

Primes in the Quadratic Intervals

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Abstract

In this note, we prove that for $n \geq 30$, there exists at least a prime number in the interval $\left(n^2, (n + f(n))^2\right]$ in which $f(n)$ is a function with the order of $O\left(\frac{n}{\ln^2 n}\right)$, and we count the number of primes in this interval. By using the result of this counting, we estimate the probability that a prime exists in the interval $(n^2, (n + 1)^2)$. Also, we show that there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, the interval $\left[(n - g(n))^2, n^2\right)$, in which $g(n) = O(n^{\frac{1}{20}})$.

2000 *Mathematics Subject Classification*: 11A41, 46N30, 11N05.

Keywords: primes, probability, distribution of primes.

1 Introduction

It seems that the story of studying intervals containing primes becomes to Bertrand, where he conjectured that for all $n \in \mathbb{N}$, $\mathbb{P} \cap (n, 2n] \neq \emptyset$. Note that, as usual, we let \mathbb{P} be the set of all prime numbers. In this area, many activities have done until now,

for example two of more recent of them are as follows:

► In 1999, P. Dusart [2] showed that for every $x \geq 3275$, we have

$$\mathbb{P} \cap \left(x, x \left(1 + \frac{1}{2 \ln^2 x} \right) \right] \neq \phi.$$

► In 2001, Baker, Harman and Pintz [1] proved that there exists real x_0 such that for all $x > x_0$ the interval $[x - x^{0.525}, x]$ contains a prime.

Ok! What we are going to do? Before answering, we introduce some notations: We denote the interval $\left(n^2, (n + f(n))^2 \right]$, by $I_{+f(n)}^2$. So, we have $I_{+1}^2 = \left(n^2, (n + 1)^2 \right]$.

Also, by $I_{-g(n)}^2$ we will mean the interval $\left[(n - g(n))^2, n^2 \right)$.

In this note, we study existence of primes in the intervals $I_{+f(n)}^2$ and $I_{-g(n)}^2$. Also, we consider the following open problem:

$$\mathbb{P} \cap I_1^2 \neq \phi \quad (n \in \mathbb{N}).$$

We study the probabilistic existence of primes in the interval I_{+1}^2 . For estimate above probability, we will need the following sharp bounds for the function $\pi(x) = \#\mathbb{P} \cap [1, x]$:

$$L(x) = \frac{x}{\ln x} \left(1 + \frac{1}{\ln x} + \frac{1.8}{\ln^2 x} \right) \leq \pi(x) \quad (x \geq 32299),$$

and

$$\pi(x) \leq U(x) = \frac{x}{\ln x} \left(1 + \frac{1}{\ln x} + \frac{2.51}{\ln^2 x} \right) \quad (x \geq 355991).$$

These results are due to P. Dusart [2].

2 Existence of Primes in $I_{+f(n)}^2$ and $I_{-g(n)}^2$

2.1 Study of $I_{+f(n)}^2$

In this part, we prove the following theorem:

Theorem 1 *If $n \geq 30$, then we have*

$$\mathbb{P} \cap I_{+f(n)}^2 \neq \phi,$$

in which,

$$f(n) = \frac{n}{8 \ln^2 n + \sqrt{8 \ln n} \sqrt{1 + 8 \ln^2 n}} = O\left(\frac{n}{\ln^2 n}\right).$$

For prove this theorem, we need the following lemma which help us to change the form of intervals.

Lemma 1 *Suppose $a, b > 0$. We have,*

$$a^2 + b^2 = \left(a + \frac{b^2}{a + \sqrt{a^2 + b^2}} \right)^2.$$

Proof. Let $a^2 + b^2 = (a + \frac{b}{M})^2$, with $M > 0$. Solving this quadratic equation with respect to M , we have

$$M = \frac{a}{b} + \sqrt{1 + \left(\frac{a}{b}\right)^2},$$

and this yields the result. □

Now, proof of theorem 1:

Proof. For $30 \leq n \leq 57$ we can check the result by computer. For $n \geq 58 = \lceil \sqrt{3275} \rceil$, according to P. Dusart [2], there exists at least a prime p such that

$$n^2 < p \leq n^2 \left(1 + \frac{1}{2 \ln^2(n^2)} \right).$$

Now, by lemma 1, we have

$$n^2 \left(1 + \frac{1}{2 \ln^2(n^2)} \right) = n^2 + \left(\frac{n}{\sqrt{8} \ln n} \right)^2 = (n + f(n))^2,$$

such that,

$$f(n) = \frac{n}{8 \ln^2 n + \sqrt{8} \ln n \sqrt{1 + 8 \ln^2 n}} = O\left(\frac{n}{\ln^2 n}\right).$$

This completes the proof. □

Note 1 *The truth of theorem 1, holds also for*

$$n = 2, 4, 6, 9, 10, 14, 15, 16, 17, 20, 21, 22, 24, 25, 26, 27, 28.$$

2.2 Study of $I_{-g(n)}^2$

Theorem 2 *There exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$ we have*

$$\mathbb{P} \cap I_{-g(n)}^2 \neq \phi,$$

in which,

$$g(n) = n - \sqrt{n^2 - n^{1.05}} = O(n^{\frac{1}{20}}).$$

Proof. We know that [1], there exists real x_0 such that for all $x > x_0$ we have

$$\mathbb{P} \cap [x - x^{0.525}, x] \neq \phi.$$

Let $x = n^2$. So, for $n > n_0 = \lceil \sqrt{x_0} \rceil$ we have $\mathbb{P} \cap [n^2 - n^{1.05}, n^2] \neq \phi$. Now, let $n^2 - n^{1.05} = (n - g(n))^2$. This completes the proof. \square

Note 2 We can see that $g(n) \sim \frac{1}{2}n^{\frac{1}{20}}$. Beside, we have the following bounds for $g(n)$:

$$\frac{1}{2}n^{\frac{1}{20}} < g(n) < n^{\frac{1}{20}}.$$

which hold for all $n > 1$. Also, for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$ we have

$$\frac{1}{2}n^{\frac{1}{20}} < g(n) < \frac{1}{2 - \epsilon}n^{\frac{1}{20}}.$$

3 How Many Primes?

3.1 Counting Primes in $I_{+f(n)}^2$

Let $F(n)$ is the number of primes in $I_{+f(n)}^2$; i.e.

$$F(n) = \#\mathbb{P} \cap I_{f(n)}^2 = \pi\left((n + f(n))^2\right) - \pi(n^2).$$

By using Prime Number Theorem we can see that:

$$F(n) \sim \frac{1}{2} \left(\frac{(n + f(n))^2}{\ln(n + f(n))} - \frac{n^2}{\ln n} \right) \quad (n \rightarrow \infty).$$

Beside, by considering asymptotic behavior of $f(n)$ we yield:

$$F(n) \sim \frac{1}{32} \left(\frac{\left(n + \frac{n}{\ln^2 n}\right)^2}{\ln\left(n + \frac{n}{\ln^2 n}\right)} - \frac{n^2}{\ln n} \right) \quad (n \rightarrow \infty).$$

Theorem 1, asserts that for $n \geq 58$ we have $F(n) > 0$. By using P. Dusart's bounds on $\pi(x)$ we can yield the following bounds for $F(n)$:

$$L\left((n + f(n))^2\right) - U(n^2) \leq F(n) \leq U\left((n + f(n))^2\right) - L(n^2),$$

which holds for all $n \geq \max\left\{\left\lceil \sqrt{355991} \right\rceil, \left\lceil \sqrt{32299} \right\rceil\right\} = 597$.

But, since $\lim_{n \rightarrow \infty} L\left((n + f(n))^2\right) - U(n^2) = -\infty$, we replace above lower bound by trivial one, 1. So, we have

$$1 \leq F(n) \leq U\left((n + f(n))^2\right) - L(n^2) \quad (n \geq 597).$$

About sharp lower and upper bounds for $F(n)$, we have the following conjecture which supported by some computational evidences:

Conjecture 1 *For every $\epsilon > 0$, there exists $n_0(\epsilon) \in \mathbb{N}$, such that for every $n \geq n_0$ we have*

$$\frac{1}{32 - \epsilon} \left(\frac{\left(n + \frac{n}{\ln^2 n}\right)^2}{\ln \left(n + \frac{n}{\ln^2 n}\right)} - \frac{n^2}{\ln n} \right) \leq F(n) \leq \frac{1}{32 + \epsilon} \left(\frac{\left(n + \frac{n}{\ln^2 n}\right)^2}{\ln \left(n + \frac{n}{\ln^2 n}\right)} - \frac{n^2}{\ln n} \right).$$

3.2 Probabilistic Existence of Primes in I_1^2

Estimating of $F(n)$ can be useful in the following theorem.

Theorem 3 *The probability that the interval I_1^2 contains a prime is*

$$1 - \left(\frac{\left(n + f(n)\right)^2 - (n + 1)^2}{\left(n + f(n)\right)^2 - n^2} \right)^{F(n)}.$$

Note that $f(n)$ and $F(n)$ are defined in above.

Proof. There are $F(n)$ primes between n^2 and $(n + f(n))^2$. Since $n^2 < (n + 1)^2 < (n + f(n))^2$, and because these primes distributed randomly, the probability that all of these primes are between $(n + 1)^2$ and $(n + f(n))^2$ is equal to

$$\left(\frac{\left(n + f(n)\right)^2 - (n + 1)^2}{\left(n + f(n)\right)^2 - n^2} \right)^{F(n)},$$

and this yields the result. □

References

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