds if and only if the barycentric coordinates of point P is (x,y,z).

In this short note, we give a new Klamkin type inequality on the triangle. In the final, an open problem is posed.

2. Main Result

In order to prove Theorem2.1 below, we require the following lemma.

Lemma 2.1. *For any point P inside triangle ABC, we have*

$$4K_1^2=b^2+c^2-a^2+R_2^2+R_3^2-R_1^2 \tag{2.1}$$

Proof. As the following fig.1, we have

$$2AB^2+2BP^2-AP^2=4BX^2, \tag{2.2}$$

$$2AC^2+2CP^2-AP^2=4CX^2, \tag{2.3}$$

$$2AB^2+2AC^2-BC^2=4AD^2, \tag{2.4}$$

$$2PB^2+2CP^2-BC^2=4PD^2, \tag{2.5}$$

$$2BX^2+2CX^2-BC^2=4XD^2, \tag{2.6}$$

$$2AD^2+2PD^2-AP^2=4XD^2. \tag{2.7}$$

From (2.2)-(2.7), we immediately obtain (2.1). This is proved. □

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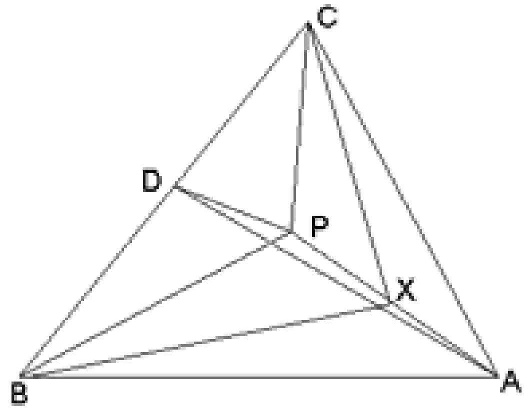

[fig.1.]

Theorem 2.1. *If $x, y, z \in \mathbb{R}$, for any point P inside triangle ABC, then*

$$(x+y+z)\left(xK_1^2+yK_2^2+zK_3^2\right) \geqslant a^2yz+b^2zx+c^2xy. \tag{2.8}$$

The equality in (2.8) holds if and only if the barycentric coordinates of point P is $(y+z-x, z+x-y, x+y-z)$.

Proof. Alter $x \to y+z-x, y \to z+x-y, z \to x+y-z$, then inequality (1.1) is equivalent to

$$\begin{aligned}
&\sum(y+z-x)\cdot\sum(y+z-x)R_1^2 \geqslant \sum a^2(z+x-y)(x+y-z)\\
\Leftrightarrow &\sum x\cdot\sum(y+z-x)R_1^2 \geqslant \sum a^2\left[-(x+y+z)(y+z-x)+4yz\right]\\
\Leftrightarrow &\sum x\cdot\sum(y+z-x)\left(R_1^2+a^2\right) \geqslant 4\sum a^2yz\\
\Leftrightarrow &\sum x\cdot\sum x\left(R_2^2+R_3^2-R_1^2+b^2+c^2-a^2\right) \geqslant 4\sum a^2yz
\end{aligned}$$

From (2.1) and another two formulas for K_2, K_3, combining last inequality above, we get immediately inequality (2.8). The proof of Theorem 2.1 is completed. □

Using Theorem 2.1, we can give the following corollary.

Corollary 2.1. *For any point P inside triangle ABC, we have*

$$K_1^2+K_2^2+K_3^2 \geqslant \frac{1}{3}\left(a^2+b^2+c^2\right), \tag{2.9}$$

$$aK_1^2+bK_2^2+cK_3^2 \geqslant abc, \tag{2.10}$$

$$K_1^2\cos^2\frac{A}{2}+K_2^2\cos^2\frac{B}{2}+K_3^2\cos^2\frac{C}{2} \geqslant s^2\cdot\frac{\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}}{\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}}. \tag{2.11}$$

The equalities in (2.9), (2.10) and (2.11) hold if and only if the barycentric coordinates of point P are respectively $(1,1,1), (b+c-a, c+a-b, a+b-c)$ and $\left(\frac{1}{b+c-a}, \frac{1}{c+a-b}, \frac{1}{a+b-c}\right)$.

3. An Identity Involving K_1, K_2 and K_3

From Lemma 2.1, we easy obtain:

$$4\left(K_1^2+K_2^2+K_3^2\right) = a^2+b^2+c^2+R_1^2+R_2^2+R_3^2 \tag{3.1}$$

Now, we give another identity involving K_1, K_2 and K_3.

Theorem 3.1. *For any point P inside triangle ABC, we have*

$$a^2b^2c^2 + a^2R_2^2R_3^2 + b^2R_3^2R_1^2 + c^2R_1^2R_2^2 = 4\left(a^2R_1^2K_1^2 + b^2R_2^2K_2^2 + c^2R_3^2K_3^2\right) \tag{3.2}$$

Proof. Utilizing the facts that

$$\begin{aligned}&\sum a^2(R_3+R_1)^2(R_1+R_2)^2 - 2\prod(R_2+R_3)\sum a^2R_1\\&= \sum a^2R_1^4 + \sum a^2R_2^2R_3^2 - \sum a^2R_1^2\left(R_2^2+R_3^2\right),\end{aligned}$$

$$\sum b^2c^2(R_2+R_3)^2 - 2\sum b^2c^2R_2R_3 = \sum b^2c^2\left(R_2^2+R_3^2\right) = \sum a^2\left(b^2+c^2\right)R_1^2,$$

and

$$a^2R_1 + b^2R_2 + c^2R_3 - (R_2+R_3)(R_3+R_1)(R_1+R_2) = \prod\sqrt{(R_2+R_3)^2 - a^2},$$

we find

$$\begin{aligned}&\left(\sum a^2R_1\right)^2 + \prod(R_2+R_3)^2 - 2\sum a^2R_1\cdot\prod(R_2+R_3) = \prod\left[(R_2+R_3)^2 - a^2\right]^2,\\&\Leftrightarrow \sum a^4R_1^2 + 2\sum b^2c^2R_2R_3 - 2\sum a^2R_1\cdot\prod(R_2+R_3)\\&\quad= \sum b^2c^2(R_2+R_3)^2 - a^2b^2c^2 - \sum a^2(R_3+R_1)^2(R_1+R_2)^2,\\&\Leftrightarrow a^2b^2c^2 + \sum a^4R_1^2 + \sum a^2R_1^4 + \sum a^2R_2^2R_3^2 - \sum a^2R_1^2\left(R_2^2+R_3^2\right) - \sum a^2\left(b^2+c^2\right)R_1^2 = 0,\\&\Leftrightarrow a^2b^2c^2 + \sum a^2R_2^2R_3^2 = \sum a^2R_1^2\left(b^2+c^2-a^2+R_2^2+R_3^2-R_1^2\right).\end{aligned}$$

From (3.1), the last identity above is (3.2). Theorem 3.1 is proved. □

4. An Open Problem

In the final, we conjecture that: For any point P inside triangle ABC, the following inequalities hold

$$aR_1K_1 + bR_2K_2 + cR_3K_3 \geqslant abc, \tag{4.1}$$

$$aK_2K_3 + bK_3K_1 + cK_1K_2 \geqslant abc, \tag{4.2}$$

$$K_1 + K_2 + K_3 + r_1 + r_2 + r_3 \geqslant h_a + h_b + h_c, \tag{4.3}$$

where h_a, h_b, h_c the altitudes from A, B, C of triangle ABC and r_1, r_2, r_3 the distances from P to BC, CA, AB on any point P inside the triangle ABC.

References

[1] M.S.Klamkin *Geometric Inequalities via the Polar Moment of Inertia.* Math. Mag. 48(1975), 44-46.

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