THE BEST CONSTANT FOR A GEOMETRIC INEQUALITY

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Abstract. In this paper, we prove that the best constant for the geometric inequality $\frac{11\sqrt{3}}{5R+12r+k(2r-R)} \le \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ is a root of one polynomial by the method of mathematical analysis and linear algebra.

1. Introduction and Main Results

In 1993, Shi-Chang Shi strengthened the familiar geometric inequality(in triangle)

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \le \frac{\sqrt{3}}{2r} \tag{1.1}$$

to be

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \le \frac{1}{\sqrt{3}}\left(\frac{1}{r}+\frac{1}{R}\right) \tag{1.2}$$

in [1]. After several months, Ji Chen obtained the following beautiful and strong inequality chain in [2].

$$\frac{11\sqrt{3}}{5R+12r} \le \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \le \frac{1}{\sqrt{3}}\left(\frac{5}{4R}+\frac{7}{8r}\right) \tag{1.3}$$

And in the same year, Xi-Ling Huang posed the following interesting inequality problem in [3].

Problem 1.1. *Determine the best constant k for the inequality that makes it holding.*

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \le \frac{1}{\sqrt{3}}\left[\frac{1}{R}+\frac{1}{r}+\frac{1}{k}\left(\frac{2}{R}-\frac{1}{r}\right)\right] \tag{1.4}$$

In 1996, Sheng-Li Chen solved Problem 1.1 completely in [4]. He obtained the following theorem.

Theorem 1.1. *The best constant k for the inequality* (1.2) *is* $2(1+\sqrt[3]{2}+\sqrt[3]{4})$.

In the same year, Xue-Zhi Yang strengthened the inequality

$$\frac{11\sqrt{3}}{5R+12r} \le \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \tag{1.5}$$

to be

$$\frac{243\sqrt{3}}{110R+266r} \le \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \tag{1.6}$$

in [5]. In this paper we will determine the best constant for the next inequality

$$\frac{11\sqrt{3}}{5R+12r+k(2r-R)} \le \frac{1}{a}+\frac{1}{b}+\frac{1}{c}, \tag{1.7}$$

where $0 < k < 5$. And we obtain the following theorem.

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Theorem 1.2. *The maximum of k which makes the inequality* (1.7) *holding is the root on open interval* $(0, \frac{1}{15})$ *of the following equation*

$$405k^5 + 6705k^4 + 129586k^3 + 1050976k^2 + 2795373k - 62181 = 0.$$

And its approximation is 0.02206078402*.*

In fact, let $k = \frac{5}{243} \approx 0.020576131687 < 0.02206078402$, we immediately find that the inequality (1.7) is just the inequality (1.6).

2. Lemmas

In order to prove Theorem 1.1, we require several lemmas. The second was got by Sheng-Li Chen in [6](see also [7]).

Lemma 2.1. *If* $0 < k < 5$*, then the inequality*

$$\frac{11\sqrt{3}}{5R + 12r + k(2r - R)} \le \frac{1}{\sqrt{3}} \left(\frac{5}{4R} + \frac{7}{8r} \right) \tag{2.1}$$

holds if and only if $0 < k \le \frac{5}{12}$*.*

Proof. Owing to $0 < k < 5$, it's obvious that $5R + 12r + k(2r - R) > 0$. So the inequality (2.1) is equivalent to

$$7(5 - k)R^2 + (4k - 130)Rr + (20k + 120)r^2 \ge 0. \tag{2.2}$$

Setting $\frac{R}{2r} = x$, then with Euler's Inequality $R \ge 2r$, we have $x \ge 1$. And the inequality (2.2) is equivalent to

$$28(5 - k)x^2 + 2(4k - 130)x + 20k + 120 \ge 0.$$

That is

$$4(x - 1)[(35 - 7k)x - 5k - 30] \ge 0. \tag{2.3}$$

Considering that $x \ge 1$, the inequality (2.3) holds if and only if $(35 - 7k)x - 5k - 30 \ge 0 (x \ge 1)$. Namely, $k \le \frac{5(7x-6)}{7x+5}$ or $k \le \min \frac{5(7x-6)}{7x+5} (x \ge 1)$.
Define the function

$$f(x) = \frac{5(7x - 6)}{7x + 5} (x \ge 1).$$

Calculating the derivative for $f(x)$, we get

$$f'(x) = -\frac{35(7x - 6)}{(7x + 5)^2} + \frac{35}{7x + 5} = \frac{385}{(7x + 5)^2} > 0.$$

So the function $f(x)$ is strictly monotone increasing on interval $[1, +\infty)$. Then $f(x) \ge f(1) = \frac{5}{12}$. That is minf(x)=1 for $x \ge 1$. So $k \le \frac{5}{12}$, combining $0 < k < 5$, we immediately obtain $0 < k \le \frac{5}{12}$. Thus, Lemma 2.1 was proved. ∎

Lemma 2.2. [6] *The same exponential inequality in triangle which form like* $p \ge (>) f(R, r)$ *holds if and only if it holds by setting* $R = 2$*,* $r = 1 - x^2$*,* $p = \sqrt{(1 - x)(3 + x)^3}$*, where* $0 \le x < 1$*. And the form like* $p \le (<) f(R, r)$ *holds if and only if it holds by setting the same substitution, where* $-1 < x \le 0$*.*

Proof. It's well known that the following two inequalities

$$p^2 \ge 2R^2 + 10Rr - r^2 - 2(R - 2r)\sqrt{r(R - 2r)} \tag{2.4}$$

$$p^2 \le 2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{r(R - 2r)} \tag{2.5}$$

hold in any triangle ABC.
Now we prove the inequality (2.4) with equality holding if and only if ΔABC is an isosceles triangle which top angle is more than or equal to 60°, and the inequality (2.5) with equality holding if and only if ΔABC is an isosceles triangle which top angle is less than or equal to 60°.
Let A be the top angle of isosceles triangle ABC, and let

$$t = \sin\frac{A}{2}(= \cos B = \cos C) \in (0, 1),$$

then

$$\sin\frac{B}{2} = \sin\frac{C}{2} = \sqrt{\frac{1-t}{2}}.$$

With known identities

$$r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}, p = R(\sin A + \sin B + \sin C).$$

in triangle. We easily get

$$r = 2Rt(1-t), p = 2R(1+t)\sqrt{1-t^2}. \tag{2.6}$$

We put the identities (2.6) into the inequality (2.4) and (2.5), with simple calculation, we can find the inequality (2.4) with equality holding if and only if $t \in [\frac{1}{2}, 1)$ or $A \geq 60^\circ$; the inequality (2.5) with equality holding if and only if $t \in (0, \frac{1}{2}]$ or $A \leq 60^\circ$.
Then we prove the following two propositions.

Proposition 2.1. *For every triangle ABC,there are isosceles triangle $A_1B_1C_1$ with top angle $A_1 \geq 60^\circ$ and isosceles triangle $A_2B_2C_2$ with top angle $A_1 \leq 60^\circ$ make*

$$R_1 = R_2 = R, r_1 = r_2 = r; p_1 \leq p \leq p_2.$$

And $p = p_1$ holding if and only if ΔABC is an isosceles triangle with top angle $A \geq 60^\circ$, $p = p_2$ holding if and only if ΔABC is an isosceles triangle with top angle $A \leq 60^\circ$.

Proof. Denote $\bigodot O$ be the circumcircle of ΔABC, then there are inscribed isosceles triangles $A_1A_2A_3$ and $A_2B_2C_2$ of $\bigodot O$ which satisfied the next two identities:

$$\frac{A_1}{2} = \arcsin\frac{1}{2}\left(1 + \sqrt{1 - \frac{2r}{R}}\right),$$

$$\frac{A_2}{2} = \arcsin\frac{1}{2}\left(1 - \sqrt{1 - \frac{2r}{R}}\right).$$

Then $A_1 \geq 60^\circ$, $A_2 \leq 60^\circ$ and

$$\sin\frac{A_1}{2}(1 - \sin\frac{A_1}{2}) = \frac{r}{2R}, \tag{2.7}$$

$$\sin\frac{A_2}{2}(1 - \sin\frac{A_2}{2}) = \frac{r}{2R}. \tag{2.8}$$

For isosceles triangles $A_1A_2A_3$ with top angle is A_1 and $A_2B_2C_2$ with top angle is A_2, we have

$$\sin\frac{A_1}{2}(1 - \sin\frac{A_1}{2}) = \frac{r_1}{2R_1}, \tag{2.9}$$

$$\sin\frac{A_2}{2}(1 - \sin\frac{A_2}{2}) = \frac{r_2}{2R_2}. \tag{2.10}$$

From (2.7) to (2.10), we get $\frac{r}{R} = \frac{r_1}{R_1} = \frac{r_2}{R_2}$. And it's easy to see that $R = R_1 = R_2$, so $r_1 = r_2 = r$. Denote $\varphi(R,r)$ be the right of (2.4), then $p^2 \geq \varphi(R,r) = \varphi(R_1,r_1) = p_1^2$, so $p \leq p_1$. With the same way we can prove $p \leq p_2$. ∎

Proposition 2.2. *(i)If the inequality $p \geq (>) f_1(R,r)$ holds for any isosceles triangle whose top angle is more than or equal to $60°$, then the inequality $p \geq (>) f_1(R,r)$ holds for any triangle.*
(ii)If the inequality $p \leq (<) f_1(R,r)$ holds for any isosceles triangle whose top angle is less than or equal to $60°$, then the inequality $p \leq (<) f_1(R,r)$ holds for any triangle.

Proof. For any $\Delta A'B'C'$, with Proposition 2.1, we know there is an isosceles triangle $A_1B_1C_1$ which make

$$R_1 = R', r_1 = r', p_1 \leq p'.$$

Because the inequality $p \geq (>) f_1(R,r)$ holds for isosceles triangle $A_1B_1C_1$, we have

$$p' \geq p_1 \geq (>) f_1(R_1,r_1) = f_1(R',r').$$

Thus, the inequality $p \geq (>) f_1(R,r)$ holds for $\Delta A'B'C'$. In the same way we can prove (ii). ∎

From Proposition 2.2, the same exponential inequality in triangle which form like $p \geq (>) f(R,r)$ holds if and only if it holds by setting $R = 2$, $r = 4t(1-t)$, $p = 4(1+t)\sqrt{1-t^2}$. And taking $t = \frac{x+1}{2}$, we immediately get $r = 1 - x^2$, $p = \sqrt{(1-x)(3+x)^3}$, where $0 \leq x < 1$. For the same exponential inequality in triangle which form like $p \leq (<) f(R,r)$, we only need to change the range of x. Namely, we change $0 \leq x < 1$ to be $-1 < x \leq 0$.
Thus, the proof of Lemma 2.2 is completed.(The proof was given by Sheng-Li Chen in [6].) ∎

Lemma 2.3. [8] *Denote*

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n,$$
$$g(x) = b_0x^m + b_1x^{m-1} + \cdots + b_m.$$

If $a_0 \neq 0$ or $b_0 \neq 0$, then the polynomials $f(x)$ and $g(x)$ have a common root if and only if

$$R(f,g) = \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_n & 0 & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & a_{n-1} & a_n & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_0 & \cdots & \cdots & \cdots & a_n \\ b_0 & b_1 & b_2 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & b_0 & b_1 & \cdots & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & b_0 & b_1 & \cdots & b_m \end{vmatrix} = 0,$$

where $R(f,g)$ is Sylvester's Resultant of $f(x)$ and $g(x)$.

3. Proof of Theorem 1.1

Proof. With known identities $abc = 4Rrp$ and $ab + bc + ca = p^2 + 4Rr + r^2$ in triangle, we easily know the inequality (1.7) is equivalent to

$$\frac{11\sqrt{3}}{5R + 12r + k(2r - R)} \leq \frac{p^2 + 4Rr + r^2}{4Rrp}. \tag{3.1}$$

And the inequality (3.1) is equivalent to the following inequality

$$[5R + 12r + k(2r - R)]p^2 - 44\sqrt{3}Rrp + [5R + 12r + k(2r - R)](4Rr + r^2) \geq 0. \tag{3.2}$$

(i) If $\Delta(R,r) = (44\sqrt{3}Rr)^2 - 4[5R+12r+k(2r-R)]^2(4Rr+r^2) < 0$, it's obvious that the inequality (3.2) holds.

(ii) If $\Delta(R,r) = (44\sqrt{3}Rr)^2 - 4[5R+12r+k(2r-R)]^2(4Rr+r^2) \geq 0$, then the inequality (3.2) is equivalent to

$$p \geq \frac{44\sqrt{3}Rr + \sqrt{\Delta(R,r)}}{2[5R+12r+k(2r-R)]} \tag{3.3}$$

or

$$p \leq \frac{44\sqrt{3}Rr - \sqrt{\Delta(R,r)}}{2[5R+12r+k(2r-R)]}. \tag{3.4}$$

In fact, the inequality (3.4) does not hold. From (1.3) and (1.7), we have

$$\frac{11\sqrt{3}}{5R+12r+k(2r-R)} \leq \frac{1}{\sqrt{3}}\left(\frac{5}{4R}+\frac{7}{8r}\right) \tag{3.5}$$

And with Lemma 2.1, we know $0 < k \leq \frac{5}{12}$. it's easy to see that the following inequalities holds.

$$\frac{44\sqrt{3}Rr - \sqrt{\Delta(R,r)}}{2[5R+12r+k(2r-R)]} \leq \frac{44\sqrt{3}Rr}{2[5R+12r+k(2r-R)]} \leq \frac{22\sqrt{3}Rr}{[5R+12r+\frac{5}{12}(2r-R)]} \tag{3.6}$$

Now we prove the next inequality

$$p \geq \frac{22\sqrt{3}Rr}{[5R+12r+\frac{5}{12}(2r-R)]}. \tag{3.7}$$

The inequality (3.7) is equivalent to

$$p^2 \geq \frac{1452R^2r^2}{[5R+12r+\frac{5}{12}(2r-R)]^2} \tag{3.8}$$

With Gerretsen's Inequality $p^2 \geq 16Rr - 5r^2$, in order to prove the inequality (3.8), we only need to prove the following inequality.

$$16Rr - 5r^2 \geq \frac{1452R^2r^2}{[5R+12r+\frac{5}{12}(2r-R)]^2} \tag{3.9}$$

The inequality (3.9) is equivalent to

$$r(400R^3 + 387R^2 + 2436Rr - 980r^3) \geq 0. \tag{3.10}$$

With Euler's Inequality $R \geq 2r$, we easily see that the inequality (3.10) holds. So, the inequality (3.7) holds. Then the inequality (3.4) does not hold. Therefore, the inequality (3.2) is equivalent to the inequality (3.3). From Lemma 2.2, the inequality (3.2) holds if and only if the following inequality holds.

$$8(1-x)(3+x)[(2x+3)(11-6x^2-kx^2) - 11\sqrt{3}(x+1)\sqrt{(1-x)(3+x)}] \geq 0 (0 \leq x < 1). \tag{3.11}$$

The inequality (3.11) holds when $x = 0$. When $0 < x < 1$, the inequality (3.11)is equivalent to

$$k \leq \frac{(2x+3)(11-6x^2) - 11\sqrt{3}(x+1)\sqrt{(1-x)(3+x)}}{x^2(2x+3)}. \tag{3.12}$$

Define the function

$$g(x) = \frac{(2x+3)(11-6x^2) - 11\sqrt{3}(x+1)\sqrt{(1-x)(3+x)}}{x^2(2x+3)} (0 < x < 1).$$

Calculating the derivative for $g(x)$, we get

$$g'(x) = \frac{-22[\sqrt{3}(x^4+5x^3+2x^2-9x-9) + (2x+3)^2\sqrt{(1-x)(3+x)}]}{x^3(2x+3)^2\sqrt{(1-x)(3+x)}}.$$

Let $g'(x) = 0$, we get

$$3x^5 + 30x^4 + 103x^3 + 134x^2 + 48x - 18 = 0 (0 < x < 1). \tag{3.13}$$

And it's easy to see that the equation (3.13) have the only one positive root on open interval $(0, 1)$. Denote x_0 be the root of the equation (3.13). Then

$$g(x)_{min} = g(x_0) = \frac{(2x_0+3)(11-6x_0^2) - 11\sqrt{3}(x_0+1)\sqrt{(1-x_0)(3+x_0)}}{x_0^2(2x_0+3)}.$$

Therefore, the maximum of k is $g(x_0)$. Now we prove $g(x_0)$ is the root of the equation

$$405k^5 + 6705k^4 + 129586k^3 + 1050976k^2 + 2795373k - 62181 = 0.$$

It's easy to find that $g(x_0)$ is a root of the following equation.

$$x_0^2(2x_0+3)^2t^2 - 2(2x_0+3)^2(11-6x_0^2)t + 144x_0^4 + 432x_0^3 + 159x_0^2 - 132x_0 + 22 = 0$$

And we know that

$$3x_0^5 + 30x_0^4 + 103x_0^3 + 134x_0^2 + 48x_0 - 18 = 0.$$

Considering the simultaneous equations

$$\begin{cases} x_0^2(2x_0+3)^2t^2 - 2(2x_0+3)^2(11-6x_0^2)t + 144x_0^4 + 432x_0^3 + 159x_0^2 - 132x_0 + 22 = 0 \\ 3x_0^5 + 30x_0^4 + 103x_0^3 + 134x_0^2 + 48x_0 - 18 = 0 \end{cases} \tag{3.14}$$

The simultaneous equations (3.14) can be changed to the simultaneous equations as follows.

$$\begin{cases} 4(t+6)^2x_0^4 + 12(t+6)^2x_0^3 + (9t^2+20t+159)x_0^2 - 132(2t+1)x_0 - 198t + 22 = 0 \\ 3x_0^5 + 30x_0^4 + 103x_0^3 + 134x_0^2 + 48x_0 - 18 = 0 \end{cases} \tag{3.15}$$

Then,

$$R_{x_0}(f,g) = \begin{vmatrix} 4(t+6)^2 & 12(t+6)^2 & \cdots & 22-198t & 0 & \cdots & 0 \\ 0 & 4(t+6)^2 & \cdots & \cdots & 22-198t & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 4(t+6)^2 & \cdots & \cdots & 22-198t \\ 3 & 30 & \cdots & -18 & 0 & 0 & 0 \\ 0 & 3 & 30 & \cdots & -18 & 0 & 0 \\ 0 & 0 & 3 & 30 & \cdots & -18 & 0 \\ 0 & 0 & 0 & 3 & 30 & \cdots & -18 \end{vmatrix}$$

$$= 100(405t^5 + 6705t^4 + 129586t^3 + 1050976t^2 + 2795373t - 62181)(405t^5 \\ -178425t^4 - 1656374t^3 - 13317290t^2 - 100675599t - 330639021)$$

The solution of the equation $R_{x_0}(f,g) = 0$ is the union of the solution of the equation

$$405t^5 + 6705t^4 + 129586t^3 + 1050976t^2 + 2795373t - 62181 = 0, \tag{3.16}$$

and the equation

$$405t^5 - 178425t^4 - 1656374t^3 - 13317290t^2 - 100675599t - 330639021 = 0. \tag{3.17}$$

With differential calculus, it's easy to see that the equation (3.17) has no root on interval $[0, 1]$. And we can get $g(x_0) < 1$, with Lemma 2.3, we can conclude that $g(x_0)$ is the real root of the equation (3.16). Define the function

$$f(t) = 405t^5 + 6705t^4 + 129586t^3 + 1050976t^2 + 2795373t - 62181. \tag{3.18}$$

Then $f(\frac{1}{15}) = \frac{2174963624}{16875} > 0$. Therefore, the real root of the equation (3.16) is on the interval $(0, \frac{1}{15})$.

Thus, the proof of Theorem 1.2 is completed. ∎

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